Majorization と作用素不等式
(Majorization and operator inequalities)
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In this paper we deal with bounded self-adoint operators or Hermitian matrices. Let’s start with the definition of an o.m. function. Let $f$ be a real valued continuous function on an interval $I$. The functional calculus by $f$ induces a non-linear mapping on $H_n(I)$, which is the set of all Hermitian matrices on $n-$ dimensional space. If the mapping preserves the order, then $f$ is called a o.m. function. We denote the set of all o.m. by $P(I)$, and the subset of non-negative functions by $P_+(I)$. So a power function with an exponent between 0 and 1 belongs to $P_+$ on $[0, \infty)$; The inequality induced from this is called L"owner-Heinz inequality.

It seemd that only one mapping was considered so far. I tried to compare two mappings. At first We noticed that for $0 \leqq A, B$

$$A^2 \leqq B^2 \implies (A+1)^2 \leqq (B+1)^2,$$

but the converse is not valid. We posed a problem by myself to seek a pair of $u, v$ s.t.

$$0 \leqq A, B, u(A) \leqq u(B) \Rightarrow v(A) \leqq v(B).$$

And We first considered the case both $u$ and $v$ are polynomials with positive coefficients.
1 A New Majorization

To study systematically We defined the set of the inverses of o.m. functions. If the left extreme point $a$ is finite, then these two sets are identical by natural extension. Also we considered the set of a function whose logarithm is o.m. And we introduced the concept of a new majorization as follows: $h$ is said to be majorized by $k$ and denoted by

$$h \preceq k$$

if $J \subset I, \ h \circ k^{-1} \in P(k(J))$. This definition is equivalent with

$$k(A) \leq k(B) \implies h(A) \leq h(B).$$

Löwner-Heinz inequality says for $0 < a \leq 1 \leq \beta$

$$t^a \leq t \leq t^\beta \quad ([0, \infty)).$$

We list several properties of the majorization.

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(i) $k^\alpha \leq k^\beta$ for any increasing function

$$k(t) \geq 0 \text{ and } 0 < \alpha \leq \beta;$$

(ii) (transitive) $g \leq h, \ h \leq k \implies g \leq k$;

(iii) (invariant for homeomorphism) if $\tau$ is an increasing function whose range is the domain of $k$, then

$$h \leq k \iff h \circ \tau \leq k \circ \tau;$$
(iv) if the range of \(k\) is \([0, \infty)\) and \(h, k \ge 0\), then
\[ h \leq k \implies h^2 \leq k^2; \]
Remark: Consider \(t\) and \(t - 1\) on \(1 \leq t < \infty\).
\[ t - 1 \leq t \text{ but } (t - 1)^2 \neq t^2. \]

(v) if the ranges of \(k, h\) are \([0, \infty)\), then
\[ h \leq k, \quad k \leq h \iff h = ck + d \]
for real numbers \(c > 0, d\).
Remark: The range condition is indispensable: in fact, \(t \leq \frac{t}{1+t}, \frac{t}{1+t} \leq t\)
on \([0, \infty)\).

The next lemma is very significant for our study, so we named it.

**Lemma 1.1 (Product lemma)**

Suppose \(-\infty \leq a < b \leq \infty\),
\[ 0 \leq h(t), \quad 0 \leq g(t) \text{ on } [a, b). \]
If the product \(h(t)g(t)\) is increasing and the range is \([0, \infty)\) (or \((0, \infty)\) if \(a = -\infty\)),
then
\[ g \leq h g \implies h \leq h g. \]
Moreover
\[ \psi_1(h)\psi_2(g) \leq h g \quad \text{for } \psi_1, \psi_2 \in P_+[0, \infty). \]

This lemma is subtle; so we give some examples.

\( \circ \) \(1 \leq t [0, \infty), \quad t \leq 1 + t^2 [0, \infty). \)

but, \( t \not\leq t(1 + t^2) [0, \infty). \)
\( t \preceq t + 1 \quad [0, \infty) \).

but \( t^2 \not\leq (1 + t)^2 \quad [0, \infty) \).

Now we are in the position to state the main theorem.

**Theorem 1.2 (Product theorem)**

Suppose \(-\infty \leq a < b \leq \infty\). \([a, b)\) denotes \((-\infty, b)\) if \(a = -\infty\). Then

\[
\text{LP}_+[a, b) \cdot \text{P}_+^{-1}[a, b) \subset \text{P}_+^{-1}[a, b),
\]

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\text{P}_+^{-1}[a, b) \cdot \text{P}_+^{-1}[a, b) \subset \text{P}_+^{-1}[a, b).
\]

Further, let \( h_i(t) \in \text{P}_+^{-1}[a, b) \) for \(1 \leq i \leq m\),

and let \( g_j(t) \in \text{LP}_+[a, b) \) for \(1 \leq j \leq n\).

Then for \( \psi_i, \phi_j \in \text{P}_+[0, \infty) \)

\[
\prod_{i=1}^{m} h_i(t) \prod_{j=1}^{n} g_j(t) \in \text{P}_+^{-1}[a, b),
\]

\[
\prod_{i=1}^{m} \psi_i(h_i) \prod_{j=1}^{n} \phi_j(g_j) \preceq \prod_{i=1}^{m} h_i \prod_{j=1}^{n} g_j.
\]

It is easy to see the following result is the special case of the above.

**Corollary 1.3 Ando[1]**

\( f(t) \in \text{P}_+[0, \infty) \Rightarrow tf(t) \in \text{P}_+^{-1}[0, \infty) \).

He proved this by successive approximation. We could get the above theorem by using successive approximation too. \( \text{P}_+^{-1}[a, b) \) is closed in the sense that if a limit point of \( \text{P}_+^{-1}[a, b) \) is increasing and the range is \([0, \infty)\), then it belongs to \( \text{P}_+^{-1}[a, b) \). However we can construct a sequence of functions in this set which converges to \((t - 1)_+\).
2 Polynomials

Let's get back to the original problem. Now we can reach at the solution to the problem.

For non-increasing sequences \( \{a_i\}_{i=1}^{n} \) and \( \{b_i\}_{i=1}^{n} \),

\[
\begin{align*}
  u(t) &:= \prod_{i=1}^{n}(t-a_i) \quad (t \geq a_1), \\
  v(t) &:= \prod_{i=1}^{m}(t-b_i) \quad (t \geq b_1).
\end{align*}
\]

**Lemma 2.1** Suppose \( v \preceq u \) for \( u \) and \( v \).

Then \( m \leq n \).

**Theorem 2.2** Suppose \( m \leq n \).

\[
\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{k} a_i \quad (1 \leq k \leq m) \implies v \preceq u.
\]

Recall the classical definition of submajorization for two sequences \( \{a_i\}_{i=1}^{n} \) and \( \{b_i\}_{i=1}^{n} \). If they satisfies the above condition, it is said that \( \{a_i\}_{i=1}^{n} \) submajorizes \( \{b_i\}_{i=1}^{m} \).

**Corollary 2.3** Let \( \{p_n\}_{n=0}^{\infty} \) be a sequence of orthonormal polynomials with the positive leading coefficient. Consider the restricted part of \( p_n \) to \( [a_n, \infty) \), where \( a_n \) is the maximal zero of \( p_n \). Then

\[
p_{n-1} \preceq p_n.
\]

As to a polynomial with imaginary zeros, we can get similar result:
Theorem 2.4

\[ u(t) := \prod_{i=1}^{n}(t - a_i) \quad (t \geqq a_1) \]
\[ w(t) := \prod_{j=1}^{m}(t - \alpha_j) \quad (\Re\alpha_1 \leqq t < \infty) \]

where \( \Re\alpha_1 \geqq \Re\alpha_2 \geqq \cdots \geqq \Re\alpha_m, m \leqq n \). Then

\[ \sum_{j=1}^{k} \Re\alpha_j \leqq \sum_{j=1}^{k} a_j \quad (1 \leqq k \leqq m) \Rightarrow w \preceq u. \]

Theorem 2.5 Let \( p(t) \) be a real polynomial with a positive leading coefficient such that \( p(0) = 0 \) and zeros of \( p \) are all in \( \{z: \Re z \leqq 0\} \). Let \( q(t) \) be a factor of \( p(t) \). Then

\[ p(\sqrt{t})^2 \in \mathbb{P}^{-1}_+[0, \infty), \quad q(t)^2 \preceq p(t)^2, \]

that is

\[ p(A)^2 \leqq p(B)^2 \quad (0 \leqq A, B) \Rightarrow A^2 \leqq B^2, \quad q(A)^2 \leqq q(B)^2. \]

Furthermore, if \( p(0) = p'(0) = 0 \), then

\[ p(\sqrt{t}) \in \mathbb{P}^{-1}_+[0, \infty), \quad q(t) \preceq p(t), \]

that is

\[ p(A) \leqq p(B) \quad (0 \leqq A, B) \Rightarrow A^2 \leqq B^2, \quad q(A) \leqq q(B). \]

We were asked by S. Pereverzev and U. Tautenhahn if \( t^\alpha e^{-t^\beta} \in \mathcal{P}^{-1}_+(0, \infty) \).

It is clear that \( t^\alpha e^{-t^\beta} \to 0 \) as \( t \to +0 \) for \( \alpha, \beta > 0 \).
Proposition 2.6 For $0 < \beta \leq \alpha$

\[ t^\alpha \preceq t^\alpha e^{-t^{\beta}}. \]

Moreover, if $1 \leq \alpha$, then

\[ t^\alpha e^{-t^{\beta}} \in P_+^{-1}(0, \infty). \]

3 Operator Inequalities

Theorem 3.1 Let $h(t) \in P_+^{-1}[a, b)$, $g(t) \in LP_+[a, b)$ and $\tilde{h}(t) \geq 0$ on $[a, b)$.

Suppose

\[ \tilde{h} \preceq h. \]

Then the function $\varphi$ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ belongs to $P_+[0, \infty)$, and satisfies

\[ a \leq A \leq B < b \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq g(A)^{\frac{1}{2}} \tilde{h}(B)g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leq g(B)^{\frac{1}{2}} \tilde{h}(A)g(B)^{\frac{1}{2}}. \end{cases} \]

Furthermore, if $\tilde{h} \in P_+[a, b)$, then

\[ a \leq A \leq B < b \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq \tilde{h}(A)g(A), \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leq \tilde{h}(B)g(B). \end{cases} \]

Proposition 3.2 Let $h(t) \in P_+^{-1}[a, b)$, $g(t) \in LP_+[a, b)$. If $0 < \alpha < 1$, then

\[ h(t)^{\alpha}g(t)^{\alpha-1} \preceq h(t), \]

and

\[ 0 \leq A \leq B \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^{\alpha} \geq g(A)^{\frac{1}{2}} h(B)^{\alpha} g(A)^{\alpha-1} g(A)^{\frac{1}{2}}, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^{\alpha} \leq g(B)^{\frac{1}{2}} h(A)^{\alpha} g(A)^{\alpha-1} g(B)^{\frac{1}{2}}. \end{cases} \]

Furthermore, if $h(t)^{\alpha}g(t)^{\alpha-1} \in P_+[a, b)$, then

\[ a \leq A \leq B < b \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^{\alpha} \geq (h(A)g(A))^{\alpha}, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^{\alpha} \leq (h(B)g(B))^{\alpha}. \end{cases} \]
**Corollary 3.3** Let $f(t) \in P_+(0, \infty)$. Suppose $p, r, \alpha > 0$ and $s \geq 0$ satisfy

$$1 \leq p, \quad r(s - 1) \leq p, \quad \alpha \leq \frac{1+r}{p+s+r}.$$  

Then

$$0 \leq A \leq B \Rightarrow (A^{\frac{r}{2}}B^pf(B)A^{\frac{r}{2}})^{\alpha} \geq (A^{\frac{r}{2}}A^p f(A)A^{\frac{r}{2}})^{\alpha},$$

$$ (B^{\frac{r}{2}}B^pf(B)B^{\frac{r}{2}})^{\alpha} \geq (B^{\frac{r}{2}}A^p f(A)B^{\frac{r}{2}})^{\alpha}. $$   

**Example** Let $f(t) \in P_+[0, \infty)$. Suppose $p, r > 0, 0 < \alpha \leq \frac{1+r}{p+r}$. Then

$$0 \leq A \leq B \Rightarrow \left\{ \begin{array}{l}
(A^{\frac{r}{2}}B^pf(B)A^{\frac{r}{2}})^{\alpha} \geq (A^{\frac{r}{2}}A^p f(A)A^{\frac{r}{2}})^{\alpha}, \\
(B^{\frac{r}{2}}B^pf(B)B^{\frac{r}{2}})^{\alpha} \geq (B^{\frac{r}{2}}A^p f(A)B^{\frac{r}{2}})^{\alpha}.
\end{array} \right.$$  

**References**


