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On Minimal Clones and Generating Polynomials

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Abstract

A minimal clone is an atom of the lattice of clones. For a prime power k we consider the base set with k elements as a finite field GF(k) and express generating functions of minimal clones as polynomials over GF(k).

In the joint works with M. Pinsker and T. Waldhauser, we have found explicitly some polynomials generating minimal clones. In this paper, as a brief summary of those results obtained so far, some binary idempotent polynomials and ternary majority polynomials are presented which generate minimal clones.

1 Introduction

Let $E_k = \{0, 1, \ldots, k-1\}$ for a fixed $k > 1$. For an integer $n > 0$ let $\mathcal{O}_k^{(n)}$ be the set of all n-variable functions on E_k, that is, maps from $(E_k)^n$ into E_k, and let

$$\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}.$$

Denote by $\mathcal{J}_k$ the set of all **projections** e_i^n on E_k, $1 \leq i \leq n$, where e_i^n is a function in $\mathcal{O}_k^{(n)}$ defined by $e_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n \in E_k$.

Definition 1. 1 *A subset C of $\mathcal{O}_k$ is a* **clone** *on E_k if the following conditions are satisfied:*

(i) *C contains $\mathcal{J}_k$.*
(ii) *C is closed under (functional) composition.*

For every $k > 1$ the set of all clones on E_k forms a lattice ordered by inclusion. It is called the **lattice of clones** on E_k and is denoted by $\mathcal{L}_k$.

Since E. Post ([Po 41]) the structure of $\mathcal{L}_2$ is completely known. However, the structure of $\mathcal{L}_k$ for every $k \geq 3$ is extremely complex and still mostly unknown. The cardinality of the lattice of clones is known for each $k \geq 2$: $|\mathcal{L}_2| = \aleph_0$ ([Po 41]) and $|\mathcal{L}_k| = 2^{\aleph_0}$ if $3 \leq k < \aleph_0$ ([IM 59]).

Minimal clones are defined as follows:

Definition 1. 2 *A clone C on E_k is a* **minimal clone** *if it is an atom of $\mathcal{L}_k$. In other words, C is a minimal clone if it satisfies the following conditions:*

(i) *$\mathcal{J}_k \subset C$ (proper inclusion).*
(ii) *For $\forall C' \in \mathcal{L}_k$, $\mathcal{J}_k \subseteq C' \subset C \implies C' = \mathcal{J}_k$.*

Thus, a minimal clone is a clone which sits just above the least clone $\mathcal{J}_k$ in the lattice $\mathcal{L}_k$ of clones on E_k. The other 'extreme' clone is a maximal clone which is defined to be a co-atom of $\mathcal{L}_k$, that is, a clone which sits just below the greatest clone $\mathcal{O}_k$ in the lattice $\mathcal{L}_k$ of clones on E_k. The complete characterization of all maximal clones was obtained by I. G. Rosenberg [Ro70], which is one of the greatest achievements in the theory of clones. In contrast to maximal clones, however, the characterization of all minimal clones is not yet settled. The complete determinination of all minimal clones is known only for $k = 2, 3$. The problem of determinining all minimal clones is now widely recognized as one of the most challenging problems in universal algebra and discrete mathematics.

With M. Pinsker and T. Waldhauser, the author has been working on this problem for these few years ([MP 06], [MP 07a], [MP 07b] and [MW 08]). The basic idea of our work is to consider the base set E_k as Galois field GF(k) for a prime power k and to express generating functions of minimal clones as polynomials over GF(k). In this paper, summarizing the results obtained so far, we present some polynomials generating minimal clones over GF(k) for any prime power k.

2 Minimal Clones

For $F \subseteq \mathcal{O}_k$, $\langle F\rangle$ denotes the clone generated by F, that is, $\langle F\rangle$ is the least clone containing F. When F is a sigleton, i.e., $F = \{f\}$, $\langle F\rangle$ is simply denoted by $\langle f\rangle$.

Lemma 2. 1 *A minimal clone is generated by a single function. That is, for any minimal clone $C \in \mathcal{L}_k$ there exists $f \in \mathcal{O}_k$ such that $C = \langle f\rangle$.*

Complete list of minimal clones is known only for $k = 2$ and 3 ([Cs 83]). For more general cases we have the *type theorem* due to I. G. Rosenberg, which gives a rough picture of the classification of minimal clones.

Definition 2. 1 *An function f on E_k is* **minimal** *if* (i) *it generates a minimal clone and* (ii) *every function from $\langle f\rangle$ whose arity is smaller than the arity of f is a projection.*

Theorem 2. 2 ([Ro 86]) *Every minimal function belongs to one of the following five types:*

(1) *Unary functions f on E_k such that either* (i) f^2 $(= f \circ f) = f$ *or* (ii) *f is a permutation of prime order p (i.e., $f^p = id$).*

(2) *Idempotent binary functions; i.e., $f \in \mathcal{O}^{(2)}$ such that $f(x,x) = x$ for every $x \in E_k$.*

(3) *Majority functions; i.e., $f \in \mathcal{O}^{(3)}$ such that $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$ for every x, $y \in E_k$.*

(4) *Semiprojections; i.e., $f \in \mathcal{O}^{(n)}$ $(3 \le n \le k)$ such that there exists i $(1 \le i \le n)$ satisfying $f(a_1,\ldots,a_n) = a_i$ whenever $a_1,\ldots,a_n \in E_k$ are not pairwise distinct.*

(5) *If $k = 2^m$, the ternary functions $f(x,y,z) := x+y+z$ where $\langle E_k;+\rangle$ is an elementary 2-group (i.e., the additive group of an m-dimensional vector space over GF(2)).*

For $k = 3$, B. Csákány [Cs 83] determined all minimal clones by explicitly giving minimal functions generating them. There are 84 minimal clones on E_3.

3 Polynomials over a Finite Field

In the sequel, let k be a prime power and $(E_k;+,\cdot)$ be a finite field, i.e., Galois field GF(k). We shall express functions defined on E_k as polynomials defined over GF(k). Over a field GF(k) a function $f \in \mathcal{O}_k^{(2)}$, for example, can be expressed as

$$f(x,y) = \sum_{0 \le i,j < k} a_{ij}\, x^i y^j$$

where $a_{ij} \in E_k$ for $0 \le i$, $j < k$.

Note that $x^k = x$ for every $x \in$ GF(k). Hence, if $x \neq 0$ then $x^{k-1} = 1$ for $x \in$ GF(k).

4 Minimal Clones Generated by Binary Idempoint Functions

First, we consider idempotent binary functions generating minimal clones (Item (2) in Theorem 2.2). According to B. Csákány [Cs 83], there are 48 minimal clones generated by idempotent binary functions for $k = 3$.

In Csákány's paper, a binary idempotent function f is denoted by b_n if $n = f(0,1)\cdot 3^5 + f(0,2)\cdot 3^4 + f(1,0)\cdot 3^3 + f(1,2)\cdot 3^2 + f(2,0)\cdot 3 + f(2,1)$.

We proceed according to the following strategy.

<u>Our Strategy</u> :

Step 1 : Take arbitrary binary function $b(x,y) \in \mathcal{O}_3^{(2)}$ from Csákány's list of generators of minimal clones over E_3.

Step 2 : Search for a polynomial $g(x,y) \in \mathcal{O}_k^{(2)}$ for $k \ge 3$ whose counterpart for $k = 3$ is $b(x,y)$.

Step 3 : Verify that $g(x,y)$ is minimal.

4.1 Linear Polynomials

Consider binary linear polynomials $ax+by$ for a, $b \in E_k$. There is only one linear polynomial which is minimal on E_3.

Step 1: For $k = 3$, the binary linear polynomial

$$b_{624}(x, y) \;=\; 2x + 2y$$

is minimal.

Step 2: We have:

Theorem 4. 1 (*Á. Szendrei*) *Let k be a prime. Let $g(x, y)$ be a binary linear polynomial on E_k. Then g is* minimal *if and only if*

$$g(x,\, y) = a\,x + (k + 1 - a)\,y$$

for some $1 < a < k$. Moreover, all such linear polynomials generate the same minimal clone.

4.2 Monomials

Consider monomials $x^s\,y^t$ for $1 \le s \le t < k$. There is only one monomial which is minimal on E_3.

Step 1: For $k = 3$, $b_{11}(x, y) \;=\; x\,y^2$ is minimal.

Step 2: We have:

Theorem 4. 2 *Let k be a prime.*

(1) *$g(x, y) = x\,y^{k-1}$ is a minimal function.*

(2) *For any $1 < s < k - 1$, $x^s\,y^{k-s}$ is not a minimal function.*

For the proof see [MP 07a].

4.3 Some Generalizations

We shall add more minimal polynomials to our list of minimal polynomials obtained by the above procedure. For the proof of the minimality of polynomials for $k \ge 3$ we rely on the following lemma.

For $f \in \mathcal{O}_k^{(2)}$ let Γ_f be the following set of expressions:

$$\begin{aligned}\{\; &f(x, f(x, y)), \quad f(x, f(y, x)), \quad f(f(x, y), x),\\ &f(f(x, y), y), \quad f(f(x, y), f(y, x))\;\}\end{aligned}$$

Lemma 4. 3 *Let $f \in \mathcal{O}_k^{(2)}$ be a binary idempotent function which is not a projection. Suppose that for every $\gamma \in \Gamma_f$ one of the following holds:*

$$\gamma(x, y) \approx f(x, y) \quad \textit{or} \quad \gamma(x, y) \approx f(y, x)$$

Then f is minimal.

Here, by $h_1(x, y) \approx h_2(x, y)$ for $h_1, h_2 \in \mathcal{O}_k^{(2)}$ we mean $h_1(x, y) = h_2(x, y)$ for all $(x, y) \in (E_k)^2$.

For the proof of the lemma the reader is refered to [MP 07b].

(1) Generalization of b_{449}

Step 1 : (Basis)
For $k = 3$, $b_{449}(x, y) = x+y+2\,x\,y^2$ is minimal.

Step 2 : (Generalization)
For $k \ge 3$,
$$g(x, y) = x + y + (k - 1)\,x\,y^{k-1}$$
is minimal.

Step 3 : (Proof of Step 2)
It is clear that

$$g(x, y) = \begin{cases} x & \text{if } y = 0, \\ x + y + (k - 1)\,x = y & \text{if } y \ne 0. \end{cases}$$

Then for each expression in Γ_g we have the following:

(i)
$$g(x, g(x, y)) = \begin{cases} g(x, x) = x & \text{if } y = 0, \\ g(x, y) & \text{if } y \ne 0. \end{cases}$$

Hence $g(x, g(x, y)) \approx g(x, y)$.

(ii)
$$g(x, g(y, x)) = \begin{cases} g(x, y) = y & \text{if } x = 0, \\ g(x, x) = x & \text{if } x \ne 0. \end{cases}$$

Note that $g(x, y) = y$ for $x = 0$ follows from the definition of $g(x, y)$. Hence $g(x, g(y, x)) \approx g(y, x)$.

(iii)
$$g(g(x, y), x) = \begin{cases} g(x, y) = y & \text{if } x = 0, \\ x & \text{if } x \ne 0. \end{cases}$$

Hence $g(g(x, y), x) \approx g(y, x)$.

(iv)
$$g(g(x, y), y) = \begin{cases} g(x, y) = x & \text{if } y = 0, \\ y & \text{if } y \ne 0. \end{cases}$$

Hence $g(g(x,y),y) \approx g(x,y)$.

(v) Furthermore, we obtain

$$g(g(x,y),g(y,x)) = \begin{cases} 0 & \text{if } x = y = 0\,, \\ y & \text{if } x = 0,\ y \neq 0\,, \\ x & \text{if } x \neq 0,\ y = 0\,, \\ g(y,x) & \text{if } x \neq 0,\ y \neq 0\,. \end{cases}$$

Hence $g(g(x,y),g(y,x)) \approx g(y,x)$.
Therefore $g(x,y)$ is minimal by Lemma 4. 3 . □

(2) Generalization of b_{692}

Step 1 : (Basis)
For $k=3$, $b_{692}(x,y) = x + 2y^2 + x^2\,y^2$ is minimal.

Step 2 : (Generalization)
For $k \geq 3$,
$$g(x,y) = x + (k-1)y^{k-1} + x^{k-1}\,y^{k-1}$$
is minimal.

Step 3 : (Proof of Step 2)
It is easy to see that

$$g(x,y) = \begin{cases} k-1 & \text{if } \ x = 0,\ y \neq 0\,, \\ x & \text{otherwise.} \end{cases}$$

Then for each expression in Γ_g we have the following:

(i)
$$g(x,g(x,y)) = \begin{cases} k-1 & \text{if } x = 0,\ y \neq 0\,, \\ g(x,x) = x & \text{otherwise.} \end{cases}$$

Hence $g(x,g(x,y)) \approx g(x,y)$.

(ii) Similarly, we have
$$g(x,g(y,x)) \approx g(x,y).$$

(iii) $\sim$ (v): Note that $g(x,y) = 0$ iff $x = y = 0$. Then it is easy to see that

$$g(g(x,y),x) \approx g(g(x,y),y)$$
$$\approx g(g(x,y),g(y,x)) \approx g(x,y).$$

Therefore $g(x,y)$ is minimal by Lemma 4. 3 . □

(3) Generalization of b_{368}

Step 1 : (Basis)
For $k=3$, $b_{368}(x,y) = x + y^2 + 2\,x^2\,y^2$ is minimal.

Step 2 : (Generalization)
For $k \geq 3$,
$$g(x,y) \ = \ x + y^{k-1} + (k-1)x^{k-1}\,y^{k-1}$$
is minimal.

Step 3 : (Proof of Step 2)
For this $g(x,y)$ we see that

$$g(x,y) = \begin{cases} 1 & \text{if } \ x = 0\,,\ y \neq 0\,, \\ x & \text{otherwise}\,. \end{cases}$$

(i) We have

$$g(x,g(x,y)) = \begin{cases} g(0,1) = 1 & \text{if } x = 0,\ y \neq 0, \\ g(x,x) = x & \text{otherwise}\,, \end{cases}$$

which implies $g(x,g(x,y)) \approx g(x,y)$.
(ii) Also we have

$$g(x,g(y,x)) = \begin{cases} g(x,1) = x & \text{if } x \neq 0,\ y = 0, \\ g(x,y) & \text{otherwise}\,. \end{cases}$$

Hence $g(x,g(y,x)) \approx g(x,y)$.
(iii) $\sim$ (v): Note, as in (2), that $g(x,y) = 0$ iff $x = y = 0$. Then it is easy to see that

$$g(g(x,y),x) \approx g(g(x,y),y)$$
$$\approx g(g(x,y),g(y,x)) \approx g(x,y)$$

Therefore $g(x,y)$ is minimal by Lemma 4. 3 . □

4.4 More Generalizations

We shall give more examples of generalization. For the proof of these generalizations, Lemma 4. 3 is not applicable and we need other ways to prove the minimality which we omit here. (See [MW 08].)

(4) Generalization of b_{71}

Step 1 : (Basis)
For $k = 3$, $b_{71}(x,y) = x\,y^2 + 2\,x^2 + x^2\,y^2$ is minimal.

Step 2 : (Generalization)
For $k \geq 3$,
$$g(x,y) = x\,y^{k-1} + (k-1)x^{k-1} + x^{k-1}\,y^{k-1}$$
is minimal.

Observation :

$$\diamondsuit \quad g(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ k-1 & \text{if } x \neq 0,\ y = 0, \\ x & \text{if } x \neq 0,\ y \neq 0. \end{cases}$$

$$\heartsuit \quad g(x, g(x,y)) \approx x,$$
$$g(x, g(y,x)) \approx g(g(x,y),x) \approx g(g(x,y),y)$$
$$\approx g(g(x,y), g(y,x)) \approx g(x,y). \qquad \square$$

(5) Generalization of b_{41}

Step 1 : (Basis)
For $k = 3$, $b_{41}(x,y) = x^2 + x\,y^2 + 2\,x^2\,y^2$ is minimal.

Step 2 : (Generalization)
For $k \geq 3$,
$$g(x,y) = x^{k-1} + x\,y^{k-1} + (k-1)x^{k-1}\,y^{k-1}$$
is minimal.

Observation :

$$\diamondsuit \quad g(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0,\ y = 0, \\ x & \text{if } x \neq 0,\ y \neq 0. \end{cases}$$

$$\heartsuit \quad g(x, g(x,y)) \approx x,$$
$$g(x, g(y,x)) \approx g(g(x,y),x) \approx g(g(x,y),y)$$
$$\approx g(g(x,y), g(y,x)) \approx g(x,y). \qquad \square$$

(6) Generalization of b_{68}

Step 1 : (Basis)
For $k = 3$, $b_{68}(x,y) = 2x + 2\,x\,y^2$ is minimal.

Step 2 : (Generalization)
For $k \geq 3$,
$$g(x,y) = (k-1)\,x + 2\,x\,y^{k-1}$$
is minimal.

Note: In this case, simple replacement of 2 by $k-1$ does not work.

Observation :

$$\diamondsuit \quad g(x,y) = \begin{cases} (k-1)x & \text{if } y = 0, \\ x & \text{if } y \neq 0. \end{cases}$$

$$\heartsuit \quad g(x, g(x,y)) \approx g(g(x,y),y)$$
$$\approx g(g(x,y), g(y,x)) \approx x,$$
$$g(x, g(y,x)) \approx g(g(x,y),x) \approx g(x,y). \qquad \square$$

(7) Generalization of b_0

Step 1 : (Basis)
For $k = 3$, $b_0(x,y) = 2\,x^2 y + 2\,x\,y^2$ is minimal.

Step 2 : (Generalization)
For $k \geq 3$,
$$g(x,y) = (k-1) \sum_{i=1}^{k-1} x^{k-i}\,y^i$$
is minimal.

Step 3 : (Sketch of Proof) We have $f(x,x) = x$ since $(k-1)^2 = 1$. For $x \neq y$, let
$$D = \sum_{i=1}^{k-1} x^{k-i}\,y^i.$$
Then $xy^{-1}D = D$. Hence $xD = yD$ which implies $D = 0$. Therefore $f(x,y) = x$ if $x = y$ and $f(x,y) = 0$ if $x \neq y$. Now it is easy to see that f is minimal.

5 Minimal Clones Generated by Ternary Majority Functions

In this section we consider ternary majority functions generating minimal clones (Item (3) in Theorem 2.2). The results stated in this section come from [MW 08].

5.1 Majority Polynomials over GF(3)

A ternary function $f(x,y,z)$ on GF(3) can be expressed as a polynomial of the following form
$$f(x,y,z) \;=\; \sum_{0 \leq s,t,u \leq 2} a_{stu}\,x^s\,y^t\,z^u$$
where $a_{stu} \in E_3$ for all $s,t,u \in E_3$.

Proposition 5. 1 *A polynomial $f(x,y,z)$ over GF(3) is a majority function if and only if $f(x,y,z)$ is expressed as*

$$\begin{aligned} & f(x,y,z) \\ = \;& (axy + bxz + cyz) \\ & +(pxy^2 + (2p+1)x^2y) + (qyz^2 + (2q+1)y^2z) \end{aligned}$$

$$
\begin{aligned}
&+(rx^2z + (2r+1)xz^2) \ + \ xyz \\
&+2(ax^2y^2 + cy^2z^2 + bx^2z^2) \\
&+2(axyz^2 + bxy^2z + cx^2yz) \\
&+(2p+r+1)xy^2z^2 + (2q+p+1)x^2yz^2 \\
&+(2r+q+1)x^2y^2z + (a+b+c)x^2y^2z^2
\end{aligned}
$$

for some $a, b, c, p, q, r \in E_3$.

5.2 Majority Minimal Polynomials over GF(3)

According to B. Csákány [Cs 83], there are seven minimal clones on E_3 which are generated by ternary majority functions. As in [Cs 83], a ternary majority function f on E_3 will be denoted by m_t if $t = f(0,1,2)\cdot 3^5 + f(0,2,1)\cdot 3^4 + f(1,0,2)\cdot 3^3 + f(1,2,0)\cdot 3^2 + f(2,0,1)\cdot 3 + f(2,1,0)$.

The following are ternary majority functions which generate minimal clones on E_3.

1. $$\begin{aligned} m_0 = \ & 2(xy^2+x^2y) + 2(yz^2+y^2z) \\ &+2(x^2z+xz^2) \ + \ xyz \\ &+(xy^2z^2+x^2yz^2+x^2y^2z) \end{aligned}$$

2. $$\begin{aligned} m_{364} = \ & (xy+xz+yz) \\ &+2(xy^2+x^2y) + 2(yz^2+y^2z) \\ &+2(x^2z+xz^2) \ + \ xyz \\ &+2(x^2y^2+x^2z^2+y^2z^2) \\ &+2(x^2yz+xy^2z+xyz^2) \\ &+(xy^2z^2+x^2yz^2+x^2y^2z) \end{aligned}$$

3. $$\begin{aligned} m_{728} = \ & 2(xy+xz+yz) \\ &+2(xy^2+x^2y) + 2(yz^2+y^2z) \\ &+2(x^2z+xz^2) \ + \ xyz \\ &+(x^2y^2+x^2z^2+y^2z^2) \\ &+(x^2yz+xy^2z+xyz^2) \\ &+(xy^2z^2+x^2yz^2+x^2y^2z) \end{aligned}$$

4. $$\begin{aligned} m_{109} = \ & 2(xy+xz+yz) \\ &+(xy^2+yz^2+x^2z) \ + \ xyz \\ &+(x^2y^2+x^2z^2+y^2z^2) \\ &+(x^2yz+xy^2z+xyz^2) \\ &+(xy^2z^2+x^2yz^2+x^2y^2z) \end{aligned}$$

5. $$\begin{aligned} m_{473} = \ & (xy^2+yz^2+x^2z) \ + \ xyz \\ &+(xy^2z^2+x^2yz^2+x^2y^2z) \end{aligned}$$

6. $$\begin{aligned} m_{510} = \ & (xy+xz+yz) \\ &+(xy^2+yz^2+x^2z) \ + \ xyz \\ &+2(x^2y^2+x^2z^2+y^2z^2) \\ &+2(xyz^2+xy^2z+x^2yz) \\ &+(xy^2z^2+x^2yz^2+x^2y^2z) \end{aligned}$$

7. $$m_{624} = 2(xy^2+x^2y) + y^2z + x^2z + xyz$$

Note that m_{364} and m_{728} are conjugate to m_0, i.e., $m_{364} = (m_0)^{(01)}$ and $m_{728} = (m_0)^{(02)}$, and m_{473} and m_{510} are conjugate to m_{109}, i.e., $m_{473} = (m_{109})^{(02)}$ and $m_{510} = (m_{109})^{(12)}$.

5.3 Generalizations

We take the same strategy as above.

Our Strategy :

Step 1 : Take arbitrary majority function $m(x,y,z) \in \mathcal{O}_3^{(3)}$ from Csákány's list of generators of minimal clones over E_3.

Step 2 : Search for a polynomial $h(x,y,z) \in \mathcal{O}_k^{(3)}$ for $k \geq 3$ whose counterpart for $k=3$ is $m(x,y,z)$.

Step 3 : Verify that $h(x,y,z)$ is minimal.

In the sequel, we shall make use of two binary polynomials $\delta(x,y)$ and $\mu(x,y)$ defined as follows:

$$
\begin{aligned}
\delta(x,y) &= \sum_{i=1}^{k} x^{i-1}y^{k-i} \\
\mu(x,y) &= (k-1)\sum_{i=1}^{k-1} x^i y^{k-i}
\end{aligned}
$$

Note: It is verified that

$$\delta(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

and

$$\mu(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ x & \text{if } x = y. \end{cases}$$

We present, without proof, generalizations for ternary majority functions m_{624}, m_0 and m_{473}. For the details, see [MW 08].

(1) Generalization of m_{624}

Step 1 : For $k = 3$, take $m_{624}(x, y, z)$.

Step 2 : For $k \geq 3$,

$$h_{624}(x, y, z) = \mu(x, y) + z\,\delta(x, y)$$

is minimal.

(2) Generalization of m_0

Step 1 : For $k = 3$, take $m_0(x, y, z)$.

Step 2 : For $k \geq 3$,

$$\begin{aligned} h_0(x, y, z) &= \mu(x, y) + \mu(y, z) + \mu(z, x) \\ &\quad - 2 \cdot \mu(x, y) \cdot \mu(y, z) \cdot \mu(z, x) \cdot x^{k-3} \end{aligned}$$

is minimal.

(3) Generalization of m_{473}

Step 1 : For $k = 3$, take $m_{473}(x, y, z)$.

Step 2 : For $k \geq 3$,

$$\begin{aligned} h_{473}(x, y, z) &= h_0(x, y, z) \\ &\quad + \delta(x, y) \cdot \delta(y, z) \cdot \delta(x, z) \cdot p(x, y, z) \end{aligned}$$

is minimal, where $h_0(x, y, z)$ is a polynomial given in (2) and

$$\begin{aligned} p(x, y, z) &= xy^{k-1} + yz^{k-1} + zx^{k-1} \\ &\quad - (xyz)^{k-1} \cdot (x + y). \end{aligned}$$

Remark: Examples show the importance of $\delta(x, y)$ and $\mu(x, y)$ in constructing minimal polynomials.

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