<table>
<thead>
<tr>
<th>Title</th>
<th>Operator inequalities obtained from M. Uchiyama's recent results (Inequalities on Linear Operators and its Applications)</th>
</tr>
</thead>
<tbody>
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Operator inequalities obtained from M. Uchiyama’s recent results

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1 Remarks on Furuta inequality

In what follows, an operator means a bounded linear operator on a Hilbert space $H$. An operator $T$ is positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

**Theorem F** (Furuta inequality [2]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) \[ (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^pB^{\frac{r}{2}})^{\frac{1}{q}} \]

and

(ii) \[ (A^rA^\frac{p}{r}A^\frac{r}{2})^{\frac{1}{q}} \geq (A^\frac{r}{2}B^\frac{p}{r}A^\frac{r}{2})^{\frac{1}{q}} \]

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Löwner-Heinz theorem “$A \geq B \geq 0 \implies A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$” is the case $r = 0$ of Theorem F. Other proofs are given in [1][5] and also an elementary one-page proof in [3]. It is shown in [6] that the domain of $p$, $q$ and $r$ in Theorem F is the best possible for the inequalities (i) and (ii) to hold under the assumption $A \geq B$.

**Remark 1.** It was shown in [5] that $A \geq B \geq 0$ implies

\[ B^{\frac{p}{q}}(B^\frac{r}{2}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq A^{\frac{p}{q}+r} \geq B^{\frac{p}{q}+r} \]  \hspace{1cm} (1)

holds for $r \geq 0$, $p \geq 1$ and $q \geq 1$ with $(1+r)q \geq p+r$. The essential part of (1) is the first inequality, while the important condition $(1+r)q \geq p+r$ comes from the second.

**Remark 2.** Theorem F is based on the fact that

\[ (B^\frac{1}{2}XB^\frac{1}{2})^\frac{1}{q} \geq B^\delta \implies (B^\frac{1}{2}XB^\frac{1}{2})^\frac{1}{q+\epsilon} \geq B^{\delta+\epsilon} \]  \hspace{1cm} (2)

holds for $B, X \geq 0, \epsilon \in \mathbb{R}$ and $0 \leq u \leq \delta \leq \alpha$. Theorem F can be proved by applying (2) repeatedly as follows: for $A, B \geq 0$ and $p \geq 1$,
$A \geq B \iff (B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r}$ for $p \geq 1$ and $r \in [0, 1]$ by (2)

Proof of (2). The assumptions imply $(B^{\frac{1}{2}}XB^{\frac{1}{2}})^{\frac{6}{5}} \geq B^u$ by Löwner-Heinz theorem, and there exists a contraction $C$ such that $C^*(B^{\frac{1}{2}}XB^{\frac{1}{2}})^{\frac{6}{5}} = (B^{\frac{1}{2}}XB^{\frac{1}{2}})^{\frac{6}{5}}C = B^{\frac{6}{5}}$. Hence,

$$(B^{\frac{6}{5}}XB^{\frac{6}{5}})^{\frac{6}{5}} = (C^*(B^{\frac{1}{2}}XB^{\frac{1}{2}})^{\frac{6}{5}}C)^{\frac{6}{5}}$$

$$\geq C^*((B^{\frac{1}{2}}XB^{\frac{1}{2}})^{\frac{6}{5}}C)^{\frac{6}{5}} = B^{\frac{6}{5}}(B^{\frac{1}{2}}XB^{\frac{1}{2}})^{\frac{6}{5}}$$

$$\geq B^{4+u}$$

by Hansen's inequality [4] and the assumption. 

In the one-page proof ([3]), the fact

$$A \geq B \geq 0 \implies (B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r} \text{ for } p \geq 1 \text{ and } r \in [0, 1]$$

(3)

is shown at first, and then (3) is used doubtly and nestedly as

$$A \geq B \geq 0 \implies A_1 \geq B_1 \implies (B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{1+r_1}{p+r_1}} \geq B^{1+r_1}$$

where $A_1 = (B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{1+r_1}{p+r_1}}$, $B_1 = B^{1+r}$, $p_1 = \frac{p+r}{p+r_1}$, and $r_1 = 1$. We note that the value of $p_1$ is chosen in order that $h(t) = t^{p_1}$ becomes the inverse function of $\varphi(t) = t^{\frac{1}{p+1}}$. It might be remarkable that in the proof of (2), we use neither such an implication proposition with the hypothesis $A \geq B$ as (3) nor such an inverse function as $h(t)$.

2 Uchiyama's results and their generalizations

Let $\mathbb{P}_+[a, b]$ be the set of all non-negative operator monotone functions defined on $[a, b)$, and $\mathbb{P}_+^{-1}[a, b)$ the set of increasing functions $h$ defined on $[a, b)$ such that $h([a, b)) = [0, \infty)$ and its inverse $h^{-1}$ is operator monotone on $[0, \infty)$. Uchiyama [7] introduces a new concept of majorization, and shows a quite interesting result named "Product theorem."

Definition ([7]). Let $h$ be a non-decreasing function on $I$ and $k$ an increasing function on $J$. Then $h$ is said to be majorized by $k$, in symbols $h \preceq k$, if $J \subseteq I$ and the composite $h \circ k^{-1}$ is operator monotone on $k(J)$. 

Product theorem ([7]). Suppose \(-\infty < a < b \leq \infty\). Then
\[
\mathbb{P}_+[a, b) \cdot \mathbb{P}_-^{-1}[a, b) \subseteq \mathbb{P}_-^{-1}[a, b) \subseteq \mathbb{P}_+[a, b).
\]
Further, let \(h_i \in \mathbb{P}_-^{-1}[a, b)\) for \(1 \leq i \leq m\), and let \(g_j\) be a finite product of functions in \(\mathbb{P}_+[a, b)\) for \(1 \leq j \leq n\). Then for \(\psi_i, \phi_j \in \mathbb{P}_+[0, \infty)\)
\[
\prod_{i=1}^{m} h_i(t) \prod_{j=1}^{n} g_j(t) \in \mathbb{P}_-^{-1}[a, b),
\]
Furthermore, he applies Product theorem to obtain generalizations of Theorem F.

Proposition A ([7]). Let \(h \in \mathbb{P}_-^{-1}[0, \infty)\), and let \(\tilde{h}\) be a non-negative non-decreasing function on \([0, \infty)\) such that \(\tilde{h} \leq h\). Let \(g\) be a finite product of functions in \(\mathbb{P}_+[0, \infty)\). Then for the function \(\varphi\) defined by \(\varphi(h(t)g(t)) = \tilde{h}(t)g(t)\)
\[
A \geq B \geq 0 \implies \begin{cases} 
\varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}, \\
\varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}.
\end{cases}
\]

Theorem B ([7]). Let \(h \in \mathbb{P}_-^{-1}[0, \infty)\), and let \(\tilde{h}\) be a non-negative non-decreasing function on \([0, \infty)\) such that \(\tilde{h} \leq h\). Let \(g_n\) be a finite product of functions in \(\mathbb{P}_+[0, \infty)\) for each \(n\), and let the sequence \(\{g_n\}\) converge pointwise to \(g\). Suppose \(g \neq 0\) and \(g(0^+) = g(0)\). Then for the function \(\varphi\) defined by \(\varphi(h(t)g(t)) = \tilde{h}(t)g(t)\)
\[
A \geq B \geq 0 \implies \begin{cases} 
\varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}, \\
\varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}.
\end{cases}
\]

We obtain extensions of Proposition A and Theorem B by weakening their hypotheses from \(A \geq B\) to inequalities implied by it. We note that these results are slightly improved versions of those in [8] from the viewpoint of the remarks in the previous section.

Proposition 1. Let \(f_i\) be non-negative non-decreasing functions on \([0, \infty)\) and \(g_j(t) = \prod_{i=1}^{j} f_i(t)\). Let \(h, \tilde{h}\) and \(\hat{h}\) be non-negative non-decreasing functions on \([0, \infty)\) such that \(f_n(t) \leq \tilde{h}(t)g_{n-1}(t), \tilde{h} \leq h, \text{ and } h(0)g_{n-1}(0) = 0\). Then for the functions \(\psi_j\) and \(\varphi_j\) defined by \(\psi_j(h(t)g_j(t)) = \tilde{h}(t)g_j(t)\) and \(\varphi_j(h(t)g_j(t)) = \hat{h}(t)g_j(t)\), if \(A, B \geq 0\) satisfy
\[
\psi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}) \geq \tilde{h}(B)g_{n-1}(B),
\]
then
\[
\varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq f_n(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})f_n(B)^{\frac{1}{2}}
\]
holds. Furthermore,
\[
\psi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq \tilde{h}(B)g_n(B)
\]
holds if \(\tilde{h} \leq h\).
Theorem 2. Let \( \hat{h} \in P_{+}^{-1}[0, \infty) \), and let \( h \) and \( \tilde{h} \) be non-negative non-decreasing functions on \([0, \infty)\) such that \( \hat{h} \leq h \) and \( \tilde{h} \leq h \). Let \( g \) be a finite product of functions in \( P_{+}[0, \infty) \cup P_{+}^{-1}[0, \infty) \) and \( \gamma_{n} \) a finite product of functions in \( P_{+}[0, \infty) \) for each \( n \), and let the sequence \( \{g(t)\gamma_{n}(t)\} \) converge pointwise to \( \overline{g}(t) \). Suppose \( \overline{g} \neq 0 \) and \( \overline{g}(0+) = \overline{g}(0) \). Then for the functions \( \psi, \tilde{\psi}, \varphi \) and \( \overline{\varphi} \) defined by \( \psi(h(t)g(t)) = \hat{h}(t)g(t), \tilde{\psi}(h(t)\overline{g}(t)) = \hat{h}(t)\overline{g}(t), \varphi(h(t)g(t)) = \tilde{h}(t)g(t) \) and \( \varphi(h(t)\overline{g}(t)) = \overline{\tilde{h}}(t)\overline{g}(t) \), if \( A, B \geq 0 \) satisfy

\[
\psi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{3}{2}}) \geq \hat{h}(B)g(B),
\]

then

\[
g(B)^{\frac{1}{2}}\varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{3}{2}}) \geq \overline{g}(B)^{\frac{1}{2}}\varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{3}{2}})\overline{g}(B)^{\frac{1}{2}}
\]

and

\[
\tilde{\psi}(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{3}{2}}) \geq \overline{\tilde{h}}(B)\overline{g}(B)
\]

hold.

Proof of Proposition 1 \( \Rightarrow \) Proposition \( A \). Put \( \hat{h}(t) = t \) and \( f_{1}(t) = g_{1}(t) = 1 \), then

\[
\psi_{1}(g_{1}(B)^{\frac{1}{2}}h(A)g_{1}(B)^{\frac{3}{2}}) = \psi_{1}(h(A)g_{1}(A)^{\frac{3}{2}}) = \hat{h}(A)g_{1}(A) = A \geq B = h(B)g_{1}(B).
\]

By applying Proposition 1, we have

\[
\psi_{1}(g_{1}(B)^{\frac{1}{2}}h(A)g_{1}(B)^{\frac{3}{2}}) \geq h(B)g_{1}(B) \quad \Rightarrow \quad \psi_{2}(g_{2}(B)^{\frac{1}{2}}h(A)g_{1}(B)^{\frac{3}{2}}) \geq h(B)g_{2}(B)
\]

and

\[
\psi_{n}(g_{n}(B)^{\frac{1}{2}}h(A)g_{n}(B)^{\frac{3}{2}}) \geq h(B)g_{n}(B)
\]

for \( k = 1, 2, \ldots, n-1 \). Therefore

\[
\varphi_{n}(g_{n}(B)^{\frac{1}{2}}h(A)g_{n}(B)^{\frac{3}{2}}) \geq f_{n}(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{3}{2}})f_{n}(B)^{\frac{1}{2}}
\]

\[
\geq f_{n}(B)^{\frac{1}{2}}f_{n-1}(B)^{\frac{1}{2}}\varphi_{n-2}(g_{n-2}(B)^{\frac{1}{2}}h(A)g_{n-2}(B)^{\frac{3}{2}})f_{n-1}(B)^{\frac{1}{2}}f_{n}(B)^{\frac{1}{2}}
\]

\[
\geq \cdots
\]

\[
\geq f_{n}(B)^{\frac{1}{2}}\ldots f_{2}(B)^{\frac{1}{2}}\varphi_{1}(g_{1}(B)^{\frac{1}{2}}h(A)g_{1}(B)^{\frac{3}{2}})f_{2}(B)^{\frac{1}{2}}\ldots f_{n}(B)^{\frac{1}{2}}
\]

\[
= g_{n}(B)^{\frac{1}{2}}\tilde{h}(A)g_{n}(B)^{\frac{3}{2}}.
\]

\( \square \).
References


[2] T. Furuta, \( A \geq B \geq 0 \) assures \((B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}\) for \( r \geq 0, p \geq 0, q \geq 1 \) with \((1+2r)q \geq p+2r\), Proc. Amer. Math. Soc. 101 (1987), 85–88.


