数理解析研究所講究録
第 1596 巻 2008 年 105-118

$A \geq B \geq 0$ ensures $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$ for $p \geq 0, q \geq 1, r \geq 0$ with $(1+r)q \geq p+r$ and brief survey of its recent applications

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§1 Introduction

A capital letter means a bounded linear operator on a Hilbert space H. An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

Theorem LH (1934, Löwner-Heinz inequality, denoted by **(LH)** briefly).
(LH) *If $A \geq B \geq 0$ holds, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.*

This celebrated LH had been originally proved by Löwner (1934) and afterward by Heinz (1951). Many nice proofs of (LH) are known.

Although (LH) asserts that $A \geq B \geq 0$ *ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*, unfortunately *$A^\alpha \geq B^\alpha$ does not always hold for $\alpha > 1$*. The following result has been obtained from this point of view.

Theorem F. (1987, Furuta)
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

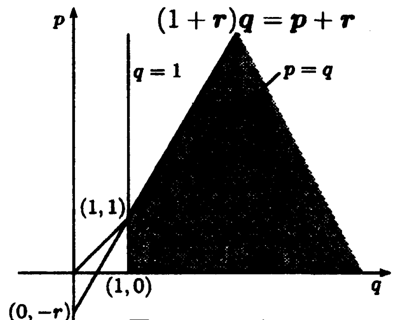


FIGURE 1

In Theorem F,(i) is equivalent to (ii). The domain drawn for p,q and r in FIGURE 1 is best possible one for Theorem F by K.Tanahashi [1]. Consider two magic boxes

$$f(\Box) = (B^{\frac{r}{2}}\Box B^{\frac{r}{2}})^{\frac{1}{q}} \text{ and } g(\Box) = (A^{\frac{r}{2}}\Box A^{\frac{r}{2}})^{\frac{1}{q}}.$$

Theorem F can be regarded as follows. Although $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general, but Theorem F asserts the following *" two order preserving operator inequalities"*

$$f(A^p) \geq f(B^p) \text{ and } g(A^p) \geq g(B^p)$$

hold whenever $A \geq B \geq 0$ under the condition p , q and r in FIGURE 1.

About 20 years have passed since appearance in 1987 of Theorem F. According to remarkable chievements of many mathematicians who have interested with operator inequalities during the 20 years, we have been finding a lot of applications of Theorem F in several branches, briefly speaking, we can devide these branches into the following three branches (A) operator inequalities, (B) norm inequalities, and (C) operator equations.

(A) OPERATOR INEQUALITIES

(A-1) Several characterizations of operators $logA \geq logB$ and its applications;

(A-2) Applications to the relative operator entropy;

(A-3) Applications to Ando-Hiai log majorization and logarithmic trace inequalities;

(A-4) Generalized Aluthge transformation on p-hyponormal operators;

(A-5) Several classes associated with log-hyponormal and paranormal operators;

(A-6) Operator functions implying order preserving inequalities.

(A-7) Applications to Kantorovich type operator inequalities.

(B) NORM INEQUALITIES

(B-1) Several generalizations of Heinz-Kato theorem;

(B-2) Generalizations of some theorem on norms;

(B-3) An extension of Kosaki trace inequality and parallel results

(C) OPERATOR EQUATIONS

(C-1) Generalizations of Pedersen-Takesaki theorem and related results.

In this short talk, as the area of applications of Theorem F is vast, I would like to confine myself to some recent applications of Theorem F of my own interest and related topics, so we would like to focus ourselves to state log majorization, logarithmic trace inequalities (A-3) and order preserving operator functions (A-6) without their proofs. We state only proof of Theorem F since Theorem F is the central position of this paper.

Lemma A. (Lemma 1 in Furuta [5]) *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

Proof of Lemma A. Let $YX^{\frac{1}{2}} = UH$ be the polar decomposition of $YX^{\frac{1}{2}}$,where U is unitary and $H = |YX^{\frac{1}{2}}|$. Then we have

$$(YXY^*)^\lambda = (UH^2U^*)^\lambda = YX^{\frac{1}{2}}H^{-1}H^{2\lambda}H^{-1}X^{\frac{1}{2}}Y^* = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*. \square$$

It easily turns out that we don't require the invertibility of A and B in the case $\lambda \geq 1$ in Lemma A which is obviously seen in the proof. Lemma A is very simple with its proof stated above, but quite useful tool in order to treat operator transformation in operator theory.

Proof of Theorem F. At first we prove (ii). In the case $1 \geq p \geq 0$, the result is obvious by Theorem L-H. We have only to consider $p \geq 1$ and $q = \frac{p+r}{1+r}$ since (ii) of Theorem F for values q larger than $\frac{p+r}{1+r}$ follows by Theorem L-H, that is, we have only to prove the following

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \textit{for any } p \geq 1 \textit{ and } r \geq 0. \tag{1.1}$$

We may assume that A and B are *invertible* without loss of generality. In the case $r \in [0,1]$, $A \geq B \geq 0$ ensures $A^r \geq B^r$ holds by Theorem L-H. Then we have

$$\begin{aligned}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} &= A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{\frac{-p}{2}}A^{-r}B^{\frac{-p}{2}})^{\frac{p-1}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}} \quad \text{by Lemma A}\\ &\leq A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{\frac{-p}{2}}B^{-r}B^{\frac{-p}{2}})^{\frac{p-1}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}}\\ &= A^{\frac{r}{2}}BA^{\frac{r}{2}} \leq A^{1+r},\end{aligned}$$

and the first inequality follows by $B^{-r} \geq A^{-r}$ and Theorem L-H since $\frac{p-1}{p+r} \in [0,1]$ holds, and the last inequality follows by $A \geq B \geq 0$, so we have the following (1.2)

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \textit{for } p \geq 1 \textit{ and } r \in [0,1]. \tag{1.2}$$

Put $A_1 = A^{1+r}$ and $B_1 = (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ in (1.2). Repeating (1.2) again for $A_1 \geq B_1 \geq 0$, $r_1 \in [0,1]$ and $p_1 \geq 1$,

$$A_1^{1+r_1} \geq (A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}$$

Put $p_1 = \frac{p+r}{1+r} \geq 1$ and $r_1 = 1$, then

$$A^{2(1+r)} \geq (A^{r+\frac{1}{2}}B^pA^{r+\frac{1}{2}})^{\frac{2(1+r)}{p+2r+1}} \quad \textit{for } p \geq 1, \textit{ and } r \in [0,1]. \tag{1.3}$$

Put $\frac{s}{2} = r + \frac{1}{2}$ in (1.3). Then $\frac{2(1+r)}{p+2r+1} = \frac{1+s}{p+s}$ since $2(1+r) = 1+s$, so that (1.3) can be rewritten as follows;

$$A^{1+s} \geq (A^{\frac{s}{2}}B^pA^{\frac{s}{2}})^{\frac{1+s}{p+s}} \quad \textit{for } p \geq 1, \textit{ and } s \in [1,3]. \tag{1.4}$$

Consequently (1.2) and (1.4) ensure that (1.2) holds for *any* $r \in [0,3]$ since $r \in [0,1]$ and $s = 2r+1 \in [1,3]$ and repeating this process, (1.1) holds for *any* $r \geq 0$, (ii) is shown.

If $A \geq B > 0$, then $B^{-1} \geq A^{-1} > 0$. Then by (ii), for each $r \geq 0$, $B^{\frac{-(p+r)}{q}} \geq (B^{\frac{-r}{2}}A^{-p}B^{\frac{-r}{2}})^{\frac{1}{q}}$ holds for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+r)q \geq p+r$. Taking inverses gives (i), so the proof of Theorem F is complete. □

This one page proof in Furuta [3] and the original one in Furuta [1], afterward, in Fijii [1] and Kamei [1].

Remark 1.1. Recall that the essential assert of Theorem F is as follows since Theorem F is obvious in case $1 \geq p \geq 0$ by Theorem L-H:

$$A \geq B \geq 0 \Longleftrightarrow A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \text{ for } p \geq 1 \text{ and } r \geq 0.$$

Theorem GF (Generalization of Theorem F). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$F(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geq t$ and $s \geq 1$, and $A^{1-t} = F(A,A,r,s) \geq F(A,B,r,s)$, that is,

(GF) $$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$.

The original proof of Theorem GF is in Furuta [5], and an alternative one is in M.Fijii-Kamei [1]. An elementary one-page proof of (GF) is in Furuta [7]. Further extensions of Theorem GF and related results are obtained by many researchers, and some of them are in Furuta [9], Furuta-Hashimoto-Ito [1], Furuta-Yanagida-Yamazaki [1], Lin [1] and Kamei [2][3]. It is originally shown in Tanahashi [2] that the exponent value $\dfrac{1-t+r}{(p-t)s+r}$ of the right hand of (GF) is best possible and alternative proofs of this fact are in Fujii-Matsumoto-Nakamoto [1], Yamazaki [1]. (GF) interpolates Theorem F and an inequality equivalent to the main result of log majorization in Ando-Hiai [1] (see Remark 2.1 in §2). Recently extensions and generalizations of Theorem F are in M.Uchiyama [2] and M.Yanagida [2].

§2 Fundamental results associated with log majorization

In this section a capital letter means $n \times n$ matrix. Following Ando and Hiai [1], let us define the *log majorization* for positive semidefinite matrices $A, B \geq 0$, denoted by $A \underset{(\log)}{\succ} B$ if

$$\prod_{i=1}^{k} \lambda_i(A) \geq \prod_{i=1}^{k} \lambda_i(B), \ k = 1,2,...,n-1, \text{ and } \prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B), \quad \text{i.e., } \det A = \det B,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq ... \geq \lambda_n(B)$ are the eigenvalues of A and B, respectively, arranged in decreasing order. When $0 \leq \alpha \leq 1$, the α-power mean of positive invertible matrices $A, B > 0$ is defined by in Kubo-Ando [1]

$$A \#_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}.$$

Further, $A\#_\alpha B$ for $A, B \geq 0$ is defined by $A\#_\alpha B = \lim_{\epsilon \downarrow 0}(A+\epsilon I)\#_\alpha(B+\epsilon I)$.

For the sake of convenience for symbolic expression, we define $A \natural_s B$, for any real number $s \geq 0$ and for $A > 0$ and $B \geq 0$, by the following

$$A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^sA^{\frac{1}{2}}.$$

$A \natural_\alpha B$ in the case $0 \leq \alpha \leq 1$ just coincides with the usual α-power mean. The following excellent and useful log majorization is shown in Ando and Hiai [1, Theorem 2.1].

Theorem A. *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,*

$$(A\#_\alpha B)^r \underset{(\log)}{\succ} A^r\#_\alpha B^r \qquad \text{for } r \geq 1. \tag{2.1}$$

Also, (2.1) can be transformed into the following matrix inequality (2.2) of Theorem B in Ando and Hiai [1,Theorem 3.5]:

Theorem B. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}} \qquad \text{for } r,p \geq 1. \tag{2.2}$$

We obtained the following extension of Theorem A in Furuta [5, Therorem 2.1] applying the method in Ando and Hiai [1] to (GF) of Theorem GF in §1.

Theorem C. *For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and for each $t \in [0,1]$,*

$$(A\#_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r}\#_\beta\,(A^{1-t}\natural_s B)$$

holds for $s \geq 1$, and $r \geq t \geq 0$, where $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}$ and $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

Remark 2.1. The inequality (GF) in Theorem GF interpolates Theorem F and Theorem B, in fact, when we put $t = 1$ and $r = s$ in (GF), we have Theorem B, and when we put $t = 0$ and $s = 1$ in (GF), we have Theorem F by Remark 1.1. Also when we put $t = 1$ and $r = s$ in Theorem C which is equivalent to (GF), we have Theorem A.

Next, we state the following result which is shown in Hiai and Petz [1, Theorem 3.5] and, recently, a new proof is given in Bebiano, Lemos and Providencia [1, Theorem 2.2].

Theorem D. *If $A, B \geq 0$, then for every $p \geq 0$*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] \geq \mathrm{Tr}[A(\log A + \log B)] \tag{2.3}$$

holds and the left hand side of (2.3) converges the right hand side as $p \downarrow 0$.

Theorem E. *If $A \geq 0$, $B > 0$, $0 \leq \alpha \leq 1$ and $p > 0$, then*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^p\natural_\alpha B^p)] + \frac{\alpha}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A] \tag{2.4}$$

holds and the left hand side of (2.4) converges the the right hand side as $p \downarrow 0$.

The inequality (2.4) is shown in Ando and Hiai [1, Theorem 5.3], and the convergence of (2.4) is shown in Bebiano, Lemos and Providencia [1, Corollary 2.2].

We extend Theorem D and Theorem E by applying the trace inequality derived from log majorization equivalent to an order preserving inequality. We show a log majorization equivalent to an order preserving operator inequality.

Theorem 2.1. *The following* (i) *and* (ii) *hold and are equivalent:*

(i) *If $A, B \geq 0$, then for each $t \in [0,1]$ and $r \geq t$*

$$A^{\frac{1}{2}}(A^{\frac{r-t}{2}}B^{p}A^{\frac{r-t}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{(p-tq)s+rq}{2ps}}\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{r}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{ps}}A^{\frac{(p-tq)s+rq}{2ps}}$$

holds for any $s \geq 1$ and $p \geq q > 0$.

(ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $r \geq t$*

$$A^{\frac{(p-tq)s+rq}{ps}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{\frac{p}{q}}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q}{ps}}$$

holds for any $s \geq 1$ and $p \geq q > 0$.

Theorem 2.2. *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$\frac{s}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^{p}A^{\frac{p}{2}})] - \frac{1}{p}\mathrm{Tr}[A\log\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{p}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}] \geq \mathrm{Tr}[A\log A] \tag{6.1}$$

holds for any $p \geq 0$ and $s \geq 1$, and the left hand side converges to the right hand side as $p \downarrow 0$.

Corollary 2.3.

(i) *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^{p}A^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A + A\log B] \tag{6.2}$$

holds and the left hand side converges to the right hand side as $p \downarrow 0$.

(ii) *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$\frac{2}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^{p}A^{\frac{p}{2}})] - \frac{1}{p}\mathrm{Tr}[A\log(B^{p}A^{p}B^{p})] \geq \mathrm{Tr}[A\log A] \tag{6.3}$$

holds and the left hand side converges to the right hand side as $p \downarrow 0$.

We remark that (i) of Corollary 2.3 is Theorem D.

Theorem 2.4. *If $A > 0$ and $B \geq 0$, then, for every positive number β,*

$$\frac{s}{p}\mathrm{Tr}[A\log(A^{p}\natural_{\beta}B^{p})] - \frac{1}{p}\mathrm{Tr}[A\log\{A^{\frac{-p}{2}}(A^{p}\natural_{\beta}B^{p})^{s}A^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A] \tag{6.4}$$

holds for any $p \geq 0$, $s \geq 1$, and the left hand side converges to the right hand side as $p \downarrow 0$.

Corollary 2.5.

(i) *If $A, B > 0$, then, for every positive number β,*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^{p}\natural_{\beta}B^{p})] + \frac{\beta}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A] \tag{6.5}$$

holds for any $p \geq 0$, and the left hand side converges to the right hand side as $p \downarrow 0$.

(ii) *If $A, B > 0$, then, for every positive number β,*

$$\frac{2}{p}\mathrm{Tr}[A\log(A^{p}\natural_{\beta}B^{p})] - \frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{-p}{2}}B^{p}A^{\frac{-p}{2}})^{\beta}A^{p}(A^{\frac{-p}{2}}B^{p}A^{\frac{-p}{2}})^{\beta}] \geq \mathrm{Tr}[A\log A] \tag{6.6}$$

holds for any $p \geq 0$ and the left hand side converges to the right hand side as $p \downarrow 0$.

We remark that, when $A \geq 0$, $B > 0$ and $\beta \in [0,1]$, (i) of Corollary 2.5 becomes Theorem E.

§3 Operator inequality implying generalized Bebiano-Lemos-Providência one

Let $A, B \geq 0$ and $0 \leq \alpha \leq 1$. The famous Araki-Cordes inequality states that $(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\alpha} \underset{(\log)}{\succ} A^{\frac{\alpha}{2}}B^{\alpha}A^{\frac{\alpha}{2}}$ holds and also Bebiano-Lemos-Providência inequality [1] asserts that

$$A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{s}A^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1+t}{2}}B^{t}A^{\frac{1+t}{2}} \textit{ holds for } s \geq t \geq 0.$$

Very recently, Fujii-Nakamoto-Tominaga [1, Theorem 2.1 and Corollary 2.2] have shown the following interesting norm inequality:

Let $A, B \geq 0$. Then

$$||A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{p+s}A^{\frac{s}{2}})A^{\frac{1}{2}}||^{\frac{p(1+s)}{p+s}} \geq ||A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}||$$

holds for all $p \geq 1$ and $s \geq 0$.

In fact, this result is essentially equivalent to the following Theorem FNT, whicg is essentially shown Fujii-Nakamoto-Tominaga [1], as an extension of both Araki-Cordes inequality and Bebiano-Lemos-Providência one:

Theorem FNT. *For every $A, B \geq 0$ and $p \geq 1$,*

$$\{A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{p+s}A^{\frac{s}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\}^{\frac{p(1+s)}{p+s}} \underset{(\log)}{\succ} A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}$$

holds for any $s \geq 0$.

As an application of (G-1) of Theorem G, we shall give an operator inequality implying generalized Bebiano-Lemos-Providência one.

Theorem 3.1 Furuta [11]. *The following* (i) *and* (ii) *hold and they are equivalent:*

(i). *For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and each $t \in [0,1]$, and any real number $q \neq 0$,*

$$\{A^{\frac{q}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\alpha}A^{\frac{q}{2}}\}^{h} \underset{(\log)}{\succ} A^{\frac{q(1-t+r)}{2}}\{A^{\frac{-qr}{2}}(A^{\frac{1+qt}{2}}BA^{\frac{1+qt}{2}})^{s}A^{\frac{-qr}{2}}\}^{\beta}A^{\frac{q(1-t+r)}{2}} \tag{3.1}$$

holds for $s \geq 1$ and $r \geq t$, where $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}$ and $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

(ii). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{3.2}$$

holds for $s \geq 1$ and $r \geq t$.

Remark 3.1. (3.1) *in* (i) *of Theorem 3.1 can be rewritten as follows: For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and each $t \in [0,1]$, and any real number $q \neq 0$,*

$$\text{(3.1')} \qquad \{A^{\frac{1+q}{2}}(A^{-1}\natural_\alpha B)A^{\frac{1+q}{2}}\}^h \underset{(\log)}{\succ} A^{\frac{1+q}{2}}\{A^{q(r-t)-1}\natural_\beta(A^{-(1+qt)}\natural_s B)\}A^{\frac{1+q}{2}}$$

holds for $s \geq 1$ *and* $r \geq t$, *where* $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r,}$ and $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

Remark 3.2. Put $q = -1$ and replace A by A^{-1} in (3.1'), then (i) of Theorem 3.1 yields the following result (a). Moreover, (a) implies (b) by putting $t = 1$ and $r = s$.

(a) *For every* $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ *and each* $t \in [0, 1]$

$$(A\natural_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r}\natural_\beta(A^{1-t}\natural_s B)$$

holds for $s \geq 1$ *and* $r \geq t$, *where* $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}$ *and* $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

(b) *For every* $A, B \geq 0$, $0 \leq \alpha \leq 1$

$$(A\natural_\alpha B)^r \underset{(\log)}{\succ} A^r\natural_\alpha B^r \qquad r \geq 1.$$

In fact (a) is Theorem C itself in §2 and (b) is Theorem A itself in §2,which is a very important result in log majorization.

Corollary 3.2. *The following* (i), (ii) *and* (iii) *hold and they are equivalent:*

(i) *For every* $A, B \geq 0$, $0 \leq \alpha \leq 1$ *and any real number* $q \neq 0$,

$$\{A^{\frac{q}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^\alpha A^{\frac{q}{2}}\}^{\frac{1+r}{1+\alpha r}} \underset{(\log)}{\succ} A^{\frac{q(1+r)}{2}}(A^{\frac{1-qr}{2}}BA^{\frac{1-qr}{2}})^{\frac{\alpha(1+r)}{1+\alpha r}}A^{\frac{q(1+r)}{2}}$$

holds for any $r \geq 0$.

(ii) *If* $A \geq B \geq 0$, *then for* $p \geq 1$,

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

holds for any $r \geq 0$.

(iii) *For every* $A, B \geq 0$, $p \geq 1$ *and any real number* $q \neq 0$,

$$\{A^{\frac{sq}{2}}(A^{\frac{s}{2}}B^{p+s}A^{\frac{s}{2}})^{\frac{1}{p}}A^{\frac{sq}{2}}\}^{\frac{p(1+r)}{p+r}} \underset{(\log)}{\succ} A^{\frac{sq(1+r)}{2}}(A^{\frac{s(1-qr)}{2}}B^{p+s}A^{\frac{s(1-qr)}{2}})^{\frac{1+r}{p+r}}A^{\frac{sq(1+r)}{2}}$$

holds for any $r \geq 0$ *and* $s \geq 0$.

Corollary 3.3. *The following* (i), (ii) *and* (iii) *hold and they are equivalent:*

(i) *For every* $A, B \geq 0$ *and* $0 \leq \alpha \leq 1$,

$$\{A^{\frac{q}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^\alpha A^{\frac{q}{2}}\}^{\frac{1+q}{\alpha+q}} \underset{(\log)}{\succ} A^{\frac{1+q}{2}}B^{\frac{\alpha(1+q)}{\alpha+q}}A^{\frac{1+q}{2}}$$

holds for any $q \geq 0$.

(ii) *If* $A \geq B \geq 0$, *then for* $p \geq 1$,

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

holds for any $r \geq 0$.

(iii) *For every* $A, B \geq 0$ *and* $p \geq 1$,

$$\{A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{p+s}A^{\frac{s}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\}^{\frac{p(1+s)}{p+s}} \underset{(\log)}{\succ} A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}$$

holds for any $s \geq 0$.

Remark 3.3. We remark that (i) of Theorem 3.1 is "*log majorization equivalent to Theorem G in matrix case* ", and (i), (iii) of Corollary 3.2 and also (i), (iii) of Corollary 3.3 are all considered as "*log majorization equivalent to an essential part of Theorem F in matrix case* ". Needless to say, (iii) of Corollary 3.3 is Theorem FNT itself. And the equivalence between (i) and (iii) in Corollary 3.3 is essentially shown in Mujii-Nakamoto-Tominaga [1].

§4 Decreasing monotonicity of order preserving operator functions associated with (GF) in Theorem GF in §1

In this chapter, we state the recent results on decreasing monotonicity of order preserving operator functions associated with (GF) and related satellite order preserving operator inequalities associated with (GF) without proofs.

Theorem 4.1 Furuta [12]. *Let* $A \geq B \geq 0$ *with* $A > 0$, $t \in [0,1]$ *and* $p \geq 1$. *Then*

$$F(\lambda,\mu) = A^{\frac{-\lambda}{2}}\{A^{\frac{\lambda}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\mu}A^{\frac{\lambda}{2}}\}^{\frac{1-t+\lambda}{(p-t)\mu+\lambda}}A^{\frac{-\lambda}{2}}$$

satisfies the following properties:

(i) $F(r,w) \geq F(r,1) \geq F(r,s) \geq F(r,s')$

holds for any $s' \geq s \geq 1$, $r \geq t$ *and* $\dfrac{1-t}{p-t} \leq w \leq 1$.

(ii) $F(q,s) \geq F(t,s) \geq F(r,s) \geq F(r',s)$

holds for any $r' \geq r \geq t$, $s \geq 1$ *and* $t-1 \leq q \leq t$.

We state several satellite inequalities of (GF) in Theorem GF as applications of Theorem 4.1.

Corollary 4.2. *If* $A \geq B \geq 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geq 1$,

(i)
$$\begin{aligned}(A^t\natural_wB^p)^{\frac{1}{(p-t)w+t}} &\geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}\\ &\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}\end{aligned}$$

(ii)
$$\begin{aligned}(A^t\natural_wB^p)^{\frac{1}{(p-t)w+t}} &\geq B \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}}A^{\frac{t-r}{2}}\\ &\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}\end{aligned}$$

hold for $s \geq 1$, $r \geq t$ *and* $\dfrac{1-t}{p-t} \leq w \leq 1$.

Corollary 4.3. *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

(i) $$A^{t-q} \natural_{\frac{1-t+q}{p-t+q}} B^p \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$$
$$\geq A^{\frac{t-r}{2}} \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{t-r}{2}}$$

(ii) $$A^{t-q} \natural_{\frac{1-t+q}{p-t+q}} B^p \geq B \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}} B^p A^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}} A^{\frac{t-r}{2}}$$
$$\geq A^{\frac{t-r}{2}} \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{t-r}{2}}$$

hold for $s \geq 1$, $r \geq t$ and $t-1 \leq q \leq t$.

Very recently, Kamei showed the following interesting result.

Theorem K [Kamei [3]]. *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^t \natural_{\frac{1-t}{p-t}} B^p \geq A^{\frac{t}{2}} F(r,s) A^{\frac{t}{2}} \text{ holds for } r \geq t \text{ and } s \geq 1.$$

Since $A^{\frac{t}{2}} F(r,s) A^{\frac{t}{2}} = A^{\frac{t-r}{2}} \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{t-r}{2}}$ holds, (i) or (ii) of Corollary 4.2 implies Theorem K. Also (i) or (ii) of Corollary 4.3 implies Theorem K.

Corollary 4.2 and Corollary 4.3 easily imply the following known satellete inequalities in [9, § 3.2.5, Corollary 2],

If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,

(i) $\{B^{\frac{t}{2}}(B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^s B^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} \geq A \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$

and

(ii) $B^{\frac{t-r}{2}}(B^{\frac{r-t}{2}} A^p B^{\frac{r-t}{2}})^{\frac{1-t+r}{p-t+r}} B^{\frac{t-r}{2}} \geq A \geq B \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}} B^p A^{\frac{r-t}{2}})^{\frac{1-t+r}{p-t+r}} A^{\frac{t-r}{2}}$

hold for $s \geq 1$, $r \geq t$ and $t \in [0,1]$.

We state *contrast among Theorem 4.1 and related results*

I would list statements (4.1)-(4.4) in the following Remark 4.1 as a concluding remark.

Remark 4.1. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p \geq 1$. Then the following properties hold.*

(4.1) $$F(r,s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is a decreasing function of r and s such that $r \geq t$ and $s \geq 1$.

(4.2) $$F(r,w) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^w A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)w+r}} A^{\frac{-r}{2}}$$

is not a decreasing function of r and w such that $r \geq t$ and $\frac{1-t}{p-t} \leq w \leq 1$, but

$$F(r,w) \geq F(r,1)$$

holds for any $r \geq t$ and $\frac{1-t}{p-t} \leq w \leq 1$.

(4.3) $$F(q,s) = A^{\frac{-q}{2}} \{A^{\frac{q}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{q}{2}}\}^{\frac{1-t+q}{(p-t)s+q}} A^{\frac{-q}{2}}$$

is not a decreasing function of q *and* s *such that* $0 \le q \le t$ and $s \ge 1$, *but*

$$F(q,s) \ge F(t,s)$$

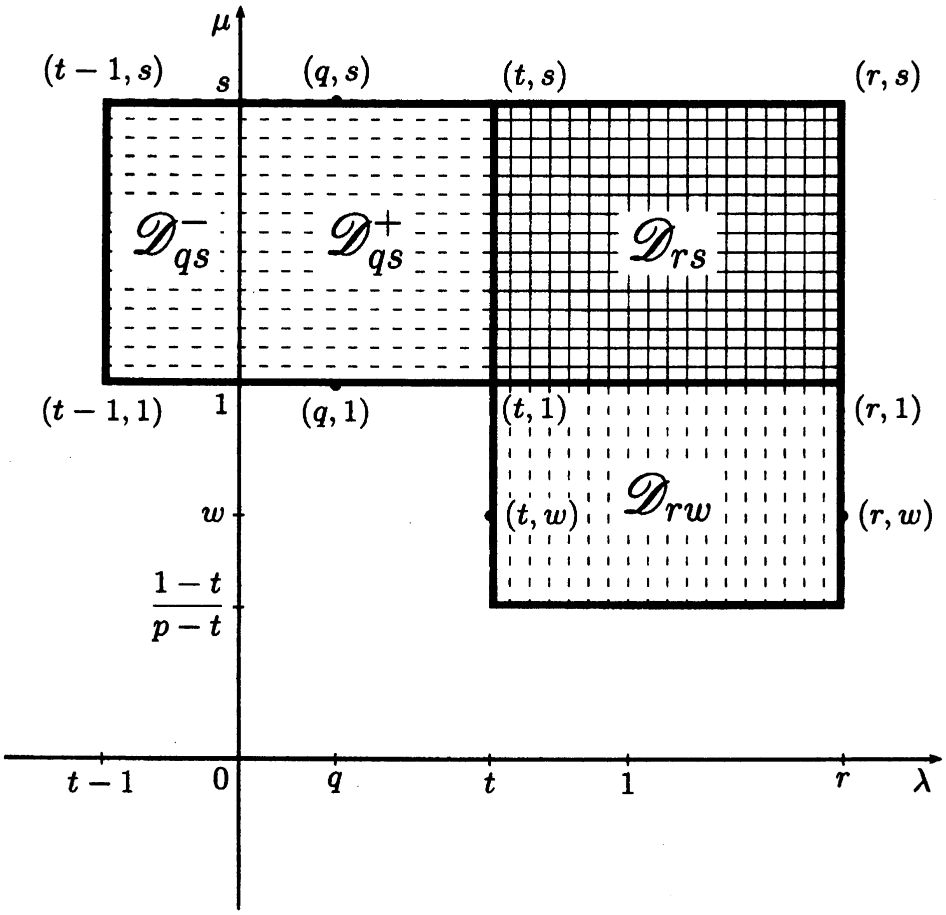
holds for any $0 \le q \le t$ *and* $s \ge 1$.

(4.4) $$F(q,s) = A^{\frac{-q}{2}}\{A^{\frac{q}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{q}{2}}\}^{\frac{1-t+q}{(p-t)s+q}}A^{\frac{-q}{2}}$$

is not a decreasing function of q *and is not an increasing of* s *such that* $t-1 \le q \le 0$ and $s \ge 1$, *but*

$$F(q,s) \ge F(t,s)$$

holds for any $t-1 \le q \le 0$ *and* $s \ge 1$.



$$t \in [0,1],\ p \ge 1,\ r \ge t,\ s \ge 1,$$

$$\frac{1-t}{p-t} \le w \le 1 \text{ and } t-1 \le q \le t$$

FIGURE 2

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