数理解析研究所講究録
第 1600 巻 2008 年 204-218

One-dimensional Schrödinger Equations and Renormalization Groups of Wegner-Houghton-Aoki type

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April 8, 2008

Abstract

We consider the low-lying spectrum of the Schrödinger equation

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+V(x)\right]\phi_n(x)=E_n\phi_n(x)$$

where $V(x)=(1/2)a_0x^2+\lambda_0x^4$ and show that the low-lying spectrum of the above equation is derived by the renormalization group methods. Taking the infinitesimal form of the transformation, the non-linear evolution equation of Houghton-Wegner-Aoki type is derived:

$$V_t(t,x)=\frac{1}{2\pi t^2}\log\left(1+t^2V_{xx}(t,x)\right)$$

where $V(0,x)=V(x)$, $E_1\leq E_2\leq E_3\leq\cdots$ and E_1 is the ground energy. We discuss low-lying spectrum by using renormalization group methods.

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1 Introduction

We consider the low-lying spectrum of the one-dimensional Schrödinger equation

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+V(x)\right]\phi_n(x) = E_n\phi_n(x) \tag{1.1}$$

$$\tag{1.2}$$

where $V(x)$ is a positive function such that $V(x)=0$ for finitely many points $x_1 < x_2 < \cdots$. In this paper we explicitly consider the following potential $V(x)$

$$V(x)=\frac{1}{2}a_0x^2+\lambda_0x^4$$

where $\lambda_0 > 0$ and a_0 may be negative. It is not difficult to extend our arguments to general $V(x)$. We are interested in describing $E_i - E_1$ by some explicit formula and we show that the difference $E_2 - E_1$ is given by the mass term of the renormalized Schroedinger operator which is obtained from the original Schroedinger operator by applications of block spin transformation.

We discuss two approaches in this paper. The first one is

(i) direct applications of the block spin transformations to the Schroedinger operators,

and the second one is

(ii) analysis of non-linear partial differential equation derived from the infinitesimal form of the block spin transformation.

If one applies the block spin transformation to the Schrödinger operator, the resultant renormalized Schrödinger equation takes the form

$$\left[-\frac{1}{2L^n}\frac{d^2}{dx^2}+L^n\left(\frac{\tau_2}{2}x^2+\tau_4x^4\right)\right]\phi_n(x)=E_n\phi_n(x) \tag{1.3}$$

neglecting higher order terms, where $L > 1$ is an integer (size of blocks) and n is the number of the iteration of the block spin transformations. Moreover τ_2 and τ_4 are strictly positive numbers obtained by the block spin transformations.

We then take the limit $n \to \infty$. This is the classical limit discussed in [11, 12], and by setting $\zeta = L^{n/2}x$ we obtain

$$\left[-\frac{1}{2}\frac{d^2}{d\zeta^2}+\frac{1}{2}\tau_2\zeta^2+\frac{1}{L^n}\tau_4\zeta^4\right]\phi_n(\zeta)=E_n\phi_n(\zeta) \tag{1.4}$$

This implies that the low-lying spectrum is given by that of the harmonic oscillator:

$$E_1 - E_0 = \sqrt{\tau_2} \tag{1.5}$$

The second purpose of this paper is the derivation of the $L \to 1$ limit of the block spin transformation. The resultant equation is a no-linear partial differential equation (PDE) called Wegner-Houghton-Aoki (WHA) equation:

$$V_t(t,x) = \frac{1}{2\pi t^2}\log(1+t^2V_{xx}(t,x)) \tag{1.6}$$

where $t = L^n$. We derive this equation by taking the $L \to 1$ limit of the block spin transformation, and discuss if the limit $\lim_{t\to\infty} V_{xx} = \tau_2$ exists or not. But so far our analysis in this direction is not yet completed.

We organize our paper as follows: in section 2, we briefly revisit block spin transformations, and discuss hierarchical approximations of it. We also derive WHA equation by taking $L \to 1$ limit in the hierarchical approximation.

In section 3, we apply the BST (of hierarchical approximation) to the one dimensional Schödinger equations and we show that they converge to the equations of harmonic oscillators

$$\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}\sigma^2x^2\right)\psi = E\psi \tag{1.7}$$

after BST and a scaling, where $\sigma^2 > 0$ is the fixed point.

In section 4, we discuss the relation between the number of iterations to bring $a_0 < 0$ to $a_n > 0$ and Simon's formula to describe $E_2 - E_1$ by Agmon's metric. We also discuss about solutions of WHA equations.

Remarks 1 *(1) Similar non-linear equations are considered in [10] which correspond to $D = 2$ and differ from ours by the non-linear term $(V_x)^2$. It is shown that the system exhibits transitions of Kosterlitz-Thouless type.*
(2) This statement must be taken with a grain of salt. This is obtained as the renormalized Gibbs measure and the decay rate of correlation functions.

2 Block Spin Transformations Revisited

We put the system in the periodic box $\Lambda \subset Z^1$ of size L^N, $L > 1$, $N >> 1$. We also set $\Lambda_n = L^{-n}\Lambda \cap Z$ and introduce the set of L points centered at the origin: $\Box = \{-(L-1)/2, -(L-1)/2+1, \cdots, (L-1)/2-1\}$ for odd L or $\Box = \{-L/2, -L/2+1, \cdots, L/2-1\}$ for even L. Moreover we denote by $[x/L]$ the integer closest to x/L.

We then consider the Gibbs measure $d\mu = d\mu_\Lambda$ defined by

$$\begin{aligned} <\phi(0)\phi(x)> &\equiv \int \phi(0)\phi(x)d\mu && \text{(2.1a)} \\ &= \frac{1}{N}\int \phi(0)\phi(x)\exp[-\mathcal{H}(\phi)]\prod d\phi(x) && \text{(2.1b)} \\ \mathcal{H}(\phi) &= \frac{1}{2} < \phi, (-\Delta + m_0^2)\phi > + \sum V(\phi(x)) && \text{(2.1c)} \\ V(\phi(x)) &= \frac{1}{2}a_0\phi^2(x) + \lambda_0\phi^4(x) && \text{(2.1d)} \end{aligned}$$

where Δ is the lattice Laplacian

$$(-\Delta f)(x) = 2f(x) - f(x+1) - f(x-1) \tag{2.2}$$

and we take $m_0^2 > 0$ arbitrarily small and take the limit $m_0 \to 0$ after all calculations.

We consider how $< \phi(0)\phi(x) >$ decreases as $|x| \to \infty$ whose decay rate is nothing but $E_1 - E_0$. To obtain the long-range behavior, it is a standard technique [14] to integrate out redundant degrees of freedom of $\{\phi(x); x \in \Lambda\}$ by introducing block spins [4] :

$$\begin{aligned} \phi_{n+1}(x) &= (C\phi_n)(x) \\ &\equiv \frac{1}{L^\alpha}\sum_{\zeta\in\square}\phi_n(Lx+\zeta) \end{aligned} \tag{2.3}$$

To start with we set $V(x) = 0$ and the covariance of ϕ_n is given by

$$< \phi_n(x)\phi_n(y) >= G_n(x-y) \tag{2.4}$$

where

$$G(x,y) = \frac{1}{-\Delta + m_0^2}(x,y) \tag{2.5a}$$

$$\begin{aligned} G_n(x,y) &= (C^n G(C^+)^n)(x,y) = L^{-2n\alpha}\sum_{\zeta,\xi} G(L^n x+\zeta, L^n y+\xi) \\ &= \frac{1}{L^{2n\alpha}}\sum_{\zeta,\xi}\int \frac{\exp[iL^n p(x-y) + ip(\zeta-\xi)]}{m_0^2 + 2(1-\cos p)}\frac{dp}{2\pi} \\ &= \frac{1}{L^{2n\alpha}}\int \frac{\exp[iL^n p(x-y)]}{m_0^2+(1-\cos p)} \times \frac{\sin^2(L^n p/2)}{\sin^2(p/2)}\frac{dp}{2\pi} \end{aligned} \tag{2.5b}$$

By rewriting $p \in (-\pi,\pi]$ as $(p+2k\pi)/L^n$ with $p \in (-\pi,\pi]$ and $k = 0,1,\cdots,L^n-1$, we get

$$\begin{aligned} G_n(x,y) &= \frac{L^{2n}}{L^{2n\alpha}}\sum_k \int_{-\pi}^{\pi} \frac{\exp[ip(x-y)]}{L^{2n}m_0^2 + 4L^{2n}\sin^2(p+2k\pi)/2L^n} \\ &\quad \times \frac{\sin^2(p/2)}{\sin^2((p+2\pi k)/2L^n)}\frac{dp}{2\pi L^n} \\ &\sim \frac{L^{3n}}{L^{2n\alpha}}\sum_k \int_{-\pi}^{\pi} \frac{\exp[ip(x-y)]}{L^{2n}m_0^2 + (p+2k\pi)^2}\frac{4\sin^2(p/2)}{(p+2\pi k)^2}\frac{dp}{2\pi} \end{aligned} \tag{2.6}$$

which implies that

$$G_n^{-1} \sim L^{2n\alpha-3n}\left(-\Delta + L^{2n}m_0^2\right) \tag{2.7}$$

The choice of $\alpha \geq 0$ is arbitrary, but $\alpha \geq 0$ should be chosen so that the analysis becomes easy. The choice $\alpha = 3/2$ leaves Δ invariant and the choice $\alpha = 1/2$ leaves m_0^2 invariant. We first leave $\alpha \geq 0$ as a free parameter.

Let

$$A_n = (G_n C^+ G_{n+1}^{-1}) \tag{2.8}$$

and introduce $Q : R^{\Lambda \setminus LZ} \to R^{\Lambda}$ such that

$$(Qf)(x) = \begin{cases} f(x) & \text{if } x \notin LZ \\ -\sum_{x \in \square} f(x) & \text{if } x \in LZ \end{cases}$$

where it is understood that $f(x) = 0$ for $x \in LZ$.

Then

$$(CQ) = 0, \quad CA_n = 1 \tag{2.9}$$

and we conversely have

$$\phi_n(x) = \sum_{\zeta} A_n(x, \zeta)\phi_{n+1}(\zeta) + \sum_{\xi \in Z \setminus LZ} Q(x, \xi) z_n(\xi) \tag{2.10}$$

This yields the following decomposition of the Gaussian variables:

$$< \phi_n, G_n^{-1}\phi_n >=< \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} > + < z, \Gamma_n^{-1} z > \tag{2.11}$$

where

$$G_{n+1}^{-1} = A_{n+1}^{+} G_n^{-1} A_{n+1}, \quad \Gamma_n^{-1} = Q^+ G_n^{-1} Q \tag{2.12}$$

2.1 Hierarchical Approximation

Since $CA_n = 1$ and $CQ = 0$, the relation (2.10) is approximately written

$$\phi_n(x) = \frac{L^{\alpha}}{L}\phi_{n+1}\left(\left[\frac{x}{L}\right]\right) + (Qz_n)(x) \tag{2.13a}$$

$$\sum_{\zeta \in \square}(Qz_n)(Lx + \zeta) = 0 \tag{2.13b}$$

See [3, 5] and references cited therein. Then

$$\sum_{x \in \Lambda_n} \phi_n^2(x) = L^{2\alpha-1} \sum_{x \in \Lambda_{n+1}} \phi_{n+1}^2(x) + \sum_{x \in \Lambda_n} (Qz_n)^2(x) \tag{2.14a}$$

$$\begin{aligned} \sum_{x \in \Lambda_n} \phi_n^4(x) &= L^{4\alpha-3} \sum_{x \in \Lambda_{n+1}} \phi_{n+1}^4(x) + 6 \sum_{x \in \Lambda_{n+1}} L^{2\alpha-2}\phi_{n+1}^2(x)(\sum_{\zeta \in \square}(Qz_n)^2(Lx + \zeta)) \\ &+3 \sum_{x \in \Lambda_{n+1}} L^{\alpha-1}\phi_{n+1}(x)(\sum_{\zeta \in \square}(Qz_n)^3(Lx + \zeta)) + \sum (Qz_n)^4(x) \end{aligned} \tag{2.14b}$$

This approximation is called the hierarchical approximation, which means that we employ an artificial (hierarchical) Laplacian Δ_{hcl} given by

$$< \phi, (-\Delta_{hcl})\phi >= \sum_n L^{-(3-2\alpha)n} \left(\sum_{[x/L]=[y/L]} (\phi_n(x) - \phi_n(y))^2 \right) \tag{2.15}$$

That is, if $L = 2$ then

$$\begin{aligned} < \phi, (-\Delta_{hcl})\phi > &= \sum_n 2^{-(3-2\alpha)n} \left(\sum_x (\phi_n(2x+1) - \phi_n(2x))^2 \right) \\ &= \sum_n 2^{-(3-2\alpha)n} \left(\sum_x 4z_n^2(2x+1) \right) \end{aligned}$$

(Note that $z_n(2x)$ is absent.) For general L, we may put

$$\begin{aligned} &< \phi_n, (-\Delta_{hcl})\phi_n > \\ = \ & L^{-(3-2\alpha)} < \phi_{n+1}, (-\Delta_{hcl})\phi_{n+1} > + L^{-(3-2\alpha)n} \sum_{x \in \Lambda_n \backslash Lx} z_n(x)^2 \end{aligned} \tag{2.16}$$

2.2 Derivation of the WHA equation

We derive a non-linear PDE as an infinitesimal form [1] of the Block Spin Transformation [14, 4] where the treatment is accurate in the small field region (of x) and seems to be reasonable in the large filed region, but high-momentum parts (say non-local parts) are completely neglected. We discuss about this shortly:

We apply the BST to $\mathcal{H}_n$ to obtain $\mathcal{H}_{n+1}$ the effective Hamiltonian at the distance scale $t = L^{-n}$. To recover PDE of WHA type, we choose $\alpha = 1/2$ so that the fluctuation field z at the distance scale at L^n has the coefficient L^{-n}. Thus we start from

$$\mathcal{H}_n = \frac{1}{2} L^{-2n} < \phi_n, (-\Delta_{hcl})\phi_n > + \sum_x V_n(\phi_n(x)) \tag{2.17}$$

Then we have

$$\begin{aligned} V_{n+1}(\phi_{n+1}) &= -\log \left[\int \exp[-\sum_x V_n(L^{-1/2}\phi_{n+1}([x/L]) + z(x))] d\mu(z) \right] \\ &= L \sum_x V_n(L^{-1/2}\phi_{n+1}(x)) \\ &\quad - \log \left[\int \exp[-\sum_x \delta V_n(L^{-1/2}\phi, z(x)] d\mu(z) \right] \end{aligned} \tag{2.18a}$$

where (denoting V_n by V for simplicity)

$$\delta V(L^{\alpha-1}\phi, z(x)) = V_\phi(L^{\alpha-1}\phi) z(x) + \frac{1}{2} V_{\phi\phi}(L^{\alpha-1}\phi) z^2(x) + O(z^3) \tag{2.19a}$$

$$d\mu(z) = \text{const.} \prod_{x \in \Lambda_n \backslash L\Lambda_{n+1}} \exp[-\frac{1}{2} L^{-2n} z^2(x)] dz(x) \tag{2.19b}$$

The infinitesimal form of the BST is derived by setting $L \to 1$ and $n \to \infty$ keeping $L^n \equiv t$ fixed. Thus the z variables carries a very thin momentum between $t^{-1} = L^{-n}$ and $(tL)^{-1} = L^{-n-1}$. Then the integral can be carried out and we have

$$\det{}^{-1/2}(L^{-2n} + V_{\phi\phi})$$

Therefore regarding $LV_n(L^{-1/2}\phi_{n+1})$ as $V_n(\phi_{n+1})$ as $L \to 1$, we have

$$-\frac{V_{n+1}(\phi_{n+1}) - V_n(\phi_{n+1})}{L^{-n-1} - L^{-n}} = -\frac{\partial}{\partial t^{-1}} V(\phi, t) = \frac{1}{2} \log\left(1 + t^2 V_{\phi\phi}\right) \tag{2.20}$$

where we set $V_n(\phi_n) = V(\phi, t)$. We then have (1.6).

Remark 2 *This is rather heuristic, and can be done in a more consistent way in momentum space, see [10]. Similar non-linear equations are considered in [10] which correspond to $D = 2$ and differ from ours by the non-linear term $(V_x)^2$. It is shown that the system exhibits transitions of Kosterlitz-Thouless type.*

3 Study by the Hierarchical BST

We apply the BST to $\mathcal{H}$ to obtain $\mathcal{H}_n$ the effective Hamiltonian at the distance scale L^n. The use of the parameter $\alpha = 3/2$ is not adequate though z variables always have the same strength $O(1)$. In fact the system is always massive, we soon get a non-zero mass term from the interaction. Put

$$\mathcal{H}_0 = \frac{1}{2} < \phi, G_0^{-1}\phi > + \sum_x V_0(\phi(x)) \tag{3.1}$$

Then we have

$$\begin{aligned} \mathcal{H}_1 &= \frac{1}{2} < \phi_1, G_1^{-1}\phi_1 > - \log\left[\int \exp[-\sum_x V_0(L^{\alpha-1}\phi_1([x/L]) + z(x))] d\mu(z)\right] \\ &= \frac{1}{2} < \phi_1, G_1^{-1}\phi_1 > + L\sum_x V_0(L^{\alpha-1}\phi_1(x)) \\ &\quad - \log\left[\int \exp[-\sum_x \delta V_0(L^{\alpha-1}\phi_1, z(x)] d\mu(z)\right] \end{aligned} \tag{3.2a}$$

We take

$$\begin{aligned} V_0(\phi) &= \frac{1}{2}a_0\phi^2 + \lambda_0\phi^4 & (3.3a) \\ d\mu_0(z) &= \prod \frac{1}{\sqrt{\pi}} \exp[-\frac{1}{2}z^2(x)] dz(x) & (3.3b) \end{aligned}$$

where $\lambda_0 > 0$ and $a_0 \in R$ may be negative (in the case of double-well potential). Then

$$\mathcal{H}_1 = \frac{1}{2} < \phi^1, G_1^{-1}\phi_1 > + \sum_x \left(\frac{1}{2}L^{2\alpha-1}a_0\phi_1^2(x) + L^{4\alpha-3}\lambda_0\phi_1^4(x)\right) + \delta V_1 \quad (3.4a)$$

$$\delta V_1 = -\log\left[\int \exp[-\sum_x \left(\frac{1}{2}a_0 z^2(x) + \lambda_0 z^4(x)\right) - 6\lambda_0 L^{2\alpha-1}\sum_x \phi_1^2(x)(\sum z(\zeta)^2)]d\mu(z)\right] \quad (3.4b)$$

3.1 case of $a_0 > 0$

Since a_0 and λ_0 are both positive, these positivity is kept through the iterations of the renormalization recursion formula, and a_n and λ_n converge to some values after suitable rescaling.

It is convenient to put $\alpha = 1$ in this case since $a_0\phi^2 + \lambda_0\phi^4$ yields $O(1)z^2$ which is large enough for the convergence of the integral. Then

$$G_n^{-1} = L^{-n}\left(-\Delta + L^{2n}m_0^2\right) \quad (3.5)$$

Thus the hierarchical approximation means the substitution

$$\phi_n(x) = \phi_{n+1}\left(\left[\frac{x}{L}\right]\right) + (Qz_n)(x)$$

and then

$$\sum_{x\in\Lambda_n} \phi_n(x) = \sum_{x\in\Lambda_{n+1}} L\phi_{n+1}(x)$$

$$\sum_{x\in\Lambda_n} \phi_n^2(x) = \sum_{x\in\Lambda_{n+1}} L\phi_{n+1}^2(x) + \sum_x (Qz_n)^2(x)$$

$$\sum_{x\in\Lambda_n} \phi_n^4(x) = \sum_{x\in\Lambda_{n+1}} L\phi_{n+1}^4(x) + \sum_{x\in\Lambda_{n+1}} 6\phi_{n+1}^2(x)(\sum_{\zeta\in\Box(Lx)} (Qz_n)^2(x)) + \sum_x (Qz_n)^4(x)$$

$$d\mu(z) = \prod_{x\in\Lambda_n\setminus LZ} \frac{L^{-n/2}}{\sqrt{\pi}} \exp[-L^{-n}z^2(x)/2]dz(x)$$

and estimate ($\alpha = 1$)

$$\begin{aligned}
\exp[-\delta V_1] &= \int \exp[-\left(\frac{1}{2}a_0 z^2 + \lambda_0 z^4\right) - 6\lambda_0\phi_1^2 z^2]d\mu(z) \\
&= \int \exp\left[-\frac{1}{2}\left((a_0+1) + 12\lambda_0\phi_1^2 + 2\lambda_0 z^2\right)z^2\right] dz \\
&= \text{const.}\left(a_0 + 1 + O(\lambda_0) + 12\lambda_0\phi_1^2\right)^{-1/2} \\
&= \text{const.}\exp\left[-\frac{12\lambda_0\phi_1^2(x)}{a_0 + O(1) + O(\lambda_0)}\right]
\end{aligned}$$

Therefore

$$\begin{aligned} a_0 \to a_1 &= La_0 + \frac{12\lambda_0}{a_0 + O(1)} \\ \lambda_0 \to \lambda_1 &= L\lambda_0 \end{aligned}$$

where we neglected higher order terms. In general we get

$$\begin{aligned} a_n \to a_{n+1} &= La_n + \frac{12\lambda_n}{a_n + O(1)} > La_n \\ \lambda_n \to \lambda_{n+1} &= L\lambda_n \end{aligned}$$

Then $\lambda_n = L^n \lambda_0$ and

$$a_{n+1} = La_n + \frac{12L^n\lambda_0}{a_n + O(1)}$$

or

$$\frac{a_{n+1}}{L^{n+1}} = \frac{a_n}{L^n} + \frac{12\lambda_0}{La_n + O(1)}$$

Then $a_{n+1} \geq L^{n+1}a_0$ and

$$\frac{a_{n+1}}{L^{n+1}} = a_0 + \sum_{k=0}^{n} \frac{12\lambda_0}{La_k + O(1)}$$

converges:

$$\tau_2 \equiv \lim \frac{a_n}{L^n}, \quad \text{and} \quad \tau_4 \equiv \lim \frac{\lambda_n}{L^n}$$

exist. Thus our effective Hamiltonian is written

$$\frac{1}{2L^n} < \phi_n, (-\Delta)\phi_n > +L^n(\tau_2\phi_n^2 + \tau_4\phi_n^4)$$

neglecting higher order terms. This corresponds to the Schroedinger equation

$$\frac{1}{2L^n}\left(-\frac{d^2}{d^2x}\right) + L^n(\tau_2x^2 + \tau_4x^4) \tag{3.6}$$

We set $\zeta = L^{n/2}x$. Then the above equation is unitarily equivalent to

$$\frac{1}{2}\left(-\frac{d^2}{d^2\zeta}\right) + (\tau_2\zeta^2 + L^{-n}\tau_4\zeta^4) \tag{3.7}$$

Thus as $L^n \to \infty$, only the mass term ζ^2 survives with the coefficients τ_2 obtained as the fixed point. This argument goes back to [11, 12].

The spectrum of the harmonic oscillator

$$-\frac{1}{2}\frac{d^2}{d^2\zeta} + \frac{\omega^2}{2}\zeta^2 \tag{3.8}$$

is $\{(n + 1/2)\omega; n \in N\}$. Thus we see that

$$E_2 - E_1 = \sqrt{2\tau_2} \tag{3.9}$$

Remark 3 *Our result may be take with a grain of salt, since only the second (E_2) energy survives in this scaling limit and the higher states vanish in this limit. Then the excited energies ($n > 2$) of the harmonic oscillator is not reliable.*

3.2 case of $a_0 < 0$

If $a_0 < 0$, the integration over $z_n(x)$ depends on the magnitude of $\phi_n^2(x)$. For large $|\phi_n(x)|$ for which the coefficient of z^2 is negative, we still set $\alpha = 1$. We later consider optimal value of α after calculations.

We again start with

$$\mathcal{H}_1 = \frac{1}{2} < \phi^1, G_1^{-1}\phi_1 > + \sum_x \left(\frac{1}{2} L a_0 \phi_1^2(x) + L\lambda_0 \phi_1^4(x) \right) + \delta V_1 \tag{3.10}$$

where

$$\begin{aligned} \exp[-\delta V_1] &= \int \exp\left[-\sum_x \frac{1}{2} \left(a_0 + 1 + 12\lambda_0 \phi_1^2(x) + 2\lambda_0 z^2(x) \right) z^2(x) \right] dz \\ &= \int \exp\left[-\sum_x 6\lambda_0 \left(\phi_1^2(x) - K_0 + \frac{1}{6} z^2(x) \right) z^2(x) \right] dz \qquad (3.11a) \\ K_0 &= \frac{-a_0 - 1}{12\lambda_0} > 0 \qquad (3.11b) \end{aligned}$$

and 1 in a_0+1 comes from $d\mu(z)$. The z integration may be separated into three regions:

(i) from the bottom of the wine bottle to large filed region:

$$\begin{aligned} L_1 &\equiv \{\phi_1(x); a_0 + 1 + 12\lambda_0 \phi_1^2(x) \geq 0\} \\ &\equiv \{\phi_1(x); \phi_1(x)^2 \geq K_1 + 1\} \end{aligned} \tag{3.12}$$

(ii) transition region:

$$T_1 = \{\phi(x); \lambda_0 |\phi_1^2 - K_0| \leq 1\} \tag{3.13}$$

(iii) small filed region:

$$\begin{aligned} S_1 &\equiv \{\phi_1(x); a_0 + 1 + 12\lambda_0 \phi_1^2(x) \leq 0\} \\ &\equiv \{\phi_1(x); \phi_1(x)^2 \leq K_0\} \end{aligned} \tag{3.14}$$

3.2.1 from the bottom to the large field region

First we consider the region $\phi_1 \in L_1$, namely the region $\sqrt{\lambda_0}(\phi_1^2 - K_0) > 1$ which contains the bottom of the wine bottle $\phi^2 = \mu_0^2 \equiv -a_0/4\lambda_0$. In this region, we have

$$\begin{aligned} \exp[-\delta V_1(\phi_1)] &= \int_{-\infty}^{\infty} \exp\left[-(6(\phi_1^2 - K_0) + \frac{z^2}{\lambda_0}) z^2 \right] dz \\ &= \frac{c}{\sqrt{\phi_1^2 - K_0}} \int \exp\left[-z^2 - \frac{z^4}{36\lambda_0(\phi_1^2 - K_0)^2} \right] dz \end{aligned}$$

Thus after suitable normalization, we obtain

$$e^{-\delta V_1} = \frac{c}{\sqrt{\phi_1^2 - K_0}} \left\{ 1 - \frac{c_1}{\lambda_0(\phi_1^2 - K_0)^2} + \frac{c_2}{\lambda_0^2(\phi_1^2 - K_0)^4} + O\left(\frac{1}{\lambda_0^3(\phi_1^2 - K_0)^6} \right) \right\}$$

where

$$c_1 = \frac{1}{48}, \qquad c_2 = \frac{35}{6\cdot 16\cdot 24}$$

These terms have no effects in the large field region, but exhibit a significant effect near the bottom of the potential $\phi_1^2 \sim \mu_0^2 \equiv -a_0/4\lambda_0$.

Set $X = \phi_1^2 - \mu_0^2$ ($\mu_0^2 = 3K_0 = a_0/4\lambda_0$) and expand δV_1 in terms of X:

$$\begin{aligned}
\frac{1}{\sqrt{\phi_1^2 - K_0}} &= \frac{1}{\sqrt{2K_0 + X}} = \exp\left[-\frac{1}{2}\log(1 + \frac{X}{2K_0})\right] \\
&= \exp\left[-\frac{X}{4K_0} + \frac{X^2}{16K_0^2}\right]
\end{aligned}$$

In the same way,

$$1 - \frac{c_1}{\lambda_0(\phi_1^2 - K_0)^2} + \frac{c_2}{\lambda_0^2(\phi_1^2 - K_0)^4} = d_0\left[1 - d_1 X - d_2 X^2 + O(X^3)\right] \tag{3.15}$$

where

$$d_1 = \frac{1}{16\lambda_0 K_0^2} - O(\lambda^{-2}K_0^{-3}), \qquad d_2 = \frac{1}{32\lambda_0 K_0^3} - O(\lambda^{-2}K_0^{-4}) \tag{3.16}$$

Then in the neighborhood of the bottom of the potential, we see that

$$\begin{aligned}
V_1(\phi_1) &= LV_0(\phi_0) + \delta V_1 \\
&= L\lambda_0 X^2 + \left(\frac{1}{4K_0} - d_1\right)X - \left(\frac{1}{8K_0^2} + d_2 - \frac{1}{2}d_1^2\right)X^2 + O(X^3) \\
&= \left(L\lambda_0 - \frac{1}{8K_0^2} + d_2 - \frac{1}{2}d_1^2\right)(\phi_1^2 - \mu_0^2 + \delta m_0^2)^2 + O(X^3)
\end{aligned} \tag{3.17}$$

where

$$\delta m_0^2 = \frac{(4K_0)^{-1} - d_1}{2L\lambda_0 + (8K_0^2)^{-1} + d_2 - d_1^2/2} \tag{3.18}$$

is a strictly positive constant, and we put

$$\mu_1^2 = \mu_0^2 - \delta m_0^2 \tag{3.19}$$

3.2.2 transition region

For ϕ_1 such that $\sqrt{\lambda_0}|\phi_1^2 - K_0| \leq 1$, we use convergent perturbative calculation to obtain

$$\begin{aligned}
\exp[-\delta V_1(\phi_1)] &= \int_{-\infty}^{\infty} \exp\left[-(6\sqrt{\lambda_0}(\phi_1^2 - K_0) + z^2)z^2\right] dz \\
&= \mathcal{N}\frac{\int e^{-z^4}\left(1 - 6\sqrt{\lambda_0}(\phi_1^2 - K_0)z^2 + \cdots\right) dz}{\int e^{-z^4} dz} \\
&= \mathcal{N}\left(1 - \tilde{c}_1\sqrt{\lambda_0}(\phi_1^2 - K_0) + \tilde{c}_2\lambda_0(\phi_1^2 - K_0)^2 + \cdots\right) \\
&= \exp\left[-(\tilde{c}_1\sqrt{\lambda_0} + (2\tilde{c}_2 - \tilde{c}_1^2)\lambda_0 K_0)\phi_1^2 + (\tilde{c}_2\lambda_0 - \frac{1}{2}\tilde{c}_1^2\lambda_0)\phi_1^4\right]
\end{aligned}$$

where we have neglected $O(\lambda_0^{3/2}|\phi^2 - K_0|^3)$ and

$$\tilde{c}_1 = 6\frac{\int e^{-z^4} z^2 dz}{\int e^{-z^4} dz} = 2.028, \quad \tilde{c}_2 = 18\frac{\int e^{-z^4} z^4 dz}{\int e^{-z^4} dz} = 4.5.$$

Then we have the following approximate recursions which hold for ϕ_1 in the intermediate region, i.e., $\sqrt{\lambda_0}|\phi_1^2 - K_0| << 1$

$$\lambda_0 \rightarrow \lambda_1 = (L - \tilde{c}_2 + \frac{1}{2}\tilde{c}_1^2)\lambda_0 \tag{3.20a}$$

$$\begin{aligned} a_0 \rightarrow a_1 &= La_0 + 2\tilde{c}_1\sqrt{\lambda_0} + 2(2\tilde{c}_2 - \tilde{c}_1^2)\lambda_0 K_1 \\ &= La_0 - \frac{1}{6}(2\tilde{c}_2 - \tilde{c}_1^2)a_0 + 2\tilde{c}_1\sqrt{\lambda_0} \qquad (3.20b) \\ &= (L - \frac{1}{6}(2\tilde{c}_2 - \tilde{c}_1^2))a_0 + 2\tilde{c}_1\sqrt{\lambda_0} \qquad (3.20c) \end{aligned}$$

$$K_0 \rightarrow K_1 = \frac{-La_0(1 - (2\tilde{c}_2 - \tilde{c}_1^2)/6L) - 2c_1\sqrt{\lambda_0}}{12L\lambda_0(1 - (2\tilde{c}_2 - \tilde{c}_1^2)/2L)} \tag{3.20d}$$

$$= K_0\left[1 + \frac{1}{3L}(2\tilde{c}_2 - \tilde{c}_1^2)\right] - \frac{\tilde{c}_1}{12\sqrt{\lambda_0}(L - (2\tilde{c}_2 - \tilde{c}_1^2)/2)} \tag{3.20e}$$

where $2\tilde{c}_2 - \tilde{c}_1^2 \sim 4.88$.

3.2.3 small field region

In the same way, for $\phi_1 \in S_1$, using $a_0 + 12\lambda_0\phi_0^2 = 12\lambda_0(\phi_0^2 - K_1) < 0$, we have

$$\begin{aligned} \exp[-\delta V_1(\phi_1)] &= \exp\left[\frac{(a_0 + 12\lambda_0\phi_1^2)^2}{16\lambda_0}\right] \\ &\times \int \exp\left[-\lambda_0\left(z^2(x) + \frac{a_0 + 12\lambda_0\phi_1^2}{4\lambda_0}\right)^2\right] d\mu(z) \\ &= \exp\left[\frac{(a_0 + 12\lambda_0\phi_1^2)^2}{16\lambda_0}\right] \\ &\times \int \exp\left[-\lambda_0\left(z^2(x) - 3(K_1 - \phi_1^2)\right)^2\right] d\mu(z) \\ &= \exp\left[\frac{(a_0 + 12\lambda_0\phi_1^2)^2}{16\lambda_0} - \log\left(K_1 - \phi_1^2(x) + O(1)\right)\right] \end{aligned}$$

$$\lambda_0 \rightarrow \lambda_1 = (L - 9)\lambda_0 \tag{3.21a}$$

$$a_0 \rightarrow a_1 = (L - \frac{3}{2})a_0 \tag{3.21b}$$

The result depends on L. For the reasonable choice of L, e.g. for $1 < L \leq 5/2$, we see $|L - 3/2| \leq 1$ and $L - 9 < -13/2$. Then $\lambda_1 < 0$, $|a_1| < |a_0|$ for $L \in (1, 5/2)$ and the function $V_n(\phi)$ may exhibit vibration in the small field region. (Namely V_n may possesses many minima.)

3.3 Iterations and Global Flow

We then iterate the renormalization group transformation, taking the forms of (3.17), (3.18) and (3.20e) as inductive assumptions.

Though our previous recursion relations are crude, we can see from (3.17),(3.18) and (3.20e) that (though the result depends on L) for a reasonable choice of L ($L = 2 \sim 3$), the potential V_n becomes tame in the neighborhood of $\phi_n^2 \sim \mu_n^2$ and μ_n^2 tends to 0 (no double-well potential) as n increases if initial $\mu_0^2 = -a_0/4\lambda_0$ ($= 3K_0$) and $-a_0$ are small, where μ_n^2 is the value of ϕ_n^2 at the minimum point of the double-well potential [7].

Theorem 4 *For small $\mu_0^2 = -a_0/4\lambda_0 > 0$ and for small $-a_0 > 0$, the value μ_n^2 tends to a negative value at some $n = n_0 > 0$. $v_n(\phi^2)$ is a single-well potential for $n > n_0$.*

On the other hand, from the general theory of one-dimensional spin systems of short range interactions, we know that the potential $V_n(x)$ tends to the high-temperature fixed point as $n \to \infty$. Even so, the intermediate V_n exhibit complicated behaviors and V_n possesses many local minima if K_0 is large. Moreover the hierarchical approximation and/or WHA type approximation may spoil this simple fact.

Theorem 5 *For any $\lambda_0 > 0$ and a_0, the value μ_n^2 tends to a negative value at some $n > 0$. $v_n(\phi^2)$ is a single-well potential for $n > n_0$.*

But we could not prove this theorem by our explicit estimates of the flow v_n since the intermediate states exhibit vibrations for small ϕ^2, see [9] for a method to avoid this phenomena.

4 $E_2 - E_1$ and Discussions

We could not see how many iterations we need to bring $a_0 < 0$ to $a_n > 0$. If it takes n_0 steps, we then expect that $L^{n_0}(E_2 - E_1) = O(1)$ or $E_2 - E_1 = O(L^{-n_0})$.

Though we cannot say anything conclusive at this level, it seems that we need more than μ_0^2 iterations to drive $a_0 < 0$ to $a_n > 0$ ($n \sim \mu_0^2$) for large μ_0 and λ_0. Thus we expect $E_2 - E_1 = O(L^{-\mu_0^2})$ (or much less).

On the other hand, if we use $O(N)$ invariant ϕ^4 model with very large N [6], then the flow $v_n(\phi)$ is easily controlled and we find $\mu_n^2 \sim -L^{-1} + \mu_0^2/L^n$. This means that it takes $n = \log \mu_0$ steps to bring $a_0 < 0$ to $a_n > 0$. But this is only for $O(N)$ ϕ^4 model with large N and it is quite plausible that N model with $N >> 1$ is different from $N = 1$ model.

Let α_i be the points which minimize the double well potential $V(x)$, $\min V(x) = V(\alpha_1) = V(\alpha_2)$. Consider the spectrum of $H = -(1/2)\Delta + \lambda V(x)$, $\lambda > 0$. Then in [12], it is shown that $(E_2 - E_1)/\lambda \sim \exp[-\rho(\alpha_1, \alpha_2)]$ for large λ, where

$$\begin{aligned}\rho(\alpha_1, \alpha_2) &= \inf_{T,\gamma} \int_0^T \left(\frac{1}{2}|\gamma^\circ(s)|^2 + V((\gamma(s)))\right) ds \\ &= \inf_{\gamma} \int_0^1 \left(\sqrt{2V(\gamma(s))}|\gamma^\circ(s)|\right) ds\end{aligned}$$

$\{\gamma(0) = \alpha_1, \gamma(T) = \alpha_2\}$ in the first equation and $\{\gamma(0) = \alpha_1, \gamma(1) = \alpha_2\}$ in the second equation. See [12]. This is the instanton solution and translated into our language in terms of λ_0 and $a_0 < 0$. It is interesting to make our calculation more precise.

Remark 6 *For $O(N)$ invariant models, the analysis of [12] does not work, and it is reasonable that our result obtained for large N is different from the above analysis.*

So there are several open problems. The first one is to solve recursion relations not only for hierarchical models but also (of course) for the full model [7]. But the latter is certainly difficult.

Therefore it is interesting to solve the non-linear renormalization evolution equation of WHA type [8]:

$$V_t(t,x) = \frac{1}{2\pi t^2}\log(1 + t^2 V_{xx}) \tag{4.1}$$

which is obtained by applying BST to the Schrödinger operator $-\frac{1}{2}\Delta + V(x)$, where $V(x) \to \infty$ as $|x| \to \infty$. Does it really converge to a harmonic oscillator system? To what extent, does the mass term approximate $E_2 - E_1$?

So far, we cannot answer to these questions. We studied if the limit

$$v(x) = V_{eff}(x) = \lim_{t\to\infty} V(t,x) \tag{4.2}$$

exists. Though computer simulations show the answer is positive [1], we could not prove the existence of $v(x)$. We just mention that If $V(x) = m^2x^2/2$, then the equation has a non-trivial solution

$$V(t,x) = \frac{m^2}{2}x^2 + a(t) \tag{4.3a}$$

$$\begin{aligned} a(t) &= \int_0^t \frac{1}{2\pi s^2}\log(1 + m^2 s^2)ds \\ &= -\frac{1}{2\pi t}\log(1 + m^2t^2) + \frac{m}{\pi}\mathrm{Tan}^{-1}mt \end{aligned} \tag{4.3b}$$

which corresponds to the harmonic oscillator.

Acknowledgements

The authors are grateful to K-I. Aoki for explaining the derivation of the non-linear PDE of renormalization groups, and Y.Teramoto and T.Nishida for useful discussions on global solutions of non-linear PDE.

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