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Extensions of the results on powers of $p$-hyponormal operators to class $wF(p, r, q)$ operators

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This report is based on “M.Ito, Parallel results to that on powers of $p$-hyponormal, log-hyponormal and class $A$ operators, to appear in Acta Sci. Math. (Szeged).”

Abstract

In this report, we shall show that inequalities

$$(T^{n+1}T^{n+1})_{n+1}^{\frac{n}{n+1}} \geq (T^{n}T^{n+1})_{n+1}^{\frac{n}{n+1}}$$

and

$$(T^{n+1}T^{n+1})_{n+1}^{\frac{n}{n+1}} \geq (T^{n+1}T^{n+1})_{n+1}^{\frac{n}{n+1}}$$

for $0 < p \leq 1$ and all positive integer $n$ hold for weaker conditions than $p$-hyponormality, that is, class $F(p, r, q)$ defined by Fujii-Nakamoto or class $wF(p, r, q)$ defined by Yang-Yuan under appropriate conditions of $p$, $r$ and $q$.

1 Introduction

In this report, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

As an extension of hyponormal operators, i.e., $T^*T \geq TT^*$, it is well known that $p$-hyponormal operators for $p > 0$ are defined by $(T^*T)^p \geq (TT^*)^p$, and also an operator $T$ is said to be $p$-quasihyponormal for $p > 0$ if $T^*[(T^*T)^p - (TT^*)^p]T \geq 0$. It is easily obtained that every $p$-hyponormal operator is $q$-hyponormal for $p > q > 0$ by Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.”

On powers of $p$-hyponormal operators, Aluthge-Wang [1] showed that “If $T$ is a $p$-hyponormal operator for $0 < p \leq 1$, then $T^n$ is $\frac{p}{n}$-hyponormal for any positive integer $n$.” As a more precise result than theirs, Furuta-Yanagida [8] obtained the following.

**Theorem 1.A** ([8]). Let $T$ be a $p$-hyponormal operator for $0 < p \leq 1$. Then

$$(T^nT^n)^{\frac{p+1}{n}} \geq \cdots \geq (T^2T^2)^{\frac{p+1}{n}} \geq (T^*T)^{p+1},$$

that is,

$$|T^n|^{\frac{2(p+1)}{n}} \geq \cdots \geq |T^2|^{p+1} \geq |T|^{2(p+1)}$$

and

$$(TT^*)^{p+1} \geq (T^2T^2)^{\frac{p+1}{2}} \geq \cdots \geq (T^nT^n)^{\frac{p+1}{n}},$$

that is,

$$|T^*|^{2(p+1)} \geq |T^2|^p \geq \cdots \geq |T^n|^{\frac{2(p+1)}{n}}$$

hold for all positive integer $n$. 

Recently, Gao-Yang [9] obtained the results on comparison of $n$th power and $(n+1)$th power of $p$-hyponormal operators for $0 < p \leq 1$.

**Theorem 1.B ([9]).** Let $T$ be a $p$-hyponormal operator for $0 < p \leq 1$. Then

\[
(T^{n+1}T^{n+1})^{\frac{n+p}{n+1}} \geq (T^nT^n)^{\frac{n+2}{n}}, \quad \text{that is,} \quad |T^{n+1}|^{\frac{n+2}{n+1}} \geq |T^n|^2
\]

and

\[
(T^nT^n)^{\frac{n+p}{n}} \geq (T^{n+1}T^{n+1})^{\frac{n+p}{n+1}}, \quad \text{that is,} \quad |T^n|^\frac{n+p}{n} \geq |T^{n+1}|^{\frac{n+p}{n+1}}
\]

hold for all positive integer $n$.

As an extension of hyponormal operators, it is also well known that invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator $T$. We remark that we treat only invertible log-hyponormal operators in this paper (see also [17]). It is easily obtained that every invertible $p$-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We note that log-hyponormality is sometimes regarded as 0-hyponormality since $\frac{X^p - I}{p} \rightarrow \log X$ as $p \rightarrow +0$ for $X > 0$. An operator $T$ is paranormal if $\|T^2x\| \geq \|Tx\|^p$ for every unit vector $x \in \mathcal{H}$. Ando [2] showed that every $p$-hyponormal operator for $p > 0$ and invertible log-hyponormal operator is paranormal. (Invertibility of a log-hyponormal operator is not necessarily required.)

Yamazaki [18] showed that “If $T$ is an invertible log-hyponormal operator, then $T^n$ is also log-hyponormal for any positive integer $n,”$ and also he obtained the following results.

**Theorem 1.C ([18]).** Let $T$ be an invertible log-hyponormal operator. Then

\[
(T^nT^n)^{\frac{1}{2}} \geq \cdots \geq (T^2T^2)^{\frac{1}{2}} \geq T^*T, \quad \text{that is,} \quad |T^n|^\frac{1}{2} \geq \cdots \geq |T^2| \geq |T|^2
\]

and

\[
TT^* \geq (T^2T^2)^{\frac{1}{2}} \geq \cdots \geq (T^nT^n)^{\frac{1}{2}}, \quad \text{that is,} \quad |T^*|^2 \geq |T^2|^2 \geq \cdots \geq |T^n|^\frac{1}{2}
\]

hold for all positive integer $n$.

**Theorem 1.D ([18]).** Let $T$ be an invertible log-hyponormal operator. Then

\[
(T^{n+1}T^{n+1})^{\frac{n+1}{n+1}} \geq T^nT^n, \quad \text{that is,} \quad |T^{n+1}|^{\frac{n+1}{n+1}} \geq |T^n|^2
\]

and

\[
T^nT^n \geq (T^{n+1}T^{n+1})^{\frac{n+1}{n+1}}, \quad \text{that is,} \quad |T^n|^2 \geq |T^{n+1}|^{\frac{2n}{n+1}}
\]

hold for all positive integer $n$. 
We remark that Theorems 1.C and 1.D correspond to Theorems 1.A and 1.B, respectively. On powers of \( p \)-hyponormal and log-hyponormal operators, related results are obtained in [7], [13], [22], [24] and so on.

On the other hand, in [6], we introduced class A defined by \(|T^2| \geq |T|^2\) where \( |T| = (T^*T)^{\frac{1}{2}}\), and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms.

As we have pointed out in [14], we have the following result by combining [20, Theorem 1] and [15, Theorem 3] as a result on powers of class A operators. We remark that Theorem 1.E in case of invertible operators was shown in [11].

**Theorem 1.E ([20][15][14]).** If \( T \) is a class A operator, then

(i) \(|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2\) and \(|T^{n^*}|^2 \geq |T^{n+1^*}|^{\frac{2n}{n+1}}\) hold for all positive integer \( n \).

(ii) \(|T^n|^\frac{2}{n} \geq \cdots \geq |T|^2\) and \(|T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^*|^\frac{2}{n}\) hold for all positive integer \( n \).

(i) (resp. (ii)) of Theorem 1.E is an extension of Theorem 1.D (resp. Theorem 1.C) since every invertible log-hyponormal operator belongs to class A.

As generalizations of class A and paranormality, Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [3] introduced class \( A(p, r) \), Yamazaki-Yanagida [19] introduced absolute-\((p, r)\)-paranormality, and Fujii-Nakamoto [4] introduced class \( F(p, r, q) \) and \((p, r, q)\)-paranormality as follows:

**Definition.**

(i) For each \( p > 0 \) and \( r > 0 \), an operator \( T \) belongs to class \( A(p, r) \) if

\[
(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.
\]

(ii) For each \( p > 0 \) and \( r > 0 \), an operator \( T \) is absolute-\((p, r)\)-paranormal if

\[
||T|^p|T^*|^r x||^r \geq ||T^*|^r x||^{p+r}
\]

for every unit vector \( x \in H \).

(iii) For each \( p > 0, r \geq 0 \) and \( q > 0 \), an operator \( T \) belongs to class \( F(p, r, q) \) if

\[
(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^\frac{2q(p+r)}{q+1}.
\]
For each $p > 0$, $r \geq 0$ and $q > 0$, an operator $T$ is $(p, r, q)$-paranormal if

$$\| |T|^p U |T|^r x\|^{\frac{1}{q}} \geq \| |T|^p x\|$$

(1.1)

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of $T$. In particular, if $r > 0$ and $q \geq 1$, then (1.1) is equivalent to

$$\| |T|^p |T^*|^r x\|^{\frac{1}{q}} \geq \| |T|^p x\|$$

for every unit vector $x \in H$ ([12]).

We remark that class $F(p, r, \frac{p+r}{r})$ equals class $A(p, r)$ and also class $F(1,1,2)$ (i.e., class $A(1,1)$) equals class $A$. Similarly, $(p, r, \frac{p+r}{r})$-paranormality equals absolute-$(p, r)$-paranormality and also $(1,1,2)$-paranormality (i.e., absolute-$(1,1)$-paranormality) equals paranormality.

Inclusion relations among these classes were shown in [3], [4], [12], [14], [15], [19] and so on (see also Theorems 3.A and 3.B). The following Figure 1 represents the inclusion relations among the families of class $F(p, r, q)$ and $(p, r, q)$-paranormality.

We can pick up inclusion relations among classes discussed in this report as follows: For $0 < \delta < p < 1$ and $0 < r < 1$,
\[ \delta \text{-hyponormal} \subset \text{class } F(p, r, \frac{r+1}{\delta+1}) \cap \text{class } F(1, 1, \frac{2}{\delta+1}) \cap \text{log-hyponormal} \subset \text{class } A(p, r) \subset \text{class } A \]

We remark that we assume invertibility on log-hyponormal operators.

In this report, as a parallel result to Theorem 1.E, we shall show that inequalities in Theorems 1.A and 1.B hold for weaker conditions than \( p \)-hyponormality, that is, class \( F(p, r, q) \) defined by Fujii-Nakamoto or class \( wF(p, r, q) \) recently defined by Yang-Yuan [23][21] (see Section 3) under appropriate conditions of \( p, r \) and \( q \).

### 2 Main results

In this section, we shall show our main results.

**Theorem 2.1.** If \( (|T^*||T^2||T^*|)^{\frac{\delta+1}{\delta}} \geq |T^*|^{2(\delta+1)} \) (i.e., \( T \) belongs to class \( F(1, 1, \frac{2}{\delta+1}) \)) for some \( 0 \leq \delta \leq 1 \), then

(i) \( |T^{n+1}|^{\frac{2(\delta+n)}{n}} \geq |T^n|^{\frac{2(\delta+n)}{n}} \) holds for all positive integer \( n \).

(ii) \( |T^n|^{\frac{2(\delta+n)}{n}} \geq \cdots \geq |T^2|^{\delta+1} \geq |T|^{2(\delta+1)} \) holds for all positive integer \( n \).

**Theorem 2.2.** If \( |T|^{2(\gamma+1)} \geq (|T||T^2||T^*|)^{\frac{r+1}{r}} \) for some \( 0 \leq \gamma \leq 1 \) holds and either

(a) \( (|T^*||T^2||T^*|)^{\frac{1}{r}} \geq |T^*|^2 \) (i.e., \( T \) belongs to class \( A \)) or

(b) \( N(|T|) \subseteq N(|T^*|) \)

holds, then

(i) \( |T^{n+1}|^{\frac{2(\gamma+n)}{n}} \geq |T^n|^{\frac{2(\gamma+n)}{n}} \) holds for all positive integer \( n \).

(ii) \( |T^*|^{2(\gamma+1)} \geq |T^2|^{\gamma+1} \geq \cdots \geq |T^n|^{\frac{2(\gamma+1)}{n}} \) holds for all positive integer \( n \).

We need the following results in order to prove Theorems 2.1 and 2.2.

**Theorem 2.A ([15]).** Let \( A \) and \( B \) be positive operators. Then for each \( p \geq 0 \) and \( r \geq 0 \),

(i) If \( (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{p+r}} \geq B^r \), then \( A^p \geq (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{p}{p+r}} \).

(ii) If \( A^p \geq (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{p}{p+r}} \) and \( N(A) \subseteq N(B) \), then \( (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{p+r}} \geq B^r \).
Theorem 2.2 ([20]). Let $A$ and $B$ be positive operators. Then

(i) If $(B^{\frac{\alpha_0}{2}} A^\alpha B^\frac{\alpha_0}{2})^\frac{1}{\alpha_0+\beta_0} \geq B^\beta$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[(B^{\frac{\alpha_0}{2}} A^\alpha B^\frac{\alpha_0}{2})^\frac{1}{\alpha_0+\beta} \geq B^\beta\]

holds for any $\beta \geq \beta_0$. Moreover,

\[A^{\frac{\alpha_0}{2}} B^{\beta_1} A^{\frac{\alpha_0}{2}} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_2} A^{\frac{\alpha_0}{2}})^{\frac{\alpha+\beta_1}{\alpha_0+\beta_2}}\]

holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq \beta_0$.

(ii) If $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

\[A^\alpha \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha}{\alpha_0+\beta_0}}\]

holds for any $\alpha \geq \alpha_0$. Moreover,

\[(B^{\frac{\alpha_0}{2}} A^{\alpha_2} B^\frac{\alpha_0}{2})^{\frac{\alpha_2+\alpha_0}{\alpha_0+\beta_0}} \geq B^{\frac{\alpha_0}{2}} A^{\alpha_1} B^\frac{\alpha_0}{2}\]

holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Lemma 2.2 ([20][16]). Let $A$, $B$ and $C$ be positive operators. Then for $p > 0$ and $0 < r \leq 1$,

(i) If $(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{r}{p+r}} \geq B^r$ and $B \geq C$, then $(C^{\frac{p}{2}} A^p C^{\frac{p}{2}})^{\frac{r}{p+r}} \geq C^r$.

(ii) If $A \geq B$, $B^r \geq (B^{\frac{p}{2}} C^p B^{\frac{p}{2}})^{\frac{r}{p+r}}$ and $N(A) = N(B)$, then $A^r \geq (A^{\frac{p}{2}} C^p A^{\frac{p}{2}})^{\frac{r}{p+r}}$.

Lemma 2.3 ([5]). Let $A > 0$ and $B$ be an invertible operator. Then

\[(B A B^*)^\lambda = B A^{\frac{1}{2}} (A^{\frac{1}{2}} B^* B A^{\frac{1}{2}})^{\lambda-1} A^{\frac{1}{2}} B^*\]

holds for any real number $\lambda$.

We remark that Lemma 2.3 holds without invertibility of $A$ and $B$ when $\lambda \geq 1$.

Proof of Theorem 2.1. Let $T = U|T|$ be the polar decomposition of $T$, and put $A_k = (T^k T^*)^\frac{1}{2} = |T^k|^\frac{1}{2}$ and $B_k = (T^k T^*)^\frac{1}{2} = |T^k|^\frac{1}{2}$ for a positive integer $k$. We remark that $T^* = U^*|T^*|$ is also the polar decomposition of $T^*$. 
Firstly we shall show $|T^2|^\delta+1 \geq |T|^2(\delta+1)$. By the hypothesis $((|T^*||T^2||T^*|)^{\frac{\delta+1}{2}} \geq |T^*|^2(\delta+1))$ for some $0 \leq \delta \leq 1$, we have

$$|T^2|^\delta+1 = (U^*|T^*||T^2||T^*|U)^{\frac{\delta+1}{2}}$$

$$= U^*(|T^*||T^2||T^*|)^{\frac{\delta+1}{2}}U$$

$$\geq U^*|T^*|^2(\delta+1)U$$

$$= |T|^2(\delta+1).$$

Next we assume that

$$|T^{n+1}|^{\frac{2(\delta+n)}{n+1}} \geq |T^n|^{\frac{2(\delta+n)}{n}}$$

that is, $A_{n+1}^{\delta+n} \geq A_{n}^{\delta+n}$ (2.1) holds for $n = 1, 2, \ldots, k$. By (2.1) and Löwner-Heinz theorem, we have

$$A_{k+1} \geq A_k \geq \cdots \geq A_2 \geq A_1$$

(2.2) since $\frac{1}{\delta+n} \in (0, 1]$ in (2.1). The hypothesis $((|T^*||T^2||T^*|)^{\frac{\delta+1}{2}} \geq |T^*|^2(\delta+1))$ can be rewritten by $(B_1^\frac{1}{2}A_1B_1^\frac{1}{2})^{\frac{\delta+1}{2}} \geq B_1^{\delta+1}$, and also this yields $A_1 \geq (A_1^\frac{1}{2}B_1A_1^\frac{1}{2})^{\frac{1}{2}}$ by Löwner-Heinz theorem and (i) of Theorem 2.A. (2.2) and $A_1 \geq (A_1^\frac{1}{2}B_1A_1^\frac{1}{2})^{\frac{1}{2}}$ ensure

$$A_k \geq (A_k^\frac{1}{2}B_1A_k^\frac{1}{2})^{\frac{1}{2}}$$

(2.3) by (ii) of Lemma 2.C since $N(A_k) = N(A_1)$ holds. We remark that $N(A_k) \subseteq N(A_1)$ holds by (2.2) and $N(A_k) = N(T^k) \supseteq N(T) = N(A_1)$ always holds. Then we get

$$A_k^k \geq (A_k^\frac{1}{2}B_1A_k^\frac{1}{2})^{k+1}$$

(2.4) by (2.3) and (ii) of Theorem 2.B. Similarly, (2.2) and $A_1 \geq (A_1^\frac{1}{2}B_1A_1^\frac{1}{2})^{\frac{1}{2}}$ ensure

$$A_{k+1} \geq (A_{k+1}^\frac{1}{2}B_1A_{k+1}^\frac{1}{2})^{\frac{1}{2}}.$$ (2.5)

Therefore we have

$$|T^{k+1}|^{\frac{2(\delta+k+1)}{k+1}} = (U^*|T^*||T^k^2||T^k|U)^{\frac{\delta+k+1}{k+1}}$$

$$= U^*(B_1^\frac{1}{2}A_k^\frac{1}{2}B_1^\frac{1}{2})^{\frac{\delta+k+1}{k+1}}U$$

$$\leq U^*B_1^\frac{1}{2}A_k^\frac{1}{2}(A_k^\frac{1}{2}B_1A_k^\frac{1}{2})^{k+1}A_k^\frac{1}{2}B_1^\frac{1}{2}U$$ by Lemma 2.D

$$\leq U^*B_1^\frac{1}{2}A_k^\frac{1}{2}A_k^\frac{1}{2}B_1^\frac{1}{2}U$$ by (2.4) and Löwner-Heinz theorem

$$= U^*B_1^\frac{1}{2}A_k^{\delta+k}B_1^\frac{1}{2}U$$

$$\leq U^*B_1^\frac{1}{2}A_k^{\delta+k}B_1^\frac{1}{2}U$$ by (2.1)

$$\leq U^*(B_1^\frac{1}{2}A_k^{\delta+k}B_1^\frac{1}{2})^{\frac{\delta+k+1}{k+1}}U$$

$$= (U^*|T^*||T^{k+1}|^2||T^*|U)^{\frac{\delta+k+1}{k+1}}$$

$$= |T^{k+2}|^{\frac{2(\delta+k+1)}{k+2}}.$$
We remark that the last inequality holds by (ii) of Theorem 2.B since (2.5) holds and $k + 1 \geq \delta + k \geq 1$.

Consequently the proof of (i) is complete. We can easily obtain (ii) by (i) and L"owner-Heinz theorem, so we omit its proof. \qed

Proof of Theorem 2.2. Let $T = U|T|$ be the polar decomposition of $T$, and put $A_k = (T^k T^*)^\frac{1}{k} = |T|^\frac{k}{2}$ and $B_k = (T^k T^*)^{\frac{1}{2}} = |T^k|^{\frac{1}{2}}$ for a positive integer $k$. We remark that $T^* U^* T^*$ is also the polar decomposition of $T^*$.

$|T|^{2(\gamma+1)} \geq (|T| |T^*| |T|)^{\gamma/2}$ and condition (b) ensure condition (a) by L"owner-Heinz theorem and (ii) of Theorem 2.A, so that we have only to prove the case where condition (a) holds.

Firstly we shall show $|T^*|^{2(\gamma+1)} \geq |T^{2*}|^{\gamma+1}$. By the hypothesis $|T|^{2(\gamma+1)} \geq (|T| |T^*| |T|)^{\gamma/2}$ for some $0 \leq \gamma \leq 1$, we have

$$|T^{2*}|^{\gamma+1} = (U |T^*| T |U^*)^{\gamma/2}$$
$$= U (|T^*| T |U^*)^{\gamma/2} U^*$$
$$\leq U |T|^{\gamma+1} U^*$$
$$= |T^*|^{2(\gamma+1)}.$$

Next we assume that

$$|T^n^*|^{2(\gamma+n)} \geq |T^{n+1*}|^{2(\gamma+n+1)}, \text{ that is, } B_n^\gamma n \geq B_{n+1}^\gamma n+1 \tag{2.6}$$

holds for $n = 1, 2, \ldots, k$. By (2.6) and L"owner-Heinz theorem, we have

$$B_1 \geq B_2 \geq \cdots \geq B_k \geq B_{k+1} \tag{2.7}$$

since $\frac{1}{\gamma+n} \in (0, 1]$ in (2.6). Condition (a) can be rewritten by $(B^\frac{1}{2} A_1 B^\frac{1}{2})^\frac{1}{2} \geq B_1$. \tag{2.7} and $(B^\frac{1}{2} A_1 B^\frac{1}{2})^\frac{1}{2} \geq B_1$ ensure

$$(B^\frac{1}{2} A_1 B^\frac{1}{2})^\frac{1}{2} \geq B_1. \tag{2.8}$$

by (i) of Lemma 2.C Then we get

$$(B^\frac{1}{2} A_1 B^\frac{1}{2})^\frac{1}{2+1} \geq B_k^\frac{1}{2} \tag{2.9}$$

by (2.8) and (i) of Theorem 2.B. Similarly, (2.7) and $(B^\frac{1}{2} A_1 B^\frac{1}{2})^\frac{1}{2} \geq B_1$ ensure

$$(B^\frac{1}{2} A_1 B^\frac{1}{2})^\frac{1}{2} \geq B_{k+1}. \tag{2.10}$$
Therefore we have

\[ \left| T^{k+1} \right|^{\frac{2(\gamma+k+1)}{k+1}} \leq (U|T||T^k|^2|T|U^*)^{\frac{\gamma+k+1}{k+1}} \]

\[ \leq U(A^\frac{1}{12}B_{k}^{k}A^\frac{1}{12})^{\frac{\gamma+k+1}{k+1}} \]

by Lemma 2.4

\[ D \geq UA^\frac{1}{12}B_{k+1}^\gamma A^\frac{1}{12} \]

by (2.6)

\[ \geq U(A^\frac{1}{12}B_{k+1}^{k+1}A^\frac{1}{12})^{\frac{(\gamma+k+1)+1}{k+1}} \]

by (i) of Theorem 2.4

We remark that the last inequality holds by (i) of Theorem 2.6 since (2.10) holds and 

\[ k+1 \geq \gamma + k \geq 1. \]

Consequently the proof of (i) is complete. We can easily obtain (ii) by (i) and Löwner-Heinz theorem, so we omit its proof.

**Remark.** By putting \( \delta = 0 \) in Theorem 2.1 and \( \gamma = 0 \) in Theorem 2.2, we get Theorem 1.6 since \( (\left| T^* \right| |T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2 \) (i.e., \( T \) belongs to class A) ensures \( |T|^2 \geq (\left| T \right| |T^*|^2|T|)^{\frac{1}{2}} \) by (i) of Theorem 2.4.

### 3 Classes \( F(p, r, q) \) and \( wF(p, r, q) \) operators

Recently, in order to continue the study of class \( F(p, r, q) \), Yang-Yuan [23][21] introduced class \( wF(p, r, q) \) operators as follows: For each \( p \geq 0, r \geq 0 \) and \( q \geq 1 \) with \( (p, r) \neq (0,0) \) and \( (p, q) \neq (0,1) \), an operator \( T \) belongs to class \( wF(p, r, q) \) if

\[ (|T^*|^2|T^2p|T^*|)^{\frac{1}{2}} \geq |T^*|^{\frac{2(p+r)}{q}} \]  

(3.1)

and

\[ |T|^{2(p+r)}(1-\frac{1}{q}) \geq (|T|^p|T^*|^r|T|^p)^{1-\frac{1}{q}}, \]  

(3.2)

denoting \( (1-q^{-1})^{-1} \) by \( q^* \) when \( q > 1 \) because \( q \) and \( (1-q^{-1})^{-1} \) are a couple of conjugate exponents. On discussions of class \( wF(p, r, q) \) (or class \( F(p, r, q) \)), we frequently consider class \( wF(p, r, \frac{p+r}{\delta+r}) \) (or class \( F(p, r, \frac{p+r}{\delta+r}) \)) by putting \( q = \frac{p+r}{\delta+r} \) as follows: For \( p \geq 0, r \geq 0 \) and \( -r < \delta \leq p \) with \( (p, r) \neq (0,0) \) and \( (p, \delta) \neq (0,0) \), an operator \( T \) belongs to class \( wF(p, r, \frac{p+r}{\delta+r}) \) if

\[ (|T^*|^2|T^2p|T^*|)^{\frac{4(p+r)}{\delta+r}} \geq |T^*|^{2(\delta+r)} \]  

(3.3)
\[ |T|^{2(-\delta+p)} \geq (|T|^{p}|T^{*}|^{2r}|T|^{p})^{\frac{\delta+r}{p+r}}. \] (3.4)

We remark that (3.1) is the definition of class $F(p, r, q)$. We also remark that class $wF(p, r, \frac{p+r}{\delta+r})$ equals class $wA(p, r)$ defined in [10], and also it was shown in [15] that class $wA(p, r)$ (i.e., class $wF(p, r, \frac{p+r}{\delta+r})$) coincides with class $A(p, r)$. On inclusion relations of classes $A(p, r), F(p, r, q)$ and $wF(p, r, q)$, the following results were obtained.

**Theorem 3.A.**

(i) For invertible operator $T$, $T$ is log-hyponormal if and only if $T$ belongs to class $A(p, r)$ for all $p > 0$ and $r > 0$ ([3]).

(ii) If $T$ belongs to class $A(p_0, r_0)$ for $p_0 > 0$, $r_0 > 0$, then $T$ belongs to class $A(p, r)$ for any $p \geq p_0$ and $r \geq r_0$ ([15]).

We note that log-hyponormality can be regarded as class $A(0, 0)$ by Theorem 3.A.

**Theorem 3.B.**

(i) For a fixed $\delta > 0$, $T$ is $\delta$-hyponormal if and only if $T$ belongs to class $F(2\delta p, 2\delta r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., $T$ belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(\delta+r)q \geq p+r$ ([4]).

(ii) For each $p > 0$ and $r > 0$, $T$ is $p$-quasihyponormal if and only if $T$ belongs to class $F(p, r, 1)$ ([12]).

(iii) If $T$ belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then $T$ belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$ ([4]).

(iv) If $T$ belongs to class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 \geq 0$ and $0 \leq \delta \leq p_0$, then $T$ belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$ ([14]).

(v) If $T$ belongs to class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 \geq 0$ and $-r_0 < \delta \leq p_0$, then $T$ belongs to class $F(p_0, r, \frac{p_0+r}{\delta+r})$ for any $r \geq r_0$ ([12]).

**Theorem 3.C ([23]).**

(i) If $T$ belongs to class $wF(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then $T$ belongs to class $wF(p_0, r_0, q)$ for any $q \geq q_0$ with $r_0q \leq p_0 + r_0$.

(ii) If $T$ belongs to class $wF(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$, $q_0 \geq 1$ and $N(T) \subseteq N(T^*)$, then $T$ belongs to class $wF(p_0, r_0, q)$ for any $q$ such that $q^* \geq q_0^*$ with $p_0q^* \leq p_0 + r_0$. 


(iii) If $T$ belongs to class $wF(p_0, r_0, \frac{p_0 + r_0}{\delta + r_0})$ for $p_0 > 0, r_0 \geq 0$ and $-r < \delta \leq p_0$, then $T$ belongs to class $wF(p, r, \frac{p + r}{\delta + r})$ for any $p \geq p_0$ and $r \geq r_0$.

(iv) If $p > 0$, $r \geq 0$, $q \geq 1$ with $rq \leq p + r$, then class $wF(p, r, q)$ coincides with class $F(p, r, q)$. In other words, if $p > 0$, $r \geq 0$, $0 \leq \delta \leq p$ and $\delta + r \neq 0$, then class $wF(p, r, \frac{p + r}{\delta + r})$ coincides with class $F(p, r, \frac{p + r}{\delta + r})$.

In this section, firstly we shall get a relation between $p$-hyponormality and class $wF(p, r, q)$ (or class $F(p, r, q)$). We remark that Theorem 3.1 is a parallel result to (i) of Theorem 3.A.

**Theorem 3.1.**

(i) For a fixed $\delta > 0$, $T$ is $\delta$-hyponormal (i.e., $T$ belongs to class $F(p_0, 0, \frac{p_0}{\delta})$ for some $p_0 \geq \delta$) if and only if $T$ belongs to class $F(p, r, \frac{p + r}{\delta + r})$ for all $p \geq \delta$ and $r \geq 0$.

(ii) For a fixed $\delta < 0$, $T$ is $(-\delta)$-hyponormal (i.e., $T$ belongs to class $wF(0, r_0, \frac{r_0}{\delta + r_0})$ for some $r_0 > -\delta$) if and only if $T$ belongs to class $wF(p, r, \frac{p + r}{\delta + r})$ for all $p \geq 0$ and $r > -\delta$.

For $0 < \delta < p < 1$ and $0 < -\delta' < r < 1$, inclusion relations among class $wF(p, r, q)$ and other classes can be expressed as the following diagram. We remark that we assume invertibility on log-hyponormal operators, and also $N(T) \subseteq N(T^*)$ is required in (*).

$$
\begin{align*}
\delta\text{-hyponormal} & \subset \text{class } F(p, r, \frac{p + r}{\delta + r}) & \subset \text{class } F(1, 1, \frac{2}{\delta + 1}) \\
\cap & \cap & \cap \\
\log\text{-hyponormal} & \subset \text{class } A(p, r) & \subset \text{class } A \\
\cup & \cup (*) & \cup (*) \\
(-\delta')\text{-hyponormal} & \subset \text{class } wF(p, r, \frac{p + r}{\delta + r}) & \subset \text{class } wF(1, 1, \frac{2}{\delta + 1})
\end{align*}
$$

Next we shall obtain the following corollaries led by Theorems 2.1 and 2.2, and also Theorems 1.A and 1.B follow from these corollaries.

**Corollary 3.2.** If $T$ belongs to class $F(p, r, \frac{p + r}{\delta + r})$ for some $0 \leq \delta \leq 1$, $0 < p \leq 1$ and $0 \leq r \leq 1$ such that $-r < \delta \leq p$, then

(i) $|T^{n+1}|^{\frac{2\delta + n}{n+1}} \geq |T^n|^{\frac{2\delta + n}{n}}$ holds for all positive integer $n$.

(ii) $|T^n|^{\frac{2\delta + n}{n}} \geq \cdots \geq |T^2|^{\delta + 1} \geq |T|^{2(\delta + 1)}$ holds for all positive integer $n$.  

Corollary 3.3. If $T$ belongs to class $wF(p, r, \frac{r + \delta}{\delta + r})$ for some $-1 \leq \delta \leq 0$, $0 \leq p \leq 1$ and $0 \leq r \leq 1$ such that $-r < \delta < p$, and $T$ satisfies $N(T) \subseteq N(T^*)$, then

(i) $|T^n|^\frac{2(\delta+n)}{n} \geq |T^{n+1}|^\frac{2(\delta+n+1)}{n+1}$ holds for all positive integer $n$.

(ii) $|T^n|^{2(-\delta+1)} \geq |T^{2}|^{-\delta+1} \geq \cdots \geq |T^n|^\frac{2(\delta+1)}{n}$ holds for all positive integer $n$.

We omit proofs of the results in this section.

References


[15] M. Ito and T. Yamazaki, *Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{\prime}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\prime}{p+r}}$ and their applications*, Integral Equations and Operator Theory, 44 (2002), 442–450.


