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Global asymptotic stability for a class of difference equations

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1 Introduction

Consider the following nonlinear difference equation with variable coefficients:

\[ x_{n+1} = qx_n - \sum_{j=0}^{m} a_j f_j(x_{n-j}), \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (1.1)

where \( 0 < q \leq 1 \), \( a_j \geq 0 \), \( 0 \leq j \leq m \) and \( \sum_{j=0}^{m} a_j > 0 \). We now assume that

\[ \left\{ \begin{array}{l}
  f(x) \in C(-\infty, +\infty) \text{ is a strictly monotone increasing function,} \\
  f(0) = 0, \quad 0 < \frac{f(x)}{f(x)^{1/2}} \leq 1, \quad x \neq 0, \quad 1 \leq j \leq m, \quad \text{and} \\
  \text{if } f(x) \neq x, \text{ then } \lim_{x \to -\infty} f(x) \text{ is finite, otherwise } f(x) = x.
\end{array} \right. \]  \hspace{1cm} (1.2)

The above difference equation has been studied by many literatures (see for example, [1]-[9] and references therein).

**Definition 1.1** The solution \( y^* \) of (1.1) is called uniformly stable, if for any \( \epsilon > 0 \) and non-negative integer \( n_0 \), there is a constant \( \delta = \delta(\epsilon) > 0 \) such that \( \sup \{ |y_{n_0-i} - y^*| \mid 0 \leq i \leq m \} < \delta \), implies that the solution \( \{y_n\}_{n=0}^{\infty} \) of (1.1) satisfies \( |y_n - y^*| < \epsilon \), \( n = n_0, n_0 + 1, \ldots \).

**Definition 1.2** The solution \( y^* \) of (1.1) is called globally attractive, if every solution of (1.1) tends to \( y^* \) as \( n \to \infty \).

**Definition 1.3** The solution \( y^* \) of (1.1) is called globally asymptotically stable, if it is uniformly stable and globally attractive.

In this paper, we study "semi-contractive" functions and global asymptotic stability of difference equations. In Section 2, we first define semi-contractivity of functions and show the related results on the global asymptotic stability of difference equations.

2 Semi-contractive function

Assume that

\[ g(z_0, z_1, \cdots, z_m) \in C(R^{m+1}) \quad \text{and} \quad g(y, y, \cdots, y) = y \] has a unique solution \( y = y^* \).  \hspace{1cm} (2.1)
Definition 2.1  The function \( g(z_0,z_1,\cdots,z_m) \) is said to be semi-contractive at \( y^* \), if

(i) for any constants \( z < y^* \) and \( z_i \geq z \), \( 0 \leq i \leq m \), there exists a constant \( y^* < \tilde{z} < +\infty \) such that \( g(z_0,z_1,\cdots,z_m) \leq \tilde{z} \), and for any \( z \leq z_i \leq \tilde{z} \), \( 0 \leq i \leq m \), there exists a constant \( \tilde{z} > z \) such that \( \tilde{z} \geq g(z_0,z_1,\cdots,z_m) \), or

(ii) for any constants \( \tilde{z} > y^* \) and \( z_i \leq \tilde{z} \), \( 0 \leq i \leq m \), there exists a constant \( y^* > \tilde{z} > -\infty \) such that \( g(z_0,z_1,\cdots,z_m) \geq \tilde{z} \), and for any \( z \leq z_i \leq \tilde{z} \), \( 0 \leq i \leq m \), there exists a constant \( \tilde{z} < \tilde{z} \) such that \( \tilde{z} \leq g(z_0,z_1,\cdots,z_m) \).

Lemma 2.1  If \( g(y) \in C(R) \) is a strictly monotone decreasing function such that \( g(g(y)) > y \) for any \( y < y^* \), then \( g(z) \) is semi-contractive for \( y^* \).

Lemma 2.2  Assume (2.1) and that each \( g_i(z_0,z_1,\cdots,z_m) \), \( 0 \leq i \leq m \) is semi-contractive for \( y^* \). Then for any \( b_{n,i} \geq 0 \), \( n \geq 0 \), \( 0 \leq i \leq m \) such that \( \sum_{i=0}^{m} b_{n,i} = 1 \) and \( \lim_{n \to \infty} b_{n,i} = b_i \), \( 0 \leq i \leq m \), it holds that \( \sum_{i=0}^{m} b_{n,i} g_i(z_0,z_1,\cdots,z_m) \) is semi-contractive for \( y^* \).

Corollary 2.1  Assume (2.1) and that \( g(z_0,z_1,\cdots,z_m) \) is semi-contractive for \( y^* \). Then for any \( 0 \leq q_n < 1 \), \( g_n(z_0,z_1,\cdots,z_m) \) and \( k \) such that

\[
\begin{align*}
\lim_{n \to \infty} q_n &= q < 1, \text{ and } 0 \leq k \leq m, \\
\lim_{n \to \infty} g_n(z_0,z_1,\cdots,z_m) &= g(z_0,z_1,\cdots,z_m) \text{ for any } z_0,z_1,\cdots,z_m \in (-\infty, +\infty),
\end{align*}
\]

it holds that \( q_nz_k + (1-q_n)g_n(z_0,z_1,\cdots,z_m) \) is semi-contractive for \( y^* \).

Corollary 2.2  Assume that each \( g_i(z) \in C(R) \) and \( g_i(y) = y \) has a unique solution \( y = y^* \), \( 0 \leq i \leq m \), and each \( g_i(z_i) \), \( 0 \leq i \leq m \) is semi-contractive for \( y^* \), then for any \( b_{n,i} \geq 0 \), \( n \geq 0 \), \( 0 \leq i \leq m \) such that \( \sum_{i=0}^{m} b_{n,i} = 1 \) and \( \lim_{n \to \infty} b_{n,i} = b_i \), \( 0 \leq i \leq m \), it holds that \( \sum_{i=0}^{m} b_{n,i} g_i(z_i) \) is semi-contractive for \( y^* \). In particular, for any \( 0 \leq q_n < 1 \) and \( k \) such that \( \lim_{n \to \infty} q_n = q < 1 \) and \( 0 \leq k \leq m \), it holds that \( q_nz_k + (1-q_n) \sum_{i=0}^{m} b_{n,i} g_i(z_i) \) is semi-contractive for \( y^* \).

Remark 2.1  If \( g(z_0,z_1,\cdots,z_m) \) is semi-contractive for any \( z_i > 0 \), \( 0 \leq i \leq m \), then there are cases that we may restrict our attention only to \( z_i > 0 \), \( 0 \leq i \leq m \) and the unique positive solution \( y^* > 0 \) of \( g(y^*,y^*,\cdots,y^*) = y^* \), whether or not \( g(y,y,y) = y \) has other solutions \( y < 0 \).

Example 2.1  Examples of semi-contractive function \( g(z_0,z_1,\cdots,z_m) \) for \( y^* \).

(i) \( g(z_0,z_1,\cdots,z_m) = z_0 e^{c(1 - z_0)} \), \( y^* = 1 \) and \( c \leq 2 \) (see [1]).

(ii) \( g(z_0,z_1,\cdots,z_m) = z_0 \exp(c(1 - \sum_{i=0}^{m} a_i z_i)) \), \( y^* = 1/(\sum_{i=0}^{m} a_i) \) and \( c \leq 2 \), where \( a_0 > 0 \), \( a_i \geq 0 \), \( 1 \leq i \leq m \) and \( \sum_{i=1}^{m} a_i /a_0 \leq 2/e \).

This is equivalent to \( h(u_0,u_1,\cdots,u_m) = u_0 - c \sum_{i=0}^{m} b_i (e^{u_i} - 1) \) is semi-contractive for \( u^* = 0 \) and \( c \leq 2 \), where \( z_i = y^* e^{u_i} \), \( b_0 = y^* a_0 > 0 \), \( b_i = y^* a_i \geq 0, 1 \leq i \leq m \), \( \sum_{i=0}^{m} b_i = 1 \), and \( (\sum_{i=1}^{m} b_i) /b_0 \leq 2/e \) (see [8]).

(iii) \( g(z_0,z_1,\cdots,z_m) = c(1 - e^{z_m}) \), \( y^* = 0 \) and \( c \leq 1 \) (see [3]).

(iv) \( g(z_0,z_1,\cdots,z_m) = \frac{e^{z_m}}{1 + e^{z_m}} \), \( x^* = ((c - 1)/b)^{1/p} \) and \( c \leq \frac{p}{p-2} \), where \( p > 2 \) and \( b > 0 \) (see [1]).

We consider the following difference equation

\[
y_{n+1} = q_n y_{n-k} + (1-q_n)g_n(y_n,y_{n-1},\cdots,y_{n-m}), \quad n = 0,1,\cdots, (2.3)
\]
where we assume (2.1) and
\[
\begin{aligned}
0 \leq q_n < 1, \quad & \lim_{n \to \infty} q_n = q < 1, \quad k \in \{0, 1, \cdots, m\}, \quad \text{and} \\
\lim_{n \to \infty} g_n(z_0, z_1, \cdots, z_m) = g(z_0, z_1, \cdots, z_m) \quad \text{for any} \quad z_0, z_1, \cdots, z_m \in (-\infty, +\infty).
\end{aligned}
\]

(2.4)

**Theorem 2.1** If \(g(z_0, z_1, \cdots, z_m)\) is semi-contractive for \(y^*\), then \(y^*\) of (2.3) is globally asymptotically stable for any \(0 \leq q < 1\).

**Corollary 2.3** Assume that there exists a constant \(0 \leq q_0 < 1\) and some \(0 \leq k \leq m\) such that \(g_0z_k + (1 - q_0)g(z_0, z_1, \cdots, z_m)\), is semi-contractive for \(y^*\). Then, for any \(q_0 \leq q_n < 1\) and \(g_n(z_0, z_1, \cdots, z_m)\) which satisfy (2.4), the solution \(y^*\) of (2.3) is globally asymptotically stable.

**Remark 2.2** (i) The corresponding continuous case (2.3) is the following differential equation
\[
\begin{aligned}
y'(t) &= -p(t)\{y(t) - \frac{1}{q_0}g_n(y(n), y(n-1), \cdots, y(n-m))\}, \quad n = t < n + 1, \quad n = 0, 1, 2, \cdots, \\
p(t) > 0, \quad q_n = e^{-\int_{n-1}^{n} p(t) dt} < 1.
\end{aligned}
\]

(ii) In Theorem 2.1, a semi-contractivity condition is a delays and \(q_n\)-independent condition for the solution \(y^*\) of (2.3) to be globally asymptotically stable.

By Theorem 2.1 and Example 2.1, we obtain the following result:

**Example 2.2** Examples of delays and \(q\)-independent stability conditions.

(i) Ricker model \(y_{n+1} = qy_n + (1-q) y_{n-m} e^{(1-y_{n-m})}, \quad n = 0, 1, 2, \cdots\). The positive equilibrium \(y^* = 1\) is globally asymptotically stable, if \(c \leq 2\) (see [1]).

(ii) Ricker model with delayed-density dependence \(y_{n+1} = qy_n + (1-q) y_n \exp\{c(1-\sum_{i=0}^{m} a_i y_{n-i})\}\). The positive equilibrium \(y^* = 1/(\sum_{i=0}^{m} a_i)\) is globally asymptotically stable, if \(c \leq 2\), where \(a_0 > 0, \quad a_i \geq 0, \quad 1 \leq i \leq m\) and \(\sum_{i=0}^{m} a_i \leq 2/e\) (see [8]).

(iii) Wazewska-Czyżewska and Lasota model \(y_{n+1} = qy_n + (1-q) \sum_{i=0}^{m} b_i e^{-\gamma y_{n-i}}, \quad n = 0, 1, 2, \cdots, \)

where \(\gamma > 0, \quad b_i \geq 0, \quad 0 \leq i \leq m,\) and \(\sum_{i=0}^{m} b_i = 1\).

The positive equilibrium \(y^*\) is the positive solution of the equation \(y^* = ce^{-\gamma y^*}\). Put \(x_n = \gamma (y^* - y_n)\). Then, this equation is equivalent to
\[
x_{n+1} = qx_n - (1-q)\gamma y^* \sum_{i=0}^{m} b_i (e^{x_{n-i}} - 1), \quad \text{where} \quad b_i \geq 0, \quad 0 \leq i \leq m, \quad \sum_{i=0}^{m} b_i = 1.
\]

(2.5)

Thus, the positive equilibrium \(y^*\) is globally asymptotically stable, if \(c \leq e/\gamma\) which is equivalent that the zero solution of (2.5) is globally asymptotically stable if \(\gamma y^* \leq 1\) (see [3]).

(iv) Bobwhite quail population model \(y_{n+1} = qy_n + (1-q) \frac{qy_{n-m}}{1+b y_{n-m}}, \quad n = 0, 1, 2, \cdots, \)

where \(c > 1, \quad b > 0\). The positive equilibrium \(y^* = ((c-1)/b)^{1/p}\) is globally asymptotically stable, if \(c \leq \frac{p}{p-2}\) for \(p > 2\) (see [1]).

We have the following counter example:

**Example 2.3** Examples of \(q\)-dependent and delay-dependent stability conditions.

(i) A model in hematopoiesis \(y_{n+1} = qy_n + (1-q) e^{2(1-y_{n-m})}, \quad n = 0, 1, 2, \cdots, \)

The equilibrium \(y^* = 1\) is globally asymptotically stable if \(q \in [1/3, 1]\), and 2-cycle if \(q \in [0, 1/3\) (see [2]).

(ii) A delayed model in hematopoiesis \(y_{n+1} = qy_n + (1-q) e^{2(1-y_{n-m})}, \quad n = 0, 1, 2, \cdots, \)

The characteristic equation takes the form \(\lambda^3 - q\lambda^2 = -2(1-q)\). Then for \(q = q_2 = \frac{3-\sqrt{2}}{2}\)
0.633975 \cdots > 1/3$, the roots are \(-1 < \lambda_1 < 0, |\lambda_2| = |\lambda_3| = 1\). For \(q_2 < q < 1\), the equilibrium \(y^* = 1\) is locally attractive but it becomes unstable for \(q = q_2\), and Hopf bifurcation occurs (see [2]).

(iii) Ricker's equation with delayed-density dependence \(y_{n+1} = y_n \exp\{c_n(1-\sum_{i=0}^{m} b_{n,i}y_{n-i})\}, n = 0,1,\cdots\), which is equivalent to \(x_{n+1} = x_n - c_n \sum_{i=0}^{m} b_{n,i}(e^{x_{n-i}} - 1)\), \(n = 0,1,\cdots\), where \(c_n, b_{n,i} > 0, \sum_{i=0}^{m} b_{n,i} = 1\) and \(y_n = e^{x_n}\).

The positive equilibrium \(y^* = 1\) is globally asymptotically stable if \(\lim\sup_{n\to\infty} \sum_{i=n}^{n+m} r_i < \frac{3}{2} + \frac{1}{2(m+1)}\) (see [7]).

(iv) A model of the growth of bobwhite quail populations \(y_{n+1} = qy_n + (1-q)c \sum_{i=0}^{m} b_{i}e^{-\gamma y_{n-i}}\), \(n = 0,1,\cdots\), where \(c, p > 0\).

If \(c \leq 1\), then for any \(0 < q < 1, \lim_{n \to \infty} y_n = 0\). If \(c > 1\), then the positive equilibrium \(y^* = (c-1)^{1/p}\) of the model exists. Moreover, if \(p \leq \frac{2c}{(c-1)(1-q)}\) for \(m = 0\), or \(p < \frac{c}{(c-1)^{1/(1-q)} \frac{3m+4}{2(m+1)^2}}\) for \(m \geq 1\), then the positive equilibrium \(y^*\) is globally asymptotically stable (see [4]).

3 Delays-independent stability conditions for (1.1)

After setting
\[
r_1 = a_0, r_2 = \sum_{i=1}^{m} a_i, r = r_1 + r_2, \quad \varphi(x) = qx - r_1 f(x), \quad \hat{z}(q) = \frac{(-1+\sqrt{1+4q})}{2q},
\]
we have the following result.

Theorem 3.1 Assume that \(f(x) = f_0(x) = e^x - 1\) and \(0 < q < 1\), and suppose that
\[
r_1 < q, \quad r \leq q + (1-q) \ln(q/r_1) \quad \text{and} \quad (q/r_1)^{q}e^{-q} - (r_1 - r_2) + (1-q) \geq 0,
\]
or
\[
\begin{cases}
  r_1 \leq q, \quad r > q + (1-q) \ln(q/r_1), \quad qr_2 \leq r_1, \\
  r - r_2(q/r_1)^{q}e^{-q} - (1-q)(\ell - 1) \geq 0 \quad \text{and} \quad \ell = \ln \frac{r - q(1-q) \ln(q/r_1)}{r_2} \leq 0,
\end{cases}
\]
or
\[
\begin{cases}
  r_1 > q, \quad r \leq 1 + q, \quad r - r_2(q/r_1)^{q}e^{-q} - (1-q)(\ln(q/r_1) - 1) \geq 0, \\
  \text{and} \quad \frac{\ell(q/r_1)}{q(1-q)} \leq \frac{\ell(q)}{q(1-q)}.
\end{cases}
\]

Then, the zero solution of (1.1) is globally asymptotically stable.

Numerical result 3.1 Assume that \(f(x) = f_0(x) = e^x - 1\) and \(0 < q < 1\).

(i) The last inequality in (3.4) can be eliminated from (3.4).

(ii) Under the condition \(\frac{a_2}{a_1} < \frac{2}{e}\) and \(r \leq 1 + q\), the third inequality of (3.4) is satisfied, and hence the zero solution of (1.1) is globally asymptotically stable.

Example 3.1 Wazewska-Czyzewska and Lasota model (see [9]).

\[
y_{n+1} = qy_n + (1-q)c \sum_{i=0}^{m} b_i e^{-\gamma y_{n-i}}, \quad \text{where} \quad c, \gamma > 0, \quad b_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{m} b_i = 1.
\]

(3.5) is equivalent to (2.5). For equation (3.5), the positive equilibrium of (3.5), say \(y^*\), is globally asymptotically stable, if \(\gamma y^* \leq 1\) (see [3] and Example 2.2 iii)). For the case \(\gamma y^* > 1\), by using
the generalized Yorke condition, [6, Theorem 8] extended these to \( \gamma y^* \leq (1 + q^{m+1})/(1 - q^{m+1}) \) with some restricted conditions "\( V_k(q) < 0, W_k(q) < 0 \). Note that the last condition contains the restriction (\( q + q^2 + \cdots + q^m \))q^{m} \leq 1 \) for \( 0 < q < 1 \). On the other hand, by applying Theorem 3.1 and Numerical result 3.1 to (2.5) for \( \alpha_i = (1 - q)\gamma y^* b_i \), \( 0 \leq i \leq m \), we obtain another sufficient condition, for example, \( \sum_{i=1}^{m} b_i \leq 2b_0 \) and \( \gamma y^* \leq (1 + q)/(1 - q) \) for the solution \( y^* \) of (3.5) to be globally asymptotically stable. Note that \( e^x - 1 < x/(1-x) \) for \( 0 < x < 1 \) and \( \sum_{i=1}^{m} b_i \leq \sum_{i=1}^{m} a_i > 0 \) for \( 0 < q < 1 \). Thus, compared with [6, Proof of Theorem 2] (and [1]-[9] and references therein), one can see that our results offer new stability conditions to (3.5).

4 Semi-contractivity with a sign condition

For \( 0 \leq q < 1 \), consider the following nonautonomous equation

\[
x_{n+1} = qx_n - \sum_{j=0}^{m} a_{n,j}f_j(x_{n-j}), \quad n = 0, 1, \ldots ,
\]

(4.1)

where \( 0 < q \leq 1 \), \( a_{n,j} \geq 0 \), \( 0 \leq j \leq m \), \( n = 0, 1, \ldots \), and \( \sum_{j=0}^{m} a_{n,j} > 0 \), and we assume that there is a function \( f(x) \) such that (1.2) holds.

For (4.1) and any \( 0 \leq l_n \leq m \), we can derive the following equation.

\[
\begin{align*}
x_{n+1} &= \{q^{l_n+1}x_{n-l_n} + (1 - q)\sum_{k=0}^{l_n} q^k \sum_{j=0}^{m-k} a_{n-k,j}f_j(x_{n-k-j})\} \\
&\quad - \sum_{k=1}^{l_n} q^k \sum_{j=m-k+1}^{m} a_{n-k,j}f_j(x_{n-k-j}), \quad n = 2m, 2m + 1, \ldots .
\end{align*}
\]

(4.2)

Similar to the proofs of [5, Lemmas 2.3 and 2.4], we have the following two lemmas for (4.1).

Lemma 4.1 Let \( \{x_n\}_{n=0}^{\infty} \) be the solution of (4.1). If there exists an integer \( n \geq m \) such that \( x_{n+1} \geq 0 \) and \( x_{n+1} > x_n \), then there exists an integer \( \underline{g}_n \in [n - m, n] \) such that

\[
x_{\underline{g}_n} = \min_{0 \leq j \leq m} x_{n-j} < 0.
\]

(4.3)

If there exists an integer \( n \geq m \) such that \( x_{n+1} \leq 0 \) and \( x_{n+1} < x_n \), then there exists an integer \( \overline{g}_n \in [n - m, n] \) such that

\[
x_{\overline{g}_n} = \max_{0 \leq j \leq m} x_{n-j} > 0.
\]

(4.4)

After setting

\[
\begin{align*}
\bar{r}_1 &= \sup_{n \geq m} q^k \sum_{j=0}^{m-k} a_{n-k,j}, \quad \bar{r}_2 = \sup_{n \geq m} q^k \sum_{j=m-k+1}^{m} a_{n-k,j}, \\
\bar{r} &= \bar{r}_1 + \bar{r}_2, \quad \bar{q}(x) = \bar{q}x - \bar{r}_1 f(x), \quad \bar{q} = q^{m+1}, \quad \bar{z} = (-1 + \sqrt{1 + 4q})/(2q),
\end{align*}
\]

(4.5)

and

\[
\bar{g}(z_0, z_1, \ldots, z_m; \bar{q}) = \bar{q}(z_0) + \sum_{k=1}^{m} q^k \sum_{j=m-k+1}^{m} a_{n-k,j}g(z_j),
\]

(4.6)

we are able to prove the following results.
If there exists an integer \( n \geq m \) such that \( x_{n+1} \geq 0 \) and \( x_{n+1} > x_n \), then by (4.3) and (4.2) with \( l_n = n - \overline{g}_n \), we have that
\[
x_{n+1} \leq \phi(x_{\overline{g}_n}) - \overline{r}_2 f(L_n), \quad L_n = \min_{0 \leq j \leq 2m} x_{n-j}.
\] (4.7)

If there exists an integer \( n \geq m \) such that \( x_{n+1} \leq 0 \) and \( x_{n+1} < x_n \), then by (4.4) and (4.2) with \( l_n = n - \underline{g}_n \), we have that
\[
x_{n+1} \geq \phi(x_{\underline{g}_n}) - \overline{r}_2 f(R_n), \quad R_n = \max_{0 \leq j \leq 2m} x_{n-j}.
\] (4.8)

**Lemma 4.2** Suppose that the solution \( x_n \) of (4.1) is oscillatory about 0. If for some real number \( L < 0 \), there exists a positive integer \( n_L \geq 2m \) such that \( x_n \geq L \) for \( n \geq n_L \), then for any integer \( n \geq n_L + 2m \),
\[
x_{n+1} \leq R_L \text{ for } n \geq n_L + 2m, \quad \text{and } x_{n+1} \geq S_L \text{ for } n \geq n_L + 4m,
\] (4.9)

where \( R_L = \max_{Lz} \varphi(x) - r_2 f(L) > 0 \) and \( S_L = \min_{0 \leq x \leq R_L} \varphi(x) - r_2 f(R_L) < 0 \). Moreover, if \( S_L > L \) for any \( L < 0 \), then \( \lim_{n \to \infty} x_n = 0 \).

Assume that \( g(z_0, z_1, \cdots, z_m) \) is continuous for \( (z_0, z_1, \cdots, z_m) \in R^{m+1} \) and \( g(y^*, y^*, \cdots, y^*) = y^* \) has a unique solution \( y^* \).

**Definition 4.1** The function \( g(z_0, z_1, \cdots, z_m) \) is said to be semi-contractive with a sign condition \( z_0 \) for \( y^* \), if
(i) for any constants \( \overline{z} < y^* \) and \( z_i \geq \overline{z}, \) \( 0 \leq i \leq m \) with \( z_0 \leq y^* \), there exists a constant \( y^* < \overline{z} < +\infty \) such that \( g(z_0, z_1, \cdots, z_m) \leq \overline{z} \) and for any \( \overline{z} \leq z_i \leq \overline{z}, \) \( 0 \leq i \leq m \) with \( z_0 \geq y^* \), there exists a constant \( \overline{z} \geq \overline{z} \) such that \( \overline{z} \leq g(z_0, z_1, \cdots, z_m) \),
or
(ii) for any constants \( \overline{z} > y^* \) and \( z_i \leq \overline{z}, \) \( 0 \leq i \leq m \) with \( z_0 \leq y^* \), there exists a constant \( y^* > \overline{z} > -\infty \) such that \( g(z_0, z_1, \cdots, z_m) \geq \overline{z} \) and for any \( \overline{z} \leq z_i \leq \overline{z}, \) \( 0 \leq i \leq m \) with \( z_0 \leq y^* \), there exists a constant \( \overline{z} \leq \overline{z} \) such that \( \overline{z} \geq g(z_0, z_1, \cdots, z_m) \).

Then by (4.7), (4.8) and Lemma 4.2, we can obtain the following result.

**Theorem 4.1** If \( \overline{g}(z_0, z_1; \overline{q}) = \phi(z_0) - \overline{r}_2 f(z_1) \) is semi-contractive with a sign condition \( z_0 \) for \( x^* = 0 \), then the zero solution of (4.1) is globally asymptotically stable.

Note that if \( \overline{g}(z_0, z_1; \overline{q}) = \phi(z_0) - \overline{r}_2 f(z_1) \) is semi-contractive with a sign condition \( z_0 \) for \( x^* = 0 \), then the zero solution \( x^* = 0 \) of (4.1) is uniformly stable and hence \( x^* = 0 \) is globally asymptotically stable.

For the special case \( f(x) = e^x - 1 \), we establish the following sufficient conditions for \( 0 < q < 1 \) which are some extensions of the result in [5] for \( q = 1 \).

**Theorem 4.2** Suppose that \( f(x) = e^x - 1 \) and that one of the following conditions is fulfilled:
\[
\begin{cases}
\overline{r}_2 \leq 1 \quad \text{and} \quad \frac{\overline{r}_2 e^{\overline{r}_2}}{1 - \overline{q}} < \frac{e^2}{1 - \overline{q}} \quad \text{if} \quad \overline{r}_1 \leq \overline{q}, \\
\overline{r} \leq 1 + \overline{q} \quad \text{and} \quad \frac{\overline{r} e^{\overline{r}}}{\overline{q} / \overline{r}_1} e^{\overline{r} - \overline{q}} < \frac{e^2}{1 - \overline{q}} \quad \text{if} \quad \overline{r}_1 > \overline{q},
\end{cases}
\] (4.10)
or
\[
\begin{cases}
\overline{r}_2 \leq 1, \quad \overline{r}_2 e^{\overline{r}_2} > \frac{e^2}{1 - \overline{q}} \quad \text{and} \quad G_3(\delta) > 0 \quad \text{if} \quad \overline{r}_1 \leq \overline{q}, \\
\overline{r} \leq 1 + \overline{q}, \quad \frac{\overline{r} e^{\overline{r}}}{\overline{q} / \overline{r}_1} e^{\overline{r} - \overline{q}} > \frac{e^2}{1 - \overline{q}} \quad \text{and} \quad G_1(\alpha) > 0 \quad \text{if} \quad \overline{r}_1 > \overline{q},
\end{cases}
\] (4.11)
with \[
\begin{align*}
G_1(x) &= \bar{q} \left( \bar{q} \ln(\bar{q} / \bar{r}_1) + \bar{r} - \bar{q} \bar{r}_2 e^x \right) + \bar{r} - \bar{q} (\bar{q} / \bar{r}_1)^{\bar{k}} e^{\bar{r} - \bar{q} \bar{r}_2 e^x} - x, \\
G_3(x) &= (\bar{r}_1 + (1 + \bar{q}) \bar{r}_2) - \bar{q} \bar{r}_2 e^x - \bar{r} e^{\bar{r}_2 - \bar{r} e^x} - x,
\end{align*}
\]
where \( \alpha \) and \( \delta \) are the lowest solutions of \( G_1(x) = 0 \) and \( G_3(x) = 0 \), respectively, and \( \bar{z} \) is a positive solution of \( \bar{q} z^2 + z - 1 = 0 \). Then, the solution \( x^* = 0 \) of (4.1) is globally asymptotically stable.

As an immediate consequence we have the following corollary.

**Corollary 4.1** Assume that \( f(x) = e^x - 1 \) and that

\[
\bar{r} \leq 1 + \bar{q} \quad \text{and} \quad \bar{r}_1 \geq \bar{q} \bar{r}_2.
\]

If

\[
(i) \quad \frac{\bar{r}}{\bar{q}} (\bar{q} / \bar{r}_1)^{\bar{q}} e^{\bar{r} - \bar{q}} \leq \frac{e^{\bar{x}}}{1 - \bar{x}}, \quad \text{or} \quad (ii) \quad \frac{\bar{r}}{\bar{q}} (\bar{q} / \bar{r}_1)^{\bar{q}} e^{\bar{r} - \bar{q}} > \frac{e^{\bar{x}}}{1 - \bar{x}} \quad \text{and} \quad G_1(\alpha) > 0,
\]

then, the zero solution of (4.1) is globally asymptotically stable.

**Example 4.1** Consider a model \( x_{n+1} = q x_n - \sum_{i=0}^{m} a_i (e^{-x_{n-i}} - 1) \), \( n = 0, 1, 2, \ldots \), where \( a_i \geq 0, 0 \leq i \leq m \), and \( \sum_{i=0}^{m} a_i > 0 \). This equation is equivalent to (2.5), if \( \sum_{i=0}^{m} a_i = (1 - q) \gamma y^* \) and \( 0 < q < 1 \). By Corollary 4.1, the zero solution \( x^* = 0 \) is globally asymptotically stable for \( \bar{r} \leq 1 + \bar{q} \), if for the setting (4.5) and \( \bar{r}_1 = \bar{q} (1 + \bar{r}_2 (1 - \bar{x}) e^{\bar{x} - 1})^{1/\bar{q}} \), it holds that \( \frac{\bar{r}}{\bar{q}} \leq \frac{1 + \bar{z}}{1} - 1 \). Since \( e^x - 1 < x/(1 - x) \) for \( 0 < x < 1 \) and we do not need the restriction \( (q + q^2 + \cdots + q^m) q^m \leq 1 \) for \( 0 < q \leq 1 \) in [6, Theorem 2], our results improve some of [6, Theorem 8] (see [5]).

**References**


