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Operator inequalities obtained from M. Uchiyama’s recent results

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1 Remarks on Furuta inequality

In what follows, an operator means a bounded linear operator on a Hilbert space $H$. An operator $T$ is positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

Theorem F (Furuta inequality [2]).
If $A \geq B \geq 0$, then for each $r \geq 0$,
(i) $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{1}q \geq (B^{\frac{r}{2}}B^{\frac{r}{2}})^{1}q$
and
(ii) $(A^{\frac{r}{2}}B^{\frac{r}{2}}A^{\frac{r}{2}})^{1}q \geq (A^{\frac{r}{2}}B^{\frac{r}{2}}A^{\frac{r}{2}})^{1}q$
hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

Löwner-Heinz theorem “$A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$” is the case $r = 0$ of Theorem F. Other proofs are given in [1][5] and also an elementary one-page proof in [3]. It is shown in [6] that the domain of $p$, $q$ and $r$ in Theorem F is the best possible for the inequalities (i) and (ii) to hold under the assumption $A \geq B$.

Remark 1. It was shown in [5] that $A \geq B \geq 0$ implies

$$B^{\frac{r}{2}}(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{1}q \geq A^{\frac{2}{(1+r)q} - r} \geq B^{\frac{2}{(1+r)q} - r}$$  \hspace{1cm} (1)

holds for $r \geq 0$, $p \geq 1$ and $q \geq 1$ with $(1 + r)q \geq p + r$. The essential part of (1) is the first inequality, while the important condition $(1 + r)q \geq p + r$ comes from the second.

Remark 2. Theorem F is based on the fact that

$$B^{\frac{1}{2}}(B^{\frac{1}{2}}X\frac{1}{2}B^{\frac{1}{2}})^{1}q \geq B^{\delta} \implies (B^{\frac{1}{2}}X\frac{1}{2}B^{\frac{1}{2}})^{1}q \geq B^{\delta + u}$$  \hspace{1cm} (2)

holds for $B, X \geq 0$, $\delta \in \mathbb{R}$ and $0 \leq u \leq \delta \leq \alpha$. Theorem F can be proved by applying (2) repeatedly as follows: for $A, B \geq 0$ and $p \geq 1$,
\[ A \geq B \iff (B^{1/2}A^{p}B^{1/2})^{1/2} \geq B^{1+r} \]
\[ \iff (B^{1/2}A^{p}B^{1/2})^{1/2} \geq B^{1+r} \quad \text{for } u_1 \in [0,1] \text{ by (2)} \]
\[ \iff (B^{1/2}A^{p}B^{1/2})^{1/2} \geq B^{1+1} \]
\[ \iff (B^{1/2}A^{p}B^{1/2})^{1/2} \geq B^{1+1} \quad \text{for } u_2 \in [0,2] \text{ by (2)} \]
\[ \iff (B^{1/2}A^{p}B^{1/2})^{1/2} \geq B^{1+1} \]
\[ \iff (B^{1/2}A^{p}B^{1/2})^{1/2} \geq B^{1+1} \quad \text{for } u_3 \in [0,4] \text{ by (2)} \]
\[ \Rightarrow \cdots \]

**Proof of (2).** The assumptions imply \((B^{1/2}X^{1/2}B^{1/2})^{1/2} \geq B^{u}\) by Löwner-Heinz theorem, and there exists a contraction \(C\) such that \(C^\ast(B^{1/2}X^{1/2}B^{1/2})^{1/2} = (B^{1/2}X^{1/2}B^{1/2})^{1/2} \leq C = B^{1/2}\) Hence,
\[
(B^{1/2}X^{1/2}B^{1/2})^{1/2} \geq C^\ast((B^{1/2}X^{1/2}B^{1/2})^{1/2} \geq C \quad \text{by Hansen's inequality [4]}
\]
\[
= B^{1/2}(B^{1/2}X^{1/2}B^{1/2})^{1/2}B^{1/2}
\]
\[
\geq B^{1/2+u} \quad \text{by the assumption.} \]

In the one-page proof ([3]), the fact
\[ A \geq B \geq 0 \implies (B^{1/2}A^{p}B^{1/2})^{1/2} \geq B^{1+r} \text{ for } p \geq 1 \text{ and } r \in [0,1] \quad (3) \]
is shown at first, and then (3) is used doubly and nestedly as
\[ A \geq B \geq 0 \implies A_1 \geq B_1 \implies (B^{1/2}_1A^{p_1}B^{1/2}_1)^{1/2} \geq B^{1+r}_1 \]
where \(A_1 = (B^{1/2}_1A^{p_1}B^{1/2}_1)^{1/2}\), \(B_1 = B^{1+r_1}, p_1 = \frac{p + r}{1 + r}\) and \(r_1 = 1\). We note that the value of \(p_1\)
is chosen in order that \(h(t) = t^{p_1}\) becomes the inverse function of \(\varphi(t) = t^{1+r}\). It might be remarkable that in the proof of (2), we use neither such an implication proposition with the hypothesis \(A \geq B\) as (3) nor such an inverse function as \(h(t)\).

2 Uchiyama’s results and their generalizations

Let \(\mathbb{P}_+[a,b]\) be the set of all non-negative operator monotone functions defined on \([a,b]\),
and \(\mathbb{P}_+^{-1}[a,b]\) the set of increasing functions \(h\) defined on \([a,b]\) such that \(h([a,b]) = [0,\infty)\)
and its inverse \(h^{-1}\) is operator monotone on \([0,\infty)\). Uchiyama [7] introduces a new concept of majorization, and shows a quite interesting result named “Product theorem.”

**Definition** ([7]). Let \(h\) be a non-decreasing function on \(I\) and \(k\) an increasing function on \(J\). Then \(h\) is said to be majorized by \(k\), in symbols \(h \preceq k\), if \(J \subseteq I\) and the composite \(h \circ k^{-1}\) is operator monotone on \(k(J)\).
Product theorem ([7]). Suppose $-\infty < a < b \leq \infty$. Then

$$P_+[a,b] \cdot P_+^{-1}[a,b] \subseteq P_+^{-1}[a,b], \quad P_+^{-1}[a,b] \cdot P_+[a,b] \subseteq P_+^{-1}[a,b].$$

Further, let $h_i \in P_+^{-1}[a,b]$ for $1 \leq i \leq m$, and let $g_j$ be a finite product of functions in $P_+[a,b]$ for $1 \leq j \leq n$. Then for $\psi_i, \phi_j \in P_+[0,\infty)$

$$\prod_{i=1}^{m} h_i(t) \prod_{j=1}^{n} g_j(t) \in P_+^{-1}[a,b],$$

Furthermore, he applies Product theorem to obtain generalizations of Theorem F.

Proposition A ([7]). Let $h \in P_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \leq h$. Let $g_n$ be a finite product of functions in $P_+[0,\infty)$ for each $n$, and let the sequence $\{g_n\}$ converge pointwise to $g$. Suppose $g \neq 0$ and $g(0+) = g(0)$. Then for the function $\varphi$ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$

$$A \geq B \geq 0 \implies \begin{cases} \varphi(g(B)^{1}h(A)g(B)^{1}) \geq g(B)^{1}\tilde{h}(A)g(B)^{1}, \\ \varphi(g(A)^{1}h(B)g(A)^{1}) \leq g(A)^{1}\tilde{h}(B)g(A)^{1}. \end{cases}$$

Theorem B ([7]). Let $h \in P_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \leq h$. Let $g_n$ be a finite product of functions in $P_+[0,\infty)$ for each $n$, and let the sequence $\{g_n\}$ converge pointwise to $g$. Suppose $g \neq 0$ and $g(0+) = g(0)$. Then for the function $\varphi$ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$

$$A \geq B \geq 0 \implies \begin{cases} \varphi(g(B)^{1}h(A)g(B)^{1}) \geq g(B)^{1}\tilde{h}(A)g(B)^{1}, \\ \varphi(g(A)^{1}h(B)g(A)^{1}) \leq g(A)^{1}\tilde{h}(B)g(A)^{1}. \end{cases}$$

We obtain extensions of Proposition A and Theorem B by weakening their hypotheses from $A \geq B$ to inequalities implied by it. We note that these results are slightly improved versions of those in [8] from the viewpoint of the remarks in the previous section.

Proposition 1. Let $f_i$ be non-negative non-decreasing functions on $[0,\infty)$ and $g_j(t) = \prod_{i=1}^{j} f_i(t)$. Let $h, \hat{h}$ and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that $f_n(t) \preceq \hat{h}(t)g_{n-1}(t), \tilde{h} \preceq h$ and $h(0)g_{n-1}(0) = 0$. Then for the functions $\psi_j$ and $\varphi_j$ defined by $\psi_j(h(t)g_j(t)) = \hat{h}(t)g_j(t)$ and $\varphi_j(h(t)g_j(t)) = \tilde{h}(t)g_j(t)$, if $A, B \geq 0$ satisfy

$$\psi_{n-1}(g_{n-1}(B)^{1}h(A)g_{n-1}(B)^{1}) \geq \hat{h}(B)g_{n-1}(B),$$

then

$$\varphi_{n}(g_{n}(B)^{1}h(A)g_{n}(B)^{1}) \geq f_n(B)^{1}\varphi_{n-1}(g_{n-1}(B)^{1}h(A)g_{n-1}(B)^{1})f_n(B)^{1}$$

holds. Furthermore,

$$\psi_{n}(g_{n}(B)^{1}h(A)g_{n}(B)^{1}) \geq \hat{h}(B)g_{n}(B)$$

holds if $\hat{h} \leq h$. 
Theorem 2. Let \( \hat{h} \in \mathbb{P}_+^{-1}[0, \infty) \), and let \( h \) and \( \tilde{h} \) be non-negative non-decreasing functions on \([0, \infty)\) such that \( \hat{h} \leq h \) and \( \hat{h} \leq \tilde{h} \). Let \( g \) be a finite product of functions in \( \mathbb{P}_+[0, \infty) \cup \mathbb{P}_+^{-1}[0, \infty) \) and \( \gamma_n \) a finite product of functions in \( \mathbb{P}_+[0, \infty) \) for each \( n \), and let the sequence \( \{g(t)\gamma_n(t)\} \) converge pointwise to \( \bar{g}(t) \). Suppose \( \bar{g} \neq 0 \) and \( \bar{g}(0+) = \bar{g}(0) \). Then for the functions \( \psi, \tilde{\psi}, \varphi \) and \( \tilde{\varphi} \) defined by \( \psi(h(t)g(t)) = \hat{h}(t)g(t) \), \( \tilde{\psi}(h(t)\bar{g}(t)) = \hat{h}(t)\bar{g}(t) \), \( \varphi(h(t)g(t)) = \bar{h}(t)g(t) \) and \( \tilde{\varphi}(h(t)\bar{g}(t)) = \tilde{h}(t)\bar{g}(t) \), if \( A, B \geq 0 \) satisfy

\[
\psi(g(B)^{1/2}h(A)g(B)^{1/2}) \geq \hat{h}(B)g(B),
\]

then

\[
g(B)^{1/2}\varphi(g(B)^{1/2}h(A)\bar{g}(B)^{1/2})g(B)^{1/2} \geq \bar{g}(B)^{1/2}\varphi(g(B)^{1/2}h(A)g(B)^{1/2})g(B)^{1/2}
\]

and

\[
\tilde{\psi}(\bar{g}(B)^{1/2}h(A)\bar{g}(B)^{1/2}) \geq \hat{h}(B)\bar{g}(B)
\]

hold.

Proof of Proposition 1 \( \implies \) Proposition A. Put \( \hat{h}(t) = t \) and \( f_1(t) = g_1(t) = 1 \), then

\[
\psi_1(g_1(B)^{1/2}h(A)g_1(B)^{1/2}) = \psi_1(h(A)g_1(A)^{1/2}) = \hat{h}(A)g_1(A) = A \geq B = h(B)g_1(B).
\]

By applying Proposition 1, we have

\[
\psi_1(g_1(B)^{1/2}h(A)g_1(B)^{1/2}) \geq h(B)g_1(B) \implies \psi_2(g_2(B)^{1/2}h(A)g_2(B)^{1/2}) \geq h(B)g_2(B) \implies \psi_3(g_3(B)^{1/2}h(A)g_3(B)^{1/2}) \geq h(B)g_3(B) \implies \cdots \implies \psi_{n-1}(g_{n-1}(B)^{1/2}h(A)g_{n-1}(B)^{1/2}) \geq h(B)g_{n-1}(B)
\]

since \( \hat{h}(t) = t \leq h(t) \), and

\[
\psi_k(g_k(B)^{1/2}h(A)g_k(B)^{1/2}) \geq h(B)g_k(B)
\]

\[
\implies \varphi_{k+1}(g_{k+1}(B)^{1/2}h(A)g_{k+1}(B)^{1/2}) \geq f_{k+1}(B)^{1/2}\varphi_k(g_k(B)^{1/2}h(A)g_k(B)^{1/2})f_{k+1}(B)^{1/2}
\]

for \( k = 1, 2, \ldots, n-1 \). Therefore

\[
\varphi_n(g_n(B)^{1/2}h(A)g_n(B)^{1/2}) \geq f_n(B)^{1/2}\varphi_{n-1}(g_{n-1}(B)^{1/2}h(A)g_{n-1}(B)^{1/2})f_n(B)^{1/2}
\]

\[
\geq f_n(B)^{1/2}f_{n-1}(B)^{1/2}\varphi_{n-2}(g_{n-2}(B)^{1/2}h(A)g_{n-2}(B)^{1/2})f_{n-1}(B)^{1/2}f_n(B)^{1/2}
\]

\[
\geq \cdots
\]

\[
\geq f_n(B)^{1/2}f_{n-1}(B)^{1/2}\cdots f_2(B)^{1/2}\varphi_1(g_1(B)^{1/2}h(A)g_1(B)^{1/2})f_2(B)^{1/2}\cdots f_n(B)^{1/2}
\]

\[
= g_n(B)^{1/2}\hat{h}(A)g_n(B)^{1/2}.
\]

\(\square\)
References


[2] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101 (1987), 85–88.


