 BLOCK MATRIX OPERATORS FOR $p$-HYPONORMALITY

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ABSTRACT. We introduce a new model of block matrix operator $M(\alpha, \beta)$ induced by two sequences $\alpha$ and $\beta$ and characterize its $p$-hyponormality. The model induces a measurable transformation $T$ on the set of nonnegative integers $\mathbb{N}_0$ with point mass and composition operator $C_T$ on $l^2 := l^2(\mathbb{N}_0)$. The techniques via composition operators will be used to treat $p$-hyponormality of $M(\alpha, \beta)$ and provide some interesting theorems about $p$-hyponormality. Finally, we apply our results to obtain examples of $p$-hyponormal making distinct as usual.

1. Introduction and Preliminaries. This was talked at the 2008 RIMS conference: Inequalities on linear operators and its applications, which was held at Kyoto University on January 30-February 1 in 2008.

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$, $p \in (0, \infty)$. If $p = 1$, $T$ is hyponormal and if $p = \frac{1}{2}$, $T$ is semi-hyponormal ([Xi]). In particular, $T$ is said to be $\infty$-hyponormal if it is $p$-hyponormal for all $p > 0$ ([MS]). The Löwner-Heinz inequality implies that every $p$-hyponormal operators are $q$-hyponormal operators for $q \leq p$ and many operator theorists have studied properties in operators in those classes; for examples, spectral theory, operator inequalities, and invariant subspaces, etc. (cf. [BJ], [Fur], [TY], [JKP], [JLPa]). Also, the study of gaps between subnormality and hyponormality has been studied in several areas by many operator theorists, and whose study is growing up still. The $p$-hyponormality is contained in those studies, but new models for $p$-hyponormal operators need to be developed still. And also, Jung-Lee-Park constructed examples induced by some block matrix operators in [JLP] and [JLL], in which the classes of those operators are distinct with respect to any positive real number $p$. Recently Burnap-Jung-Lambert discussed some models via composition operator $C_T$ on $L^2$ in [BJL] and [BJ], in which such classes of weak hyponormal operators are distinct for each $p$. Moreover, they used the notion of conditional expectations for studying of $p$-hyponormality of $C_T$, which will be also main tool of this note. Here are some terminologies for conditional expectation. Let $(X, \mathcal{F}, \mu)$ be a a finite measure space and let $T : X \to X$ be a transformation such that $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ T^{-1}/d\mu$ is in $L^\infty$. The composition operator $C_T$ acting on $L^2 := L^2(X, \mathcal{F}, \mu)$ is defined by $C_Tf = f \circ T$.  

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The condition $h \in L^\infty$ assures that $C_T$ is bounded. And we denote $Ef = E(f|T^{-1}\mathcal{F})$ for the conditional expectation of $f$ with respect to $T^{-1}\mathcal{F}$. Some useful results will come from [L], [BJL], and [HWh]. In particular, in the proofs and examples below, we will have need of the following special case: if $\mathcal{A}$ is the purely atomic $\sigma$-subalgebra of $\mathcal{F}$ generated by the measurable partition of $X$ into sets of positive measure $\{A_k\}_{k \geq 0}$, then

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)} \left( \int_{A_k} f(x) d\mu(x) \right) \chi_{A_k}.$$  

The interested readers can find a more extensive list of properties for conditional expectations in [BJL] and [Ra].

This article consists of five sections. In Section 2, we construct a block matrix operator induced by two sequences $\alpha$ and $\beta$, which will make distinct classes of $p$-hyponormal operators with respect to $p > 0$ later section. A block matrix operator $M(\alpha, \beta)$ induced by two sequences $\alpha$ and $\beta$ provides a measurable transformation $T$ on $\mathbb{N}_0$ with point mass measure on $\mathbb{N}_0$ and its corresponding composition operator $C_T$ on $l^2$ is equivalent to $M(\alpha, \beta)$. In Section 3, we characterize block matrix operators $M(\alpha, \beta)$ for $p$-hyponormality and construct a useful form for distinction examples. In Section 4, we discuss a flatness of $p$-hyponormality about block matrix operator $M(\alpha, \beta)$: the $\infty$-hyponormality of $M(\alpha, \beta)$ is equivalent to any[some] $p$-hyponormality under some conditions. Finally, in Section 5, we give some examples being distinct the classes of $p$-hyponormal operators.

This article will be appeared in other journal as the full version. And so we skip the detail proofs here.

2. Relationships. Let $\alpha := \{a_i^{(n)}\}_{1 \leq i \leq r}$ and $\beta := \{b_j^{(n)}\}_{1 \leq j \leq s}$ be bounded sequences of positive real numbers. Let $M = [A_{ij}]_{0 \leq i,j < \infty}$ be a block matrix operator whose blocks are $(r + s) \times (s + 1)$ matrices such that $A_{ij} = 0$, $i \neq j$, and

$$A_n := A_{nn} = \begin{pmatrix} a_1^{(n)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_s^{(n)} \end{pmatrix},$$

(2.1)

where other entries are 0 except $a_i^{(n)}$ and $b_j^{(n)}$ indicated in (2.1). Obviously such block matrix operator $M$ is bounded.

Definition 2.1. For two bounded sequences $\alpha := \{a_i^{(n)}\}_{1 \leq i \leq r}$ and $\beta := \{b_j^{(n)}\}_{1 \leq j \leq s}$, the block matrix operator $M := M(\alpha, \beta)$ satisfying (2.1) is called a block matrix operator with weight sequence $(\alpha, \beta)$.

Let $M$ be a block matrix operator with weight sequence $(\alpha, \beta)$ and let $W_{\alpha, \beta}$ be its corresponding operator on $l^2$ relative to some orthonormal bases. Then $W_{\alpha, \beta}$ has a duplicate form; for example, if we take $r = 3$, $s = 2$ and $a_i^{(n)} = b_j^{(n)} = 1$ for all $i, j, n \in \mathbb{N}$, then the
block matrix operator with \((\alpha, \beta)\) is unitarily equivalent to the following operator \(W_{\alpha, \beta}\) on \(l^2\) defined by
\[
W_{\alpha, \beta}(x_1, x_2, x_3, x_4, x_5, \ldots) = (x_1, x_1, x_1, x_2, x_2, x_3, x_4, x_4, x_5, x_6, x_7, x_7, \ldots).
\]

For arbitrary block matrix operator \(M\) with weight sequence \((\alpha, \beta)\), since \(M\) is \(p\)-hyponormal if and only if \(\alpha M\) is \(p\)-hyponormal for any positive real number \(\alpha\), we may assume \(a_1^{(0)} = 1\), which will be assumed throughout this note.

We now return to our work, in particular, consider \(X = N_0 := N \cup \{0\}\) and the power set \(\mathcal{P}(X)\) of \(X\) for the \(\sigma\)-algebra \(\mathcal{F}\). Define a non-singular measurable transformation \(T\) on \(N_0\) such that
\[
T^{-1}(k(s+1)) = \{k(r+s) + i - 1 : 0 \leq i \leq r\}, \quad k, r = 0, 1, 2, \ldots, \tag{2.2}
\]
\[
T^{-1}(k(s+1) + i) = k(r+s) + r - 1 + i, \quad 1 \leq i \leq s, \quad k = 0, 1, 2, \ldots.
\]

We write \(m(\{i\}) := m_i\) for a point mass measure on \(X\).

**Proposition 2.2.** Under the above notation, the composition operator \(C_T\) on \(l^2\) defined by \(C_T f = f \circ T\) is unitarily equivalent to the block matrix operator \(M(\alpha, \beta)\), where \(\alpha: a_i^{(n)} = \sqrt{\frac{m_{n}(r+s)+i-1}{m_{n+1}(s+r)}} (1 \leq i \leq r)\) and \(\beta: b_j^{(n)} = \sqrt{\frac{m_{n}(r+s)+r+j-1}{m_{n+1}(s+r)+j}} (1 \leq j \leq s)\), \(n \in N_0\).

**Proposition 2.3.** Let \(M(\alpha, \beta)\) be a block matrix with weight sequence \((\alpha, \beta)\), where \(\alpha := \{a_i^{(n)}\}_{0 \leq n \leq \infty}, \beta := \{b_j^{(n)}\}_{0 \leq n \leq \infty}\), and \(a_1^{(0)} = 1\). Then there exists a measurable transformation \(T\) on a \(\sigma\) finite measure space \((N_0, \mathcal{P}(N_0), m)\) such that \(M(\alpha, \beta)\) is unitarily equivalent to a composition operator \(C_T\) on \(l^2\).

### 3. Some Characterizations

Let \(T\) be a non-singular measurable transformation on \(l^2\) as in (2.2) and let \(m(\{i\}) = m_i\) be the point mass on \(N_0\).

**Theorem 3.1.** Let \(p \in (0, \infty)\). Then the following assertions are equivalent:
(i) \(C_T\) is \(p\)-hyponormal on \(l^2\);
(ii) the block matrix operator \(M(\alpha, \beta)\) as in Proposition 2.2 is \(p\)-hyponormal;
(iii) \(E(1/h^p)(n) \leq 1/(h^p \circ T)(n)\)
(iv) it holds that
\[
\frac{1}{m(T^{-1}(T(n)))} \sum_{j \in T^{-1}(T(n))} \frac{m_j^p m_j}{m(T^{-1}(j))^p} \leq \left(\frac{m_T(n)}{m(T^{-1}(T(n)))}\right)^p, \quad n \in N_0.
\]

**Remark 3.2.** By some formulas in the proof of Theorem 3.1, we have the following assertions:
(i) \(M(\alpha, \beta)\) is \(\infty\)-hyponormal if and only if \(m(T^{-1}(n))/m_n \geq m(T^{-1}(T(n))/m(T(n)))\) for all \(n \in N_0\).
(ii) \(M(\alpha, \beta)\) is quasinormal if and only if \(m(T^{-1}(n))/m_n = m(T^{-1}(T(n))/m(T(n)))\) for all \(n \in N_0\).
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To obtain more useful and simpler form for p-hyponormality of $M(\alpha, \beta)$, we consider a block matrix operator $M$ as following:

$$M(\alpha, \beta) : A \equiv A_1 = A_2 = \cdots$$ (with notation in (2.1)) with

$$\alpha : a_i^{(n)} = a_i, \ n \in \mathbb{N}_0, \ 1 \leq i \leq r;$$

$$\beta : b_j^{(n)} = b_j, \ n \in \mathbb{N}_0, \ 1 \leq j \leq s.$$  \hfill (3.1)

This type will be used usefully to obtain examples being distinct classes of p-hyponormal operators in Section 5.

**Theorem 3.3.** Let $M(\alpha, \beta)$ be as in (3.1). Then the block matrix operator $M(\alpha, \beta)$ is p-hyponormal if and only if the following two cases hold:

(i) for $n = k(r + s) + i - 1 \ (1 \leq i \leq r),$

$$\sum_{\substack{j \in T^{-1}(T(n)) \ j \equiv 0 \mod (s+1) \ \text{and} \ \sum_{\substack{1 \leq i \leq r \ a_i^2}} \sum_{\substack{1 \leq i \leq r \ a_i^2}}} \left( \frac{1}{\sum_{\substack{1 \leq i \leq r \ a_i^2}} a_i^2} \right)^p \frac{a_i^2}{\sum_{\substack{1 \leq i \leq r \ a_i^2}} a_i^2} + \sum_{\substack{j \in T^{-1}(T(n)) \ j \not\equiv 0 \mod (s+1)}} \left( \frac{1}{\sum_{\substack{1 \leq i \leq r \ a_i^2}} a_i^2} \right)^p \frac{a_j^2}{\sum_{\substack{1 \leq i \leq r \ a_i^2}} a_j^2}$$

$$\leq \left( \frac{1}{\sum_{\substack{1 \leq i \leq r \ a_i^2}} a_i^2} \right)^p, \ 1 \leq i_j \leq r, \ 1 \leq j \leq s,$$ \hfill (3.2)

(ii) for $n = k(r + s) + r + j - 1 \ (1 \leq j \leq s),$

(ii-a) $b_j^2 \leq \sum_{\substack{1 \leq i \leq r \ a_i^2}} a_i^2$ if $n \equiv 0 \mod (s+1)$

(ii-b) $b_j^2 \leq b_{t_n}^2$ if $n \not\equiv 0 \mod (s+1)$ and for some $t_n \ (1 \leq t_n \leq s).$

The following is a special case of Theorem 3.3, which provides a simple form.

**Corollary 3.4.** Let $M := M(\alpha, \beta)$ be as in (3.1) with $a_i^{(n)} = a \ (1 \leq i \leq r)$ and $b_j^{(n)} = b \ (1 \leq j \leq s).$ Then $M$ is p-hyponormal if and only if the following two cases hold:

(i) for $n = k(r + s) + i - 1 \ (1 \leq i \leq r),$

$$\frac{1}{r} \left[ \sum_{\substack{j \in T^{-1}(T(n)) \ j \equiv 0 \mod (s+1)}} \left( \frac{1}{ra^2} \right)^p \frac{1}{ra^2} + \sum_{\substack{j \in T^{-1}(T(n)) \ j \not\equiv 0 \mod (s+1)}} \frac{1}{b_j^2} \right] \leq \left( \frac{1}{ra^2} \right)^p,$$

(ii) for $n = k(r + s) + r + j - 1 \ (1 \leq j \leq s),$ $b^2 \leq ra^2$ holds.

Note that if we are under type of Theorem 3.3 (which will be called “type I”) it will be important to know which $j$ in $T^{-1}(T(n))$ have various $j \equiv t_j \mod (s+1)$ which if we are under type of Corollary 3.4 (which will be called “type II”) it is only important to know how many $j$ are of various $j \equiv t_j \mod (s+1).$ Then we have the following remark.

**Remark 3.5** (Special case of Corollary 3.4 with $r = N(s+1)$). In this case for $n = n(r + s) + i - 1, \ 1 \leq i \leq r,$ the set of $l$ in $T^{-1}(T(n))$ contains exactly $N$ elements of each modulus, $\mod (s+1).$ So under type II the test (3.2) for such $n$ becomes

$$N \left( \frac{1}{ra^2} \right)^p \frac{1}{r} + (r - N) \left( \frac{1}{b^2} \right)^p \frac{1}{r} \leq \left( \frac{1}{ra^2} \right)^p.$$
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For \(n = k(r + s) + r - 1 + j\), and under type II we either get a condition trivially satisfied for all \(p\), or \(1/(ra^2) \leq 1/b^2\), the latter only if there is at least one \(n\) so that \(n = K(r + s) + r - 1 + j\) and \(n = Q(s + 1)\). But since \(r = N(s + 1)\), this is \((K + 1)N(s + 1) + Ks + j - 1 = Q(s + 1)\) for some \(K, Q, j\), and take \(K = s + 1\) and \(j = 1\) to obtain a solution, so \(1/(ra^2) \leq 1/b^2\).

\textbf{Remark 3.6.} We can apply the idea of Theorem 3.3 to the model of general block matrix operator in the Definition 2.1 by the same method; the result formula will be slight complete than that of Theorem 3.3. We leave the exact formula to interested readers.

\section*{4. \(\infty\)-Hyponormality and Flatness}

We begin this section with the following fundamental lemma.

\textbf{Lemma 4.1.} Suppose \(p > 1\) and \(q > 1\) are relatively prime. Given any \(l_p, 0 \leq l_p \leq p - 1\), and any \(l_q, 0 \leq l_q \leq q - 1\), there exists \(n \in \mathbb{N}\) so that \(n \equiv l_p \mod p\) and \(n \equiv l_q \mod q\).

\textbf{Lemma 4.2.} Suppose that

\[
A := \begin{pmatrix}
\sqrt{y_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sqrt{y_r}
\end{pmatrix}
\quad \text{and} \quad
M := \begin{pmatrix}
A \\
A \\
\vdots \\
A
\end{pmatrix}.
\tag{4.1}
\]

Assume that \(\text{GCD}(r + s, s + 1) = 1\). If \(M\) is \(p\)-hyponormal for some \(p \in (0, \infty)\), then

\[
x_1 = x_2 = \cdots = x_s \leq \sum_{1 \leq i \leq r} y_i.
\tag{4.2}
\]

\textbf{Proposition 4.3.} Let \(A\) and \(M\) be as in (4.1). Suppose there exists \(N \in \mathbb{N}\) such that \(r = N(s + 1)\) and \(\text{GCD}(r + s, s + 1) = 1\). Then the following assertions are equivalent:

(i) \(M\) is \(p\)-hyponormal for some \(p \in (0, \infty)\);

(ii) \(M\) is \(\infty\)-hyponormal;

(iii) \(x_1 = x_2 = \cdots = x_s = \sum_{1 \leq i \leq r} y_i\).

\section*{5. Examples}

Let \(A\) and \(M\) be as in (4.1) with \(r + s = N(s + 1)\) for some \(N \in \mathbb{N}\) and we will see this is the "opposite" of \(r = N(s + 1)\) and \(\text{GCD}(r + s, s + 1) = 1\).

\textbf{Proposition 5.1.} Let \(M\) be the block matrix operator as in (4.1). Then \(M\) is \(p\)-hyponormal if and only if the following inequality holds:

\[
\sum_{j \not\equiv 0 \mod (s+1)} \left(\frac{1}{x_{t_j \mod (s+1)}}\right)^p y_{j+1} \leq \frac{1}{(\sum_{1 \leq i \leq r} y_i)^p} \sum_{j \not\equiv 0 \mod (s+1)} y_{j+1}.
\tag{5.1}
\]

The following corollaries come immediately from Proposition 5.1.

\textbf{Corollary 5.2.} Let \(M\) be the block matrix operator as in (4.1) with \(x_1 = x_2 = \cdots = x_s = x\). Then (5.1) is trivially satisfied as long as \(x \geq \sum_{1 \leq i \leq r} y_i\) with no conditions on the \(y_j\).
Corollary 5.3. Let $M$ be the block matrix operator as in (4.1) such that the $y_{j+1}$ for $j \equiv 0 \mod (s+1)$ occur only in $\sum_{1 \leq i \leq r} y_{i}$. Thus if we consider some $y_{j+1}'$ for $j \equiv 0 \mod (s+1)$, as long as $\sum_{j \equiv 0} y_{j+1}' = \sum_{j \equiv 0} y_{j+1}$, then $M'$ is p-hyponormal if and only if $M$ is p-hyponormal.

Now we close this paper with the following example.

Example 5.4. Let

$$A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & \sqrt{x_1} \\ 0 & \sqrt{x_2} \end{pmatrix}$$

Write $Y$ for $\sum_{1 \leq i \leq 4} y_{i}$. Then the condition of

$$\frac{1}{Y^p} \frac{y_1}{Y} + \frac{1}{x_1^p} \frac{y_2}{Y} + \frac{1}{x_2^p} \frac{y_3}{Y} + \frac{1}{Y^p} \frac{y_4}{Y} \leq \frac{1}{Y^p}$$

is equivalent to

$$\frac{y_2}{x_1^p} + \frac{y_3}{x_2^p} \leq \frac{y_2 + y_3}{4^p}.$$ 

Inserting the $y_i = 1, 1 \leq i \leq 4$, we get

$$\left( \frac{4}{x_1} \right)^p + \left( \frac{4}{x_2} \right)^p \leq 2,$$  \hspace{1cm} (5.2)

which is equivalent to $M$ is p-hyponormal. Note that (5.2) keeps distinct the classes of p-hyponormal operators with respect to $0 < p < \infty$. To obtain region for $\infty$-hyponormality of $M$ we use Remark 3.2 and formulas in proof of Theorem 3.3, and there are three cases, Cases 1a, 1b, and 2b, which imply that $m_{3k} \geq m_{3k}, x_1 \geq 4 \& x_2 \geq 4$, and $x_1 \geq x_1 \& x_2 \geq x_2$, respectively. Thus we obtain that

$$M \text{ is } \infty \text{-hyponormal } \iff x_1 \geq 4 \text{ and } x_2 \geq 4.$$ 

Of course, since (5.2) is equivalent to $x_2 \geq 4 \cdot (4/x_1)^p)^{-1/p}$ for $x_1 > 4 \cdot 2^{-1/p}$, taking $p \to \infty$, we may check easily the obtaining conditions $\infty$-hyponormality of $M$ are $x_1 \geq 4$ and $x_2 \geq 4$. On the other hand, applying Remark 3.2 and formulas in proof of Theorem 3.3 for quasinormality of $M$, we also obtain that $M$ is quasinormal if and only if $(x_1,x_2) = (4,4)$. 

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