Some remarks on grand Furuta inequality (Inequalities on Linear Operators and its Applications)

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Some remarks on grand Furuta inequality

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1. Introduction. Throughout this note, $A$ and $B$ are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. invertible) operator. We begin from Furuta inequality ([6],[7],[9]).

**Furuta inequality:** If $A \geq B \geq 0$, then for each $r \geq 0$,

$$A^{\frac{r}{p}} \geq (A^\frac{1}{2}B^pA^\frac{1}{2})^\frac{1}{r} \quad \text{and} \quad (B^\frac{r}{2}A^pB^\frac{r}{2})^\frac{1}{r} \geq B^\frac{r}{r+p}$$

holds for $p$ and $q$ such that $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

This yields the Löwner-Heinz inequality;

**(LH)** $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.

We had reformed (F) in terms of the $\alpha$-power mean (or generalized geometric operator mean) of $A$ and $B$ which is introduced by Kubo-Ando as follows [16]:

$$A \#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for} \quad \alpha \in [0, 1],$$

the case $\alpha \notin [0, 1]$, we use the notation $\natural$ to distinguish the operator mean.

By using the $\alpha$-power mean, Furuta inequality is given as follows:

**(F)** $A \geq B \geq 0$ implies $A^\alpha \#_{\frac{p+r}{p}} B^p \leq A$ for $p \geq 1$ and $r \geq 0$.

Based on this reformulation, we had proposed a satellite form of (F) [12],[13];

**(SF)** $A \geq B \geq 0$ implies $A^\alpha \#_{\frac{p+r}{p}} B^p \leq B \leq A$ for $p \geq 1$ and $r \geq 0$.

On the other hand, Ando and Hiai showed the next inequality [1],[11].

**Ando-Hiai inequality:** Ando-Hiai had shown the following inequality:

**(AH)** If $A \#_\alpha B \leq I$ for $A, B > 0$, then $A^r \#_\alpha B^r \leq 1$ holds for $r \geq 1$.

From this relation, they had shown the following inequality (AH$_0$). It is equivalent to the main result of log majorization and can be given as the following form:

**(AH$_0$)** $A^{-1} \#_\frac{1}{p} A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}} \leq I \Rightarrow A^{-r} \#_\frac{1}{p} (A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^r \leq I,$ $p \geq 1, r \geq 1$. 
Furuta had constructed the following inequality which interpolates \((AH_0)\) and \((F)\), we call this grand Furuta inequality ([2],[4],[8],[9]).

**Grand Furuta inequality:** If \(A \geq B \geq 0\) and \(A > 0\), then for each \(1 \leq p\) and \(t \in [0, 1]\),
\[(GF)\]
\[A^{-r} \#_{\frac{1-t+r}{(p-t)+r}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s \leq A^{1-t}\]
holds for \(t \leq r\) and \(1 \leq s\).

The satellite form of \((GF)\) is given also as follows ([2],[14]):
\[(SGF)\]
\[A^{-r+t} \#_{\frac{1-t+r}{(p-t)+r}} (A^t \natural_s B^p) \leq B (\leq A).\]

We pointed out that \((F)\) and \((AH)\) are obtained from each other and gave a generalised form of \((AH)\) ([3],[5]).

For \(\alpha \in (0, 1)\) fixed,
\[(GAH)\]
\[A \#_{\alpha} B \leq I \implies A^r \#_{\frac{r-\alpha}{(1-\alpha)+r}} B^s \leq I \text{ for } r, s \geq 1.\]

Using \((GAF)\), we modified \((GF)\) as follows [15]:

**Theorem A.** If \(A \geq B \geq 0\) and \(A > 0\), then for each \(1 \leq p\) and \(t \in [0, 1]\),
\[A^{-r+t} \#_{\frac{1-t+r}{(p-t)+r}} (A^t \natural_s B^p) \leq A^t \#_{\frac{1-t}{p-t}} B^p\]
holds for \(t \leq r\) and \(1 \leq s\).

Recently, Furuta has shown the following theorem concerning to the above theorem [10].

**Theorem F.** Let \(A \geq B \geq 0\) with \(A > 0\), \(t \in [0, 1]\) and \(p \geq 1\). Then
\[F(\lambda, \mu) = A^{-\frac{1}{2}} \{(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}} \}^{\frac{1-t+\frac{1}{2}}{p-t+w}} A^{-\frac{1}{2}}\]
satisfies the following properties:

(i) \[F(r, w) \geq F(r, 1) \geq F(r, s) \geq F(r, s')\]
holds for any \(s' \geq s \geq 1\), \(r \geq t\) and \(1 - t \leq (p - t)w \leq p - t\).

(ii) \[F(q, s) \geq F(t, s) \geq F(r, s) \geq F(r', s)\]
holds for any \( r' \geq r \geq t \), \( s \geq 1 \) and \( t - 1 \leq q \leq t \).

In this note, we observe this theorem from the \( \alpha \)-power mean.

2. Review of Theorem F. We rewrite Theorem F by the form of \( \alpha \)-power mean. Then

\[
F(\lambda, \mu) = A^{-\lambda} \#_{\frac{1-t+r}{(p-t)*+r}} (A^{-\frac{t}{2}} B^{p} A^{-\frac{t}{2}})^{\mu}
\]

and by putting \( B_{1} = (A^{-\frac{t}{2}} B^{p} A^{-\frac{t}{2}})^{\frac{1}{p-1}} \), (i) and (ii) of Theorem F are written as follows:

(i)

\[
A^{-r} \#_{\frac{1-t+r}{(p-t)*+r}} B_{1}^{(p-t)s} \geq A^{-r} \#_{\frac{1-t+r}{(p-t)*+r}} B_{1}^{(1-t)\epsilon'}
\]

and

(ii)

\[
A^{-q} \#_{\frac{1-t+q}{(p-t)*+q}} B_{1}^{(p-t)s} \geq A^{-r} \#_{\frac{1-t+q}{(p-t)*+q}} B_{1}^{(1-t)\epsilon'}
\]

We point out that Theorem A can be written more precisely,

\[
A^{-r+t} \#_{\frac{1-t+r}{(p-t)*+r}} (A^{t} \natural_{s} B^{p}) \leq (A^{t} \natural_{s} B^{p})^{\frac{1}{(p-t)*+t}} \leq B \leq A_{s}^{t} \#_{\frac{1-t+q}{(p-t)*+q}} B_{1}^{(p-t)s}
\]

Because \( A^{-r+t} \#_{\frac{1-t+r}{(p-t)*+r}} (A^{t} \natural_{s} B^{p}) \leq (A^{t} \natural_{s} B^{p})^{\frac{1}{(p-t)*+t}} \leq B \) is already shown in our proof of (SGF). So the result of Theorem A has shown the following inequality.

\[
A^{-r} \#_{\frac{1-t+r}{(p-t)*+r}} B_{1}^{(1-t)\epsilon'} \leq A^{-r+t} \#_{\frac{1}{(p-t)*+t}} B_{1}^{(p-t)s} \leq B_{1}^{1-t},
\]

and Furuta improved on the second inequality of this form to

\[
A^{-t} \#_{\frac{1-t+q}{(p-t)*+q}} B_{1}^{(p-t)s} \leq A^{-q} \#_{\frac{1-t+q}{(p-t)*+q}} B_{1}^{(p-t)s}, \; t - 1 \leq q \leq t.
\]

Furuta's process is the following:

Since \( 0 \leq t - q \leq 1 \), \( (A^{\frac{t}{2}} B_{1}^{(p-t)s} A^{\frac{t}{2}})^{\frac{1-t+q}{(p-t)*+q}} \leq A^{t-q} \) holds by (LH), and we can obtain the result as follows:

\[
A^{-t} \#_{\frac{1-t+q}{(p-t)*+q}} B_{1}^{(p-t)s} = B_{1}^{(p-t)s} \#_{\frac{1-t+q}{(p-t)*+q}} A^{-t} = B_{1}^{(p-t)s} \#_{\frac{1-t+q}{(p-t)*+q}} (B_{1}^{(p-t)s} \#_{\frac{1-t+q}{(p-t)*+q}} A^{-t})
\]
3. **Modification of Theorem F.** Furuta’s results (i) and (ii) are holds suppose $A \geq B_1$, but in Theorem F this order does not hold. We search a suitable relation between $A$ and $B_1$ by the help of (GAH).

$A \geq B \geq 0$ implies $A^t \geq B^t \geq 0$ for $t \in [0, 1]$ by (LH). This is equivalent to $A^{-t} \#_{\frac{p-t}{t}} B^{-t} \leq I$. By (GAH), we have

$$A^{-r} \#_{\frac{r}{p+r}} B^p \leq I \text{ for } p, r \geq 0.$$

That is,

$$A \geq B \geq 0 \Rightarrow A^{-r} \#_{\frac{r}{p+r}} B^p \leq I \Rightarrow A^{-r'} \#_{\frac{r'}{p+r}} B^p \leq I \text{ for } r' \geq r, s \geq 1.$$

So we begin from the assumption $A^{-r} \#_{\frac{r}{p+r}} B^p \leq I$.

**Lemma 1.** Let $A, B \geq 0$ and $A^{-r} \#_{\frac{r}{p+r}} B^p \leq I$ for $p, r \geq 0$. Then the following hold:

(i) $A^{-r} \#_{\frac{\delta}{p+r}} B^p \leq B^\delta$ for $0 \leq \delta \leq p$

and

(ii) $A^{-r} \#_{\frac{\lambda}{p+r}} B^p \leq A^\lambda$ for $-r \leq \lambda \leq 0$.

These results are already known, but these play essential roles in our following discussions. We can arrange Theorem F as follows except $F(q, s) \geq F(t, s)$ for $t-1 \leq q \leq 0$.

**Theorem 1.** Let $A, B \geq 0$ and $A^{-r} \#_{\frac{r}{p+r}} B^p \leq I$ for $p, r \geq 0$. Then

(1) $A^{-r} \#_{\frac{\mu}{p+r}} B^p \leq A^{-r} \#_{\frac{\delta}{p+r}} B^\mu$ holds for $p \geq \mu \geq \delta \geq 0$ and

(ii) $A^{-r} \#_{\frac{\lambda}{p+r}} B^p \leq A^{-t} \#_{\frac{\lambda}{p+r}} B^p$
holds for \( r \geq t \geq 0, \ -t \leq \lambda \leq p \).

**Proof.** (i) is obtained by the following calculation:

\[
A^{-r} \#_{\frac{r}{p-t+r}} B^p = A^{-r} \#_{\frac{r}{p-t+r}} (A^{-r} \#_{\frac{r}{p-t+r}} B^p) \leq A^{-r} \#_{\frac{r}{p-t+r}} B^\mu.
\]

(ii) can be shown as follows:

\[
A^{-r} \#_{\frac{r}{p-t+r}} B^p = B^p \#_{\frac{r}{p-t+r}} A^{-r} = B^p \#_{\frac{r}{p-t+r}} (B^p \#_{\frac{r}{p-t+r}} A^{-r})
\]

\[
= B^p \#_{\frac{r}{p-t+r}} (A^{-r} \#_{\frac{r}{p-t+r}} B^p) \leq B^p \#_{\frac{r}{p-t+r}} A^{-t} = A^{-t} \#_{\frac{r}{p-t+r}} B^p.
\]

4. **Applications.** Return to Theorem A, we summarize the above discussions.

**Theorem A(1).** If \( A \geq B \geq 0 \) and \( t \in [0, 1], \ p \geq t, \ r \geq t, \ 0 \leq \delta \leq (p-t)s \), then

\[
A^{-r+t} \#_{\frac{t+r}{(p-t)+t-r}} (A^t \#_B B^p) \leq (A^t \#_B B^p)^{\frac{t+s}{(p-t)+s-t}} \leq A^\alpha \#_{\frac{t+r}{(p-t)+r-t}} (A^t \#_B B^p)
\]

holds for \( \min\{\delta + t, 1\} \geq \alpha \geq 0 \).

This is equivalent to

\[
A^{-r} \#_{\frac{t+r}{(p-t)+r}} B_1^{(p-t)s} \leq A^{-t} \#_{\frac{t+s}{(p-t)+s-t}} B_1^{(p-t)s} \leq A^{-t} \#_{\frac{t+r}{(p-t)+r-t}} B_1^{(p-t)s}.
\]

If \( p \geq 1 \) and \( \delta = 1 - t, \ \alpha = t - q \), we have Furta's result (ii) containing the case \( t - 1 \leq q \leq 0 \).

Under the assumption \( A^{-r} \#_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I \), our Theorem A can be written as follows:

**Theorem A(2).** Let \( A, \ B \geq 0 \) and put \( B_1 = (A^{-\frac{t}{p-t}} B^p A^{-\frac{t}{p-t}})^{\frac{1}{p-t}} \) for \( p \geq t \geq 0 \). If \( A^{-r} \#_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I \) for \( r \geq t \geq 0 \), then for \( s \geq 1 \)

(i)

\[
A^{-r+t} \#_{\frac{t+r}{(p-t)+r}} (A^t \#_B B^p) \leq A^{-r+t} \#_{\frac{t+s}{(p-t)+s-t}} (A^t \#_B B^p)
\]

holds for \( 0 \leq \delta \leq \mu \leq (p-t)s \) and

(ii)

\[
A^{-r+t} \#_{\frac{t+r}{(p-t)+r}} (A^t \#_B B^p) \leq (A^t \#_B B^p)^{\frac{t+s}{(p-t)+s-t}}
\]

holds for \( -t \leq \lambda \leq (p-t)s \).
But this case reduces to Theorem 1 because

\[(i) \iff A^{-r} \#_{\frac{t+r}{p-t}+\ell} B_1^{(p-t)s} \leq A^{-r} \#_{\frac{\delta+r}{\mu_+}+\ell} B_1^{(p-t)s}\]

and

\[(ii) \iff A^{-t} \#_{\frac{\lambda+r}{(p-t)\cdot+t}+\ell} B_1^{(p-t)s} \leq A^{-t} \#_{\frac{\lambda+l}{(p-t)\cdot+t}+\ell} B_1^{(p-t)s}.\]

References


[6] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101(1987), 85-88.


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