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Algebraic methods for genetics (II)
(Cuntz algebra and Cuntz-Krieger algebras)

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Abstract
Cuntz algebra is introduced and the Mendelian molecular genetics is described by use of the representations. The equivalence can be described in terms of genetic frequencies and the Hardy-Weinberg law can be described by use of the mass distribution of Hausdorff mesures. Non Mendelian population genetics can be described by use of Cuntz-Krieger algebras in a completely similar manner.

INTRODUCTION
It is well known that neutral evolution happens in the level of molecular population genetics and that natural selection happen in population genetics([1],[2]). The main problem in population genetics is to write the exact relationship between these two genetics.
The main purpose of this paper is give a possibility of describing this connection in terms of algebra. This can be done by use of the fractal structure on the both genetics and the representations of Cuntz algebra in the Mendelian population genetics and those of Cuntz-Krieger algebra in the case of some special class of non-mendelian population genetics. In the non-Mendelian case we may describe intimate interfaces between life elements and their environmental effects. This will be performed in the forthcoming paper.

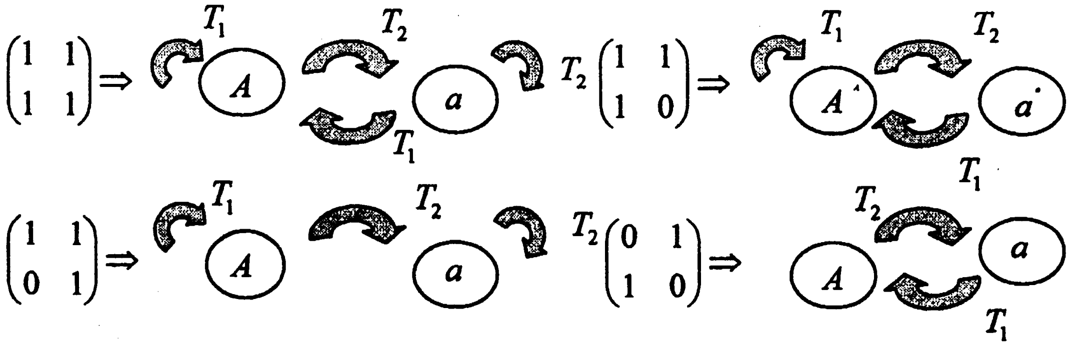
Cuntz algebra and Cuntz-Krieger algebra
A C^*-algebra which is generated by $\{S_1,S_2,...,S_N\}$ is called Cuntz algebra, when they satisfy the following commutation relations([3],[7],[8]):

$$S_i^*S_j=\delta_{ij}1(i=1,2,..,N)\quad \sum_{j=1}^{N}S_jS_j^*=1$$

A C^*-algebra which is generated by $\{S_1,S_2,...,S_N\}$ is called Cuntz-Krieger algebra of a given binary matirix A, $A=(a_{ij})\in M_N(\{0,1\})$ when they satisfy the relations:

$$S_i^*S_i=\sum_{j=1}^N a_{ij}S_jS_j^*(i=1,2,..,N),\sum_{j=1}^N S_jS_j^*=1$$

The automaton description of Cuntz and Cuntz-Krieger algebra can be given. We consider the connection matrix in the case of 2-sized matrix.



The first automaton is given by Cuntz algebra. The remained automatons are given by Cuntz-Krieger algebra. Hence we may expect to describe defects in population genetics by use of the representation of Cuntz-Krieger algebra.

Fractal structure of molecular and population genetics

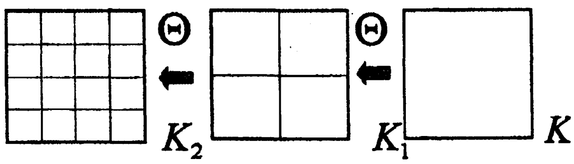
In this section we introduce fractal description of molecular population genetics and population genetics([3]).

(1) Fractal structure on population genetics

We choose a system of contraction $\sigma_j : K_0^{'} \to K_0^{'}(j=1,2,..,N)$ with contraction ratio $\lambda_j(0<\lambda_j<1)$ between a compact set $K_0^{'}$. We assume that the separation condition holds: $\sigma_j(\mathring{K}_0)\cap\sigma_i(\mathring{K}_0)=\phi(i\neq j)$ where $\mathring{K}_0$ is the open kernel of $K_0^{'}$ .Putting $K^{'}_{j_nj_{n-1}\cdots j_1}=\sigma_{j_n}$ o $\sigma_{j_{n-1}}$ o...o$\sigma_{j_1}(K)$, we can introduce a fractal set K:

$$K=\cap_{n=0}^{\infty}K_n^{'},\ K_n^{'}=\cup_{j=1}^N\sigma_j(K_{n-1}^{'})$$

We call the fractal set **fractal set of subdivision type**(or self similar fractal set). Typical examples of the fractal sets are the Cantor set and Sierpinski's gasket. We introduce a concept of evolution on a fractal set of subdivision type. Let $\Sigma(K)=\{K_{j_nj_{n-1}\cdots j_1}\}$, where $K_{j_nj_{n-1}\cdots j_1}=\sigma_{j_n}$ o...o$\sigma_{j_1}(K)$. Then the mapping $\Theta:\Sigma(K)\to\Sigma(K)$ defined by $\Theta(K_n)=K_{n+1}$ which is called evolution operator of subdivision type.



We can introduce non-integer dimension $\dim_H K(=D)$ which is called Hausdorff dimension. It is calculated by the following formula:

$$\sum_{j=1}^{N} \lambda_j^D = 1$$

We can introduce a completely σ -additive measure μ^D on K which is called Hausdorff measure. The Borel algebra is generated by $\{K_{j_n j_{n-1} \dots j_1}\}$. We have $\mu^D(K_{j_n j_{n-1} \dots j_1}) = \lambda_{j_n}^D \lambda_{j_{n-1}}^D \dots \lambda_{j_1}^D$. Next we introduce the following Hilbert space $L^2(K, d\mu^D)$ on K choosing the inner product on the characteristic function $\chi_{i_n i_{n-1} \dots i_1}$ of $K_{j_n j_{n-1} \dots j_1}$:

$$< \chi_{i_n i_{n-1} \dots i_1} \mid \chi_{i_m i_{m-1} \dots i_1} >= \delta_{nm} \delta_{i_n j_n} \dots \delta_{i_1 j_1} p_{i_n}^2 \dots p_{i_1}^2$$

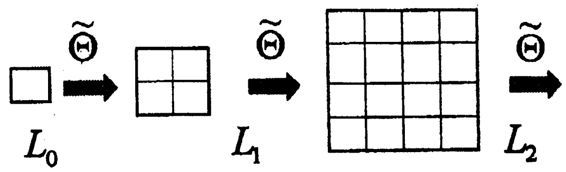
We can introduce a fractal set for the automaton which can describe the defect. The corresponding measure on the fractal set with defects will be given later.

(2) Fractal structure on molecular population genetics

We choose a system of contractions $\{\sigma_j : j = 1,2,..,N)\}$ and a compact set |0> which is called seed and putting $L_{j_n j_{n-1} \dots j_1} = \sigma_{j_n} o \sigma_{j_n} o \dots o \sigma_{j_1}(|0>)$, we can introduce a fractal set:

$$L = \overline{\cup_{n=0}^{\infty} L_n}, \; L_n = \cup_{j=1}^{N} T_j(L_{n-1})$$

which is called **fractal set of generation type**. A typical example of the fractal set is the Cauliflower. We introduce a concept of evolution of generating type. Let $\Sigma(L) = \{L_n\}$ be a set of subsets of L. Then the mapping $\tilde{\Theta} : \Sigma(L) \rightarrow \Sigma(L)$ defined by $\tilde{\Theta}(L_n) = L_{n+1}$ which is called evolution operator of generating type.



Next we introduce the following Hilbert space $L^2(L, d\tilde{\mu})$ choosing the inner product on the characteristic function $\tilde{\chi}_{i_n i_{n-1} \dots i_1}$ of $L_{j_n j_{n-1} \dots j_1}$:

$$< \tilde{\chi}_{i_n i_{n-1} \dots i_1} \mid \tilde{\chi}_{i_m i_{m-1} \dots i_1} >= \delta_{nm} \delta_{i_n j_n} \dots \delta_{i_1 j_1} p_{i_n}^2 \dots p_{i_1}^2$$

REALIZATION OF DEFECTS ON A SELF SIMILAR FRACTAL SET

We consider a system of contractions and make fractal sets of subdivision type K and generation type L respectively at first. Next we define fractal subsets K' and L' forgetting some of generators of contractions. Then we can introduce fractal sets with defects of both types, which are called **incomplete fractal sets** . The inclusion mappings are denoted by $\varpi : K' \to K$, and $\tilde{\varpi} : L' \to L$ respectively. Then we have the following lemma:

. Lemma(Realization of defects on a self similar fractal set)
Every fractal set which is defined by a system of contractions and has defects, we can realize it as a subset of a self similar fractal set. The measure and the Hilbert space can be defined by use of the inclusion mapping from those on complete fractal sets.

DUALITY BETWEEN TWO POPULATION GENETICS
We can give a correspondence between fractal sets of branch type and those of flower type which is called the duality mapping of fractal sets. We can prove the following theorem:

THEOREM I
Let L and K be fractal set of molecular genetics and that of population genetics which are generated by contractions: $\{\sigma_j : j = 1,2,..,N)\}$. Putting

$$\begin{cases} \Sigma(L) = \{L_{j_n j_{n-1} \cdots j_1} \mid L_{j_n j_{n-1} \cdots j_1} = \sigma_{j_n} o \sigma_{j_{n-1}} o \ldots o \sigma_{j_1}(q_0) \\ \Sigma(K) = \{K_{j_n j_{n-1} \cdots j_1} \mid L_{j_n j_{n-1} \cdots j_1} = \sigma_{j_n} o \sigma_{j_{n-1}} o \ldots o \sigma_{j_1}(K)\} \end{cases}$$

we can find a correspondence between in the following manner:

$$\tau : \Sigma(L) \to \Sigma(K)$$

by $\tau(L_{j_n j_{n-1} \cdots j_1}) = K_{j_n j_{n-1} \cdots j_1}$. In the following we denote the mapping by $\tau : L \to K$ simply

Proof
We can give the correspondence in the following manner:For a fractal set of tree type L we can define a fractal set of flower type K in a unique manner: Putting $K_n = \overline{L - \cup_{k=1}^{n} L_k}$, we see that $K_n = \cup_{j=1}^{N} \sigma_j(K_{n-1})$ and we can obtain a fractal set of flower type. Conversely, we can find a fractal set of branch type for a given a fractal set of flower type. For a fractal set K of flower type , we consider the defining sequence of sets $\{K'_{j_n j_{n-1} \cdots j_1}\}$. Then, putting $L_{j_n j_{n-1} \cdots j_1} = K'_{j_n j_{n-1} \cdots j_1}$,we see that $\{L_{j_n j_{n-1} \cdots j_1}\}$ become a fractal set of branch type and it determines the original fractal set K. We have to notice that the correspondence in this case is not unique. In fact. choosing a point $q_0 \in K'_0$, and putting $L_{j_n j_{n-1} \cdots j_1} = \sigma_{j_n} o \sigma_{j_{n-1}} o \ldots o \sigma_{j_1}\{q_0\}$, we can realize the original fractal set K.Hence we can define the desired duality mapping.
The duality theorem holds for a fractal set with defects in a completely similar manner.

Automaton representation of Cuntz(Cuntz-Krieger) algebra on Mendelian(Non-mendelian) population genetics

In this section we proceed to the description of evolution by use of the representations of

Cuntz/Cuntz-Krieger algebra and state the role of idempotent elements in evolution.

(1)Representations of Cuntz and Zunk algebras([3])
We can describe the fractal set of flower type in terms of a representation of the Cuntz algebra O_N ([7]). We have a representation of the Cuntz algebra on $L^2(K,d\mu^D)$:
$\pi : O_N \times L^2(K,d\mu^D) \to L^2(K,d\mu^D)$ by

$$\begin{cases} \pi(S_j)\chi_{i_n i_{n-1}\dots i_1} = \delta_{ji_n}\chi_{i_{n-1}\dots i_1} \\ \pi(S_j^*)\chi_{i_n i_{n-1}\dots i_1} = \chi_{j i_{n-1}\dots i_1} \end{cases}$$

Next we consider the central extension of the Cuntz algebra Z_N which is called the Zunk algebra. We denote the generators by $T_0,T_1,T_2,\dots,T_N$ where T_0 is the central element.

$$\begin{cases} T_i^* T_i = 1 (i = 1,2,..,N) \quad \sum_{j=1}^{N} T_j T_j^* = 1 \\ T_0 T_j = T_j T_0 (j = 0,1,2,\dots,N) \end{cases}$$

Then we have a representation of the Zunk algebra on a fractal set of branch type:
$\tilde{\pi} : Z_N \times L^2(L,d\tilde{\mu}) \to L^2(L,d\tilde{\mu})$ by

$$\begin{cases} \tilde{\pi}(T_j)\tilde{\chi}_{i_n j_{n-1}\dots i_1} = \delta_{ji_n}\tilde{\chi}_{i_{n-1}\dots i_1}, \pi(T_j^*)\tilde{\chi}_{i_n j_{n-1}\dots i_1} = \tilde{\chi}_{j i_{n-1}\dots i_1} \\ \tilde{\pi}(T_0)\tilde{\chi}_0 = \tilde{\chi}_0, \pi(T_j^*)\tilde{\chi}_{i_n i_{n-1}\dots i_1} = 0 \end{cases}$$

In the case of non-Mendelian population genetics, we can also make representations of Cuntz-Krieger algebras in a completely similar manner.
By use of the duality mapping we have the following theorem:

THEOREM II
(1) The representations of the Cuntz algebra are unitary equivalent if and only if their genetic frequencies are identical each other.
(2) By use of the duality mapping $\tau : L \to K$, we can induce the duality mapping between the representations

$$\begin{array}{ccc} \tilde{\pi} : Z_N \times L^2(L,d\tilde{\mu}) & \to & L^2(L,d\tilde{\mu}) \\ \tau\downarrow & & \tau\downarrow \\ \pi : O_N \times L^2(K,d\mu^D) & \to & L^2(K,d\mu^D) \end{array}$$

By use of the realization Lemma, we can give an analogous theorem for fractal sets with defects.

Mendelian genetics and gametic algebras([7])

Putting $S_{i_n \dots i_1} = S_{i_n} S_{i_{n-1}} \dots S_{i_1}$,we can introduce the following algebra which is called Cuntz-Mendel algebra from Cuntz algebra:

$$S_{i_n \dots i_1} * S_{j_n \dots j_1} = \frac{1}{2}(S_{i_n \dots i_1} + S_{j_n \dots j_1}).$$

THEOREM III

The Hardy-Weinberg law can be described as idempotent elements:

$$(\sum p_{j_n j_{n-1} \dots j_1} S_{j_n j_{n-1} \dots j_1})^2 = \sum p_{j_n j_{n-1} \dots j_1} S_{j_n j_{n-1} \dots j_1},$$

where we put $p_{j_n j_{n-1} \dots j_1} = p_{j_n} p_{j_{n-1}} \dots p_{j_1}$.

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