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Kyoto University
BLOCK MATRIX OPERATORS FOR \( p \)-HYPONORMALITY

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ABSTRACT. We introduce a new model of block matrix operator \( M(\alpha, \beta) \) induced by two sequences \( \alpha \) and \( \beta \) and characterize its \( p \)-hyponormality. The model induces a measurable transformation \( T \) on the set of nonnegative integers \( \mathbb{N}_0 \) with point mass and composition operator \( C_T \) on \( L^2 := L^2(\mathbb{N}_0) \). The techniques via composition operators will be used to treat \( p \)-hyponormality of \( M(\alpha, \beta) \) and provide some interesting theorems about \( p \)-hyponormality. Finally, we apply our results to obtain examples of \( p \)-hyponormal making distinct as usual.

1. Introduction and Preliminaries. This was talked at the 2008 RIMS conference: Inequalities on linear operators and its applications, which was held at Kyoto University on January 30–February 1 in 2008.

Let \( \mathcal{H} \) be a separable, infinite dimensional complex Hilbert space and let \( L(\mathcal{H}) \) be the algebra of all bounded linear operators on \( \mathcal{H} \). An operator \( T \in L(\mathcal{H}) \) is said to be \( p \)-hyponormal if \( (T^* T)^p \geq (TT^*)^p, \ p \in (0, \infty) \). If \( p = 1 \), \( T \) is hyponormal and if \( p = \frac{1}{2} \), \( T \) is semi-hyponormal ([X]). In particular, \( T \) is said to be \( \infty \)-hyponormal if it is \( p \)-hyponormal for all \( p > 0 \) ([MS]). The Löwner-Heinz inequality implies that every \( p \)-hyponormal operators are \( q \)-hyponormal operators for \( q \leq p \) and many operator theorists have studied properties in operators in those classes; for examples, spectral theory, operator inequalities, and invariant subspaces, etc. (cf. [BJ], [Fur], [TY], [JKP], [JLPa]). Also, the study of gaps between subnormality and hyponormality has been studied in several areas by many operator theorists, and whose study is growing up still. The \( p \)-hyponormality is contained in those studies, but new models for \( p \)-hyponormal operators need to be developed still. And also, Jung-Lee-Park constructed examples induced by some block matrix operators in [JLP] and [JLL], in which the classes of those operators are distinct with respect to any positive real number \( p \). Recently Burnap-Jung-Lambert discussed some models via composition operator \( C_T \) on \( L^2 \) in [BJL] and [BJ], in which such classes of weak hyponormal operators are distinct for each \( p \). Moreover, they used the notion of conditional expectations for studying of \( p \)-hyponormality of \( C_T \), which will be also main tool of this note. Here are some terminologies for conditional expectation. Let \( (X, \mathcal{F}, \mu) \) be a \( \sigma \) finite measure space and let \( T : X \to X \) be a transformation such that \( T^{-1} \mathcal{F} \subset \mathcal{F} \) and \( \mu \circ T^{-1} \ll \mu \). It is assumed that the Radon-Nikodym derivative \( h = d\mu \circ T^{-1} / d\mu \) is in \( L^\infty \). The composition operator \( C_T \) acting on \( L^2 := L^2(X, \mathcal{F}, \mu) \) is defined by \( C_T f = f \circ T \).

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The condition $h \in L^\infty$ assures that $C_T$ is bounded. And we denote $Ef = E(f|T^{-1}\mathcal{F})$ for the conditional expectation of $f$ with respect to $T^{-1}\mathcal{F}$. Some useful results will come from [L], [BJL], and [HWh]. In particular, in the proofs and examples below, we will have need of the following special case: if $\mathcal{A}$ is the purely atomic $\sigma$-subalgebra of $\mathcal{F}$ generated by the measurable partition of $X$ into sets of positive measure $\{A_k\}_{k \geq 0}$, then

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)} \left( \int_{A_k} f(x) d\mu(x) \right) \chi_{A_k}.$$  

The interested readers can find a more extensive list of properties for conditional expectations in [BJL] and [Ra].

This article consists of five sections. In Section 2, we construct a block matrix operator induced by two sequences $\alpha$ and $\beta$, which will make distinct classes of p-hyponormal operators with respect to $p > 0$ later section. A block matrix operator $M(\alpha, \beta)$ induced by two sequences $\alpha$ and $\beta$ provides a measurable transformation $T$ on $\mathcal{N}_0$ with point mass measure on $\mathcal{N}_0$ and its corresponding composition operator $C_T$ on $l^2$ is equivalent to $M(\alpha, \beta)$. In Section 3, we characterize block matrix operators $M(\alpha, \beta)$ for p-hyponormality and construct a useful form for distinction examples. In Section 4, we discuss a flatness of p-hyponormality about block matrix operator $M(\alpha, \beta)$: the $\infty$-hyponormality of $M(\alpha, \beta)$ is equivalent to any[some] p-hyponormality under some conditions. Finally, in Section 5, we give some examples being distinct the classes of p-hyponormal operators.

This article will be appeared in other journal as the full version. And so we skip the detail proofs here.

2. Relationships. Let $\alpha := \{a_i^{(n)}\}_{1 \leq i \leq r}$ and $\beta := \{b_j^{(n)}\}_{1 \leq j \leq s}$ be bounded sequences of positive real numbers. Let $M = [A_{ij}]_{0 \leq i, j < \infty}$ be a block matrix operator whose blocks are $(r + s) \times (s + 1)$ matrices such that $A_{ij} = 0$, $i \neq j$, and

$$A_n := A_{nn} = \begin{pmatrix} a_1^{(n)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_s^{(n)} \end{pmatrix},$$  \hspace{1cm} (2.1)

where other entries are 0 except $a_\ast^{(n)}$ and $b_\ast^{(n)}$ indicated in (2.1). Obviously such block matrix operator $M$ is bounded.

**Definition 2.1.** For two bounded sequences $\alpha := \{a_i^{(n)}\}_{1 \leq i \leq r}$ and $\beta := \{b_j^{(n)}\}_{1 \leq j \leq s},$ the block matrix operator $M := M(\alpha, \beta)$ satisfying (2.1) is called a block matrix operator with weight sequence $(\alpha, \beta)$.

Let $M$ be a block matrix operator with weight sequence $(\alpha, \beta)$ and let $W_{\alpha,\beta}$ be its corresponding operator on $l^2$ relative to some orthonormal bases. Then $W_{\alpha,\beta}$ has a duplicate form; for example, if we take $r = 3$, $s = 2$ and $a_i^{(n)} = b_j^{(n)} = 1$ for all $i, j, n \in \mathbb{N}$, then the
block matrix operator with \((\alpha, \beta)\) is unitarily equivalent to the following operator \(W_{\alpha, \beta}\) on \(l^2\) defined by

\[
W_{\alpha, \beta}(x_1, x_2, x_3, x_4, x_5, \cdots) = (x_1, x_1, x_1, x_2, x_2, x_4, x_4, x_5, x_5, x_7, x_7, \cdots).
\]

(3)

(3)

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For arbitrary block matrix operator \(M\) with weight sequence \((\alpha, \beta)\), since \(M\) is \(p\)-hyponormal if and only if \(\alpha M\) is \(p\)-hyponormal for any positive real number \(\alpha\), we may assume \(a_1^{(0)} = 1\), which will be assumed throughout this note.

We now return to our work, in particular, consider \(X = N_0 := N \cup \{0\}\) and the power set \(\mathcal{P}(X)\) of \(X\) for the \(\sigma\)-algebra \(\mathcal{F}\). Define a non-singular measurable transformation \(T\) on \(N_0\) such that

\[
T^{-1}(k(s + 1)) = \{k(r + i) + 1 : 0 \leq i \leq r\}, \quad k = 0, 1, 2, \cdots, \tag{2.2}
\]

\[
T^{-1}(k(s + 1) + i) = k(r + s) + r - 1 + i, \quad 1 \leq i \leq s, \quad k = 0, 1, 2, \cdots.
\]

We write \(m(\{i\}) := m_i\) for a point mass measure on \(X\).

**Proposition 2.2.** Under the above notation, the composition operator \(C_T\) on \(l^2\) defined by \(C_T f = f \circ T\) is unitarily equivalent to the block matrix operator \(M(\alpha, \beta)\), where

\[
\alpha : a_i^{(n)} = \sqrt{\frac{m_{(r+i)+1}}{m_{(r+i)}}} (1 \leq i \leq r), \quad n \in N_0,
\]

\[
\beta : b_j^{(n)} = \sqrt{\frac{m_{(r+j)+1}}{m_{(r+j)}}} (1 \leq j \leq s), \quad n \in N_0.
\]

**Proposition 2.3.** Let \(M(\alpha, \beta)\) be a block matrix with weight sequence \((\alpha, \beta)\), where

\[
\alpha := \{a_i^{(n)}\}_{0 \leq n < \infty}^{1 \leq i \leq r}, \quad \beta := \{b_j^{(n)}\}_{0 \leq n < \infty}^{1 \leq j \leq s}, \quad a_1^{(0)} = 1.
\]

Then there exists a measurable transformation \(T\) on a \(\sigma\) finite measure space \((N_0, \mathcal{P}(N_0), m)\) such that \(M(\alpha, \beta)\) is unitarily equivalent to a composition operator \(C_T\) on \(l^2\).

3. Some Characterizations. Let \(T\) be a non-singular measurable transformation on \(l^2\) as in (2.2) and let \(m(\{i\}) = m_i\) be the point mass on \(N_0\).

**Theorem 3.1.** Let \(p \in (0, \infty)\). Then the following assertions are equivalent:

(i) \(C_T\) is \(p\)-hyponormal on \(l^2\);

(ii) the block matrix operator \(M(\alpha, \beta)\) as in Proposition 2.2 is \(p\)-hyponormal;

(iii) \(E(1/h^p)(n) \leq 1/(h^p \circ T)(n)\)

(iv) it holds that

\[
\frac{1}{m(T^{-1}(T(n)))} \sum_{j \in T^{-1}(T(n))} \frac{m_j^{p}m_j}{m(T^{-1}(j))^{p}} \leq \left(\frac{m(T(n))}{m(T^{-1}(T(n)))}\right)^{p}, \quad n \in N_0.
\]

**Remark 3.2.** By some formulas in the proof of Theorem 3.1, we have the following assertions:

(i) \(M(\alpha, \beta)\) is \(\infty\)-hyponormal if and only if \(m(T^{-1}(n))/m_n \geq m(T^{-1}(T(n)))/m(T(n))\) for all \(n \in N_0\).

(ii) \(M(\alpha, \beta)\) is quasinormal if and only if \(m(T^{-1}(n))/m_n = m(T^{-1}(T(n))/m(T(n))\) for all \(n \in N_0\).
BLOCK MATRIX OPERATORS FOR P-HYPONORMALITY

To obtain more useful and simpler form for p-hyponormality of $M(\alpha, \beta)$, we consider a block matrix operator $M$ as following:

$$M(\alpha, \beta) : A \equiv A_1 = A_2 = \cdots \quad \text{(with notation in (2.1)) with}$$

\[\alpha : a_i^{(n)} = a_i, \quad n \in N_0, \quad 1 \leq i \leq r;\]

\[\beta : b_j^{(n)} = b_j, \quad n \in N_0, \quad 1 \leq j \leq s.\]

This type will be used usefully to obtain examples being distinct classes of $p$-hyponormal operators in Section 5.

**Theorem 3.3.** Let $M(\alpha, \beta)$ be as in (3.1). Then the block matrix operator $M(\alpha, \beta)$ is $p$-hyponormal if and only if the following two cases hold:

(i) for $n = k(r + s) + i - 1$ ($1 \leq i \leq r$),

\[
\sum_{\substack{j \notin T^{-1}(T(n))
\quad j \equiv 0 \mod(s+1)}} \left( \frac{1}{\sum_{1\leq i\leq r}a_i^2} I^p \right)^p + \sum_{\substack{j \in T^{-1}(T(n))
\quad j \equiv 0 \mod(s+1)}} \frac{1}{b_j^2} \sum_{1\leq i\leq r} a_i^2 \leq \left( \frac{1}{\sum_{1\leq i\leq r}a_i^2} \right)^p, \quad 1 \leq i_j \leq r, \quad 1 \leq l_j \leq s,
\]

(ii) for $n = k(r + s) + r + j - 1$ ($1 \leq j \leq s$),

(ii-a) $b_j^2 \leq \sum_{1\leq i\leq r} a_i^2$ if $n \equiv 0 \mod(s + 1)$

(ii-b) $b_j^2 \leq b_{t_n}^2$ if $n \not\equiv 0 \mod(s + 1)$ and for some $t_n$ ($1 \leq t_n \leq s$).

The following is a special case of Theorem 3.3, which provides a simple form.

**Corollary 3.4.** Let $M := M(\alpha, \beta)$ be as in (3.1) with $a_i^{(n)} = a$ ($1 \leq i \leq r$) and $b_j^{(n)} = b$ ($1 \leq j \leq s$). Then $M$ is $p$-hyponormal if and only if the following two cases hold:

(i) for $n = k(r + s) + i - 1$ ($1 \leq i \leq r$),

\[
\frac{1}{r} \left[ \sum_{j \in T^{-1}(T(n))
\quad j \equiv 0 \mod(s+1)} \left( \frac{1}{ra^2} \right)^p + \sum_{j \notin T^{-1}(T(n))
\quad j \equiv 0 \mod(s+1)} \frac{1}{b^2} \right] \leq \left( \frac{1}{ra^2} \right)^p,
\]

(ii) for $n = k(r + s) + r + j - 1$ ($1 \leq j \leq s$), $b^2 \leq ra^2$ holds.

Note that if we are under type of Theorem 3.3 (which will be called "type I") it will be important to know which $j$ in $T^{-1}(T(n))$ have various $j \equiv t_j \mod(s + 1)$ which if we are under type of Corollary 3.4 (which will be called "type II") it is only important to know how many $j$ are of various $j \equiv t_j \mod(s + 1)$. Then we have the following remark.

**Remark 3.5 (Special case of Corollary 3.4 with $r = N(s + 1)$).** In this case for $n = n(r + s) + i - 1$, $1 \leq i \leq r$, the set of $l$ in $T^{-1}(T(n))$ contains exactly $N$ elements of each modulus, mod($s + 1$). So under type II the test (3.2) for such $n$ becomes

\[
N \left( \frac{1}{ra^2} \right)^p \frac{1}{r} + (r - N) \left( \frac{1}{b^2} \right)^p \frac{1}{r} \leq \left( \frac{1}{ra^2} \right)^p.
\]
For \( n = k(r + s) + r - 1 + j \), and under type II we either get a condition trivially satisfied for all \( p \), or \( 1/(ra^2) \leq 1/b^2 \), the latter only if there is at least one \( n \) so that \( n = K(r + s) + r - 1 + j \) and \( n = Q(s + 1) \). But since \( r = N(s + 1) \), this is \((K + 1)N(s + 1) + Ks + j - 1 = Q(s + 1)\) for some \( K, Q, j \), and take \( K = s + 1 \) and \( j = 1 \) to obtain a solution, so \( 1/(ra^2) \leq 1/b^2 \).

**Remark 3.6.** We can apply the idea of Theorem 3.3 to the model of general block matrix operator in the Definition 2.1 by the same method; the result formula will be slight complete than that of Theorem 3.3. We leave the exact formula to interested readers.

4. \( \infty \)-hyponormality and Flatness. We begin this section with the following fundamental lemma.

**Lemma 4.1.** Suppose \( p > 1 \) and \( q > 1 \) are relatively prime. Given any \( l_p, 0 \leq l_p \leq p - 1 \), and any \( l_q, 0 \leq l_q \leq q - 1 \), there exists \( n \in \mathbb{N} \) so that \( n \equiv l_p \mod p \) and \( n \equiv l_q \mod q \).

**Lemma 4.2.** Suppose that

\[
A := \begin{pmatrix}
\sqrt{y_1} & & & \\
& \ddots & & \\
& & \sqrt{x_1} & \\
& & & \sqrt{x_s}
\end{pmatrix}
\quad \text{and} \quad
M := \begin{pmatrix}
A & & \\
& A & \\
& & A
\end{pmatrix} \quad (4.1)
\]

Assume that \( \text{GCD}(r + s, s + 1) = 1 \). If \( M \) is \( p \)-hyponormal for some \( p \in (0, \infty) \), then

\[
x_1 = x_2 = \cdots = x_s \leq \sum_{1 \leq i \leq r} y_i. \quad (4.2)
\]

**Proposition 4.3.** Let \( A \) and \( M \) be as in (4.1). Suppose there exists \( N \in \mathbb{N} \) such that \( r = N(s + 1) \) and \( \text{GCD}(r + s, s + 1) = 1 \). Then the following assertions are equivalent:

(i) \( M \) is \( p \)-hyponormal for some \( p \in (0, \infty) \);

(ii) \( M \) is \( \infty \)-hyponormal;

(iii) \( x_1 = x_2 = \cdots = x_s = \sum_{1 \leq i \leq r} y_i \).

5. Examples. Let \( A \) and \( M \) be as in (4.1) with \( r + s = N(s + 1) \) for some \( N \in \mathbb{N} \) and we will see this is the “opposite” of \( r = N(s + 1) \) and \( \text{GCD}(r + s, s + 1) = 1 \).

**Proposition 5.1.** Let \( M \) be the block matrix operator as in (4.1). Then \( M \) is \( p \)-hyponormal if and only if the following inequality holds:

\[
\sum_{j \not\equiv 0 \mod(s+1)} \left( \frac{1}{x_{j \mod(s+1)}} \right)^p y_{j+1} \leq 1 \sum_{1 \leq i \leq r} \left( \frac{1}{\sum_{1 \leq i \leq r} y_i} \right)^p y_{j+1} \quad \text{for some } \quad \sum_{1 \leq i \leq r} y_i \quad \text{with no conditions on } \quad y_j. \quad (5.1)
\]

The following corollaries come immediately from Proposition 5.1.

**Corollary 5.2.** Let \( M \) be the block matrix operator as in (4.1) with \( x_1 = x_2 = \cdots = x_s = x \). Then (5.1) is trivially satisfied as long as \( x \geq \sum_{1 \leq i \leq r} y_i \).
Corollary 5.3. Let $M$ be the block matrix operator as in (4.1) such that the $y_{j+1}$ for $j \equiv 0 \mod (s+1)$ occur only in $\sum_{1 \leq i \leq r} y_i$. Thus if we consider some $y_{j+1}'$ for $j \equiv 0 \mod (s+1)$, as long as $\sum_{j \equiv 0} y_{j+1}' = \sum_{j \equiv 0} y_{j+1}$, then $M'$ is $p$-hyponormal if and only if $M$ is $p$-hyponormal.

Now we close this paper with the following example.

Example 5.4. Let

$$A := \begin{pmatrix} 1 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$

and $M := \begin{pmatrix} A \\ A \\ \vdots \end{pmatrix}$.

Write $Y$ for $\sum_{1 \leq i \leq 4} y_i$. Then the condition of

$$\frac{1}{Y^p} \frac{y_1}{Y} + \frac{1}{x_1^p} \frac{y_2}{Y} + \frac{1}{x_2^p} \frac{y_3}{Y} + \frac{1}{Y^p} \frac{y_4}{Y} \leq \frac{1}{Y^p}$$

is equivalent to

$$\frac{y_2}{x_1^p} + \frac{y_3}{x_2^p} \leq \frac{y_2 + y_3}{4^p}.$$

Inserting the $y_i \equiv 1, 1 \leq i \leq 4$, we get

$$\left(\frac{4}{x_1}\right)^p + \left(\frac{4}{x_2}\right)^p \leq 2,$$

(5.2)

which is equivalent to $M$ is $p$-hyponormal. Note that (5.2) keeps distinct the classes of $p$-hyponormal operators with respect to $0 < p < \infty$. To obtain region for $\infty$-hyponormality of $M$ we use Remark 3.2 and formulas in proof of Theorem 3.3, and there are three cases, Cases 1a, 1b, and 2b, which imply that $m_{3k_1} \geq m_{3k}, x_1 \geq 4, x_2 \geq 4$, and $x_1 \geq x_1 & x_2 \geq x_2$, respectively. Thus we obtain that

$M$ is $\infty$-hyponormal $\iff x_1 \geq 4$ and $x_2 \geq 4$.

Of course, since (5.2) is equivalent to $x_2 \geq 4 \cdot (2 - (4/x_1)^p)^{-1/p}$ for $x_1 > 4 \cdot 2^{-1/p}$, taking $p \to \infty$, we may check easily the obtaining conditions $\infty$-hyponormality of $M$ are $x_1 \geq 4$ and $x_2 \geq 4$. On the other hand, applying Remark 3.2 and formulas in proof of Theorem 3.3 for quasinormality of $M$, we also obtain that $M$ is quasinormal if and only if $(x_1, x_2) = (4, 4)$. 
BLOCK MATRIX OPERATORS FOR $p$-HYPERSONALITY

REFERENCES


