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Partitions of the reals and models of \(ZF\)

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Abstract

We consider several partition relations and describe models of \(ZF\) which can be used to distinguish between them. This is an extended abstract of a talk delivered in the RIMS Symposium on Axiomatic Set Theory and Set Theoretic Topology, held at RIMS University of Kyoto, 28-30 November 2008.

1 Introduction.

We consider partitions of the Baire space \(\omega^\omega\) of all infinite sequences of natural numbers with the product topology obtained giving to \(\omega\) the discrete topology, and also partitions of its closed subspace \([\omega]^\omega\) of all infinite subsets of \(\omega\), which can be identified with the strictly increasing sequences of natural numbers. If \(A\) is an infinite set of natural numbers, we use \([A]^\omega\) to denote the set of infinite subsets of \(A\).

Definition 1 Given \(n \in \omega\), we say that a partition \(c : [\omega]^\omega \to n\) is Ramsey if there is \(H \in [\omega]^\omega\) such that \(c\) is constant on \([H]^\omega\). Such a set \(H\) is said to be homogeneous for \(c\).

One of the emblematic results in this area is the following theorem of F. Galvin and K. Prikry

Theorem 2 [5] For every \(n \in \omega\), every Borel measurable partition \(c : [\omega]^\omega \to n\) is Ramsey.

The notation

\[
\omega \rightarrow (\omega)_n^\omega
\]

is used to express that for every \(\Gamma\)-measurable \(c : [\omega]^\omega \rightarrow n\), there is \(H \in [\omega]^\omega\) such that \(c\) is constant on \([H]^\omega\). So, the Galvin-Prikry theorem is

\[
\forall n (\omega \rightarrow (\omega)_n^\omega). \\
\text{Borel}
\]
If no class $\Gamma$ is mentioned, the partition symbol refers to all functions $c : [\omega]^\omega \rightarrow n$. Also, if $n = 2$, the subindex is usually omitted.

It is well known that $\omega \rightarrow (\omega)^\omega$ implies that there are no non-principal ultrafilters on $\omega$; so, $ZFC$ proves that this partition relation is false. Nevertheless, a celebrated result of Mathias [7] shows that this partition relation is consistent with $ZF + DC$, provided that the existence of an inaccessible cardinal is consistent.

2 Infinite partitions.

It is easy to find a clopen non-Ramsey partition of $[\omega]^\omega$ into infinitely many pieces. Namely, $h : [\omega]^\omega \rightarrow \omega$ defined by $h(A) = \min(A)$. Thus, $ZF$ proves $\omega \not\rightarrow (\omega)^\omega$.

It is interesting to consider a version of $\omega \not\rightarrow (\omega)^\omega$ that requires only the existence of a set of the form $[H]^\omega$ which avoids a piece of the partition, instead of requiring that it is contained in a single piece. For this type of partition relation it is customary to use the following notation. The expression

$$\omega \rightarrow [\omega]^\omega_{K}$$

means that for every $\Gamma$-measurable $c : [\omega]^\omega \rightarrow K$, there is $H \in [\omega]^\omega$ such that $c''[H]^\omega \subsetneq K$.

It is straightforward to verify that this partition relation holds for Borel partitions, but again, the Axiom of Choice implies that there are partitions of $[\omega]^\omega$ into infinitely many pieces for which every set of the form $[H]^\omega$ meets every piece. In fact, we have the following.

**Proposition 3** If there is a non-principal ultrafilter on $\omega$, then

$$\omega \not\rightarrow [\omega]_{2^\omega}^\omega,$$

Actually, a weaker hypothesis is enough to refute the partition relation

$$\omega \rightarrow [\omega]_{2^\omega}^\omega,$$

namely, the existence of a non-principal non-meager filter on $\omega$. This result is part of ongoing work done jointly with S. Todorcevic and will appear elsewhere.

3 Homogeneous sublattices and perfect sets.

We now turn to a different type of partition property, which was first considered in [4].

We use the symbol

$$\omega \rightarrow ((\omega))_{n}^\omega$$
to express that for every $\Gamma$-measurable function $c : [\omega]^\omega \to n$, there are $A, B \in [\omega]^\omega$, with $A \subseteq B$ and $B \setminus A \in [\omega]^\omega$, such that $c$ is constant on the sublattice of subsets of $B$ given by $[A, B] = \{X \subseteq B : A \subseteq X\}$.

It is easily seen that the relation

$$\omega \rightarrow ((\omega))_n^\omega$$

Borel

follows from

$$\omega \rightarrow ((\omega))_n^\omega.$$

And just as in the case of $\omega \rightarrow (\omega)^\omega$, the existence of a non-principal ultrafilter on $\omega$ implies that $\omega \not\rightarrow (\omega)^\omega$.

The third type of partition relation we consider here is denoted by

$$\omega \rightarrow (\text{perfect})_n^\omega$$

meaning that for every $\Gamma$-measurable function $c : [\omega]^\omega \rightarrow n$, there is a perfect set $P \subseteq [\omega]^\omega$ on which $c$ is constant.

A Bernstein set is just a counterexample to $\omega \rightarrow (\text{perfect})^\omega$, this is, a set $B$ with the property that both $B$ and its complement meet every perfect set. Such a set can be obtained from a well ordering of the reals.

In his article [8] Solovay, assuming the consistency of inaccessible cardinals, constructed a model of $ZF$ where every set of reals is Lebesgue measurable, has the property of Baire, and if not countable, contains a perfect subset. Of course, the axiom of choice does not hold in this model, although the axiom of dependent choices does. In general, a model $M$ of $ZF$ is said to be a Solovay model if it is (elementary equivalent to) the model $L(\mathbb{R})$ computed in the Levy collapse of an inaccessible cardinal to $\aleph_1$. The result of Mathias mentioned above ([17]), establishes that the partition property $\omega \rightarrow (\omega)^\omega$ holds in Solovay models; therefore the same is true for the properties $\omega \rightarrow ((\omega))^\omega$, $\omega \rightarrow [\omega]^\omega_2$, and $\omega \rightarrow (\text{perfect})^\omega$ which follow from it.

Consider now the model $L(\mathbb{R})[\mathcal{U}]$ obtained adding a selective ultrafilter to a Solovay model $L(\mathbb{R})$ using the poset of infinite subsets of $\omega$ ordered by the relation of almost containment.

It was shown in [2] that $\omega \rightarrow (\text{perfect})^\omega$ holds in $L(\mathbb{R})[\mathcal{U}]$. This was done proving that in Solovay models, the following parameterized partition relation holds: for every $n \in \omega$ and every $c : [\omega]^\omega \times [\omega]^\omega \rightarrow n$, there is $H \in [\omega]^\omega$ and a perfect set $P \subseteq [\omega]^\omega$ such that $c$ is constant on the product $[H]^\omega \times P$.

Therefore, the existence of a non-principal ultrafilter on $\omega$ is a consequence of the Axiom of Choice not strong enough to produce a Bernstein set. By our previous remarks about non-principal ultrafilters, none of the other properties hold in the model $L(\mathbb{R})[\mathcal{U}]$, since in it $\mathcal{U}$ is non-principal ultrafilter on $\omega$. 


4 Cohen extensions

Adding Cohen generic reals to the constructible universe $L$, we obtain a model in which

$$\omega \rightarrow ((\omega))^\omega$$

holds but there is a $\Delta^1_2$ counterexample for $\omega \rightarrow (\omega)^\omega$.

We start from $L$, and add $\omega_1$-many Cohen generic reals using the $\omega_1$ product of Cohen forcing with finite support. In, [1] it is shown that in this extension the partition relation $\omega \rightarrow ((\omega))^\omega$ holds for projective partitions.

It follows from [6], 2.2, that in this model there is a $\Delta^1_2$ counterexample for $\omega \rightarrow (\omega)^\omega$, i.e. there is a $\Delta^1_2$ non-Ramsey set.

In fact, the relation $\omega \rightarrow ((\omega))^\omega$ holds in the extension for partitions definable with real parameters, and so, it also holds in the inner model $L(\mathbb{R})$ of all the sets in the extension that are constructible from reals. In this way we obtain a model in which $\omega \rightarrow ((\omega))^\omega$ holds but $\omega \rightarrow (\omega)^\omega$ does not.

The model obtained adding $\omega_2$-many Cohen generic reals to $L$ offers additional features. For example, in this model there is a non-meager non-principal filter on $\omega$. Taking the appropriate inner model we obtain a model in which $\omega \rightarrow ((\omega))^\omega$ holds, but $\omega \rightarrow [\omega]_{2^\omega}$ fails.

5 Conclusion.

Sumarizing, we have that $\omega \rightarrow (\omega)^\omega$ implies both $\omega \rightarrow ((\omega))^\omega$ and $\omega \rightarrow [\omega]_{2^\omega}$, the first implication being strict.

Each of the properties $\omega \rightarrow ((\omega))^\omega$ and $\omega \rightarrow [\omega]_{2^\omega}$ imply $\omega \rightarrow (\text{perfect})^\omega$, and both implications are strict. The partition relation $\omega \rightarrow [\omega]_{2^\omega}$ is not implied by $\omega \rightarrow ((\omega))^\omega$.

Question: What is the exact relationship between the propereties $\omega \rightarrow (\omega)^\omega$ and $\omega \rightarrow [\omega]_{2^\omega}$? (See [3]).

References


