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Microlocal \(a\ priori\) estimates and the Cauchy problem

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1. Microlocal \(a\ priori\) estimates.

Microlocal analysis has made a great contribution to the recent development of the theory of partial differential equations, and many authors have proved that microlocal analysis is a powerful and useful tool in studying partial differential equations. In particular, it leads us to studies on standard forms of pseudodifferential operators and clarifies essential parts to be analyzed. However, for general problems one only obtains results modulo \(C^\infty\). To remove "modulo \(C^\infty\)" one must make an effort according to the problems. In the studies of the Cauchy problem one must distinguish the time variable from other variables, and only limited application of microlocal analysis was permitted. Our purpose is to show that microlocal analysis is fairly applicable to the studies of uniqueness and well-posedness of the Cauchy problem. In [7] we showed that microlocal \(a\ priori\) estimates (Carleman type estimates) give theorems on propagation of singularities. Here we shall show that similar microlocal estimates also give theorems on uniqueness and well-posedness of the Cauchy problem. Let

\[ P(x,\xi) = \xi^n + \sum_{|a| \leq m, a_1 < m} a_\alpha(x)\xi^\alpha, \]

where \(a_\alpha(x) \in C^\infty(\mathbb{R}^n)\).

### Microlocal \(a\ priori\) estimate at \(z^0 \in S^*\mathbb{R}^n(\simeq \mathbb{R}^n \times S^{n-1})\) \((0 < A \leq \infty)\)

\[ P(x,\xi;\gamma) \in S^\infty_{20}, \text{ satisfies } (E;\mathbb{R}^n, \{t_\tau(x,\xi)\}_{\tau \in \mathcal{T}}, A) \iff \exists \psi_j(x,\xi) \in C^\infty(\mathbb{R}^n) \]

\[ \exists \psi_j(x,\xi) \in C^\infty(T^*\mathbb{R}^n \setminus \{0\}) \quad (j = 1, 2) \quad \text{ (pos. homo. of deg. 0), } \exists \ell_j \in \mathbb{R} \quad (1 \leq j \leq 3) \quad \text{s.t. } \]

\[ \psi_j(x,\xi) = 1 \quad \text{in a conic nbd of } z^0 \quad (j = 1, 2) \quad \text{and } \forall \tau \in \mathcal{T}, \forall b \in \mathbb{R}, 1 \leq \exists K < A, \exists \gamma_0 \geq 1, \exists C > 0 \text{ satisfying } \]

\[ \| (D)_{\lambda}^\gamma v \| \leq C \{ \| (D)_{\lambda}^\gamma P_A(x, D; \gamma)v \| + \| (D)_{\lambda}^\gamma (1 - \psi_{1,h}(x, D)v) \| \} \]

if \(v \in H^\infty(\mathbb{R}^n), \gamma \geq \gamma_0, h = K \gamma \) and \(A(x,\xi) = (t_\tau(x,\xi) - b) \log(\xi)(1 - \Theta(4|\xi|/h))\) \(\psi_2(x,\xi), \) where \(\Theta(t) \in C^\infty(\mathbb{R}), \Theta(t) = 1\) \((|t| \leq 1), \) \(\supp \Theta \subset (-2,2), \psi_{1,h}(x,\xi) = (1 - \Theta(|\xi|/h))\psi_1(x,\xi), P_A(x, D; \gamma)v = (e^{-A})(x, D)P(x, D; \gamma)(e^A)(x, D)v, \langle \xi \rangle = (\gamma^2 + |\xi|^2)^{1/2} \) and \(\| \cdot \|\) denotes the \(L^2\)-norm.

(P-1) \(P(x,\xi)\) (=the principal part of \(P(x,\xi)\)) is hyperbolic w.r.t. \(\vartheta = (1, 0, \ldots, 0) \in \mathbb{R}^n\).

Put \(P(x, D; \gamma)u = e^{-\gamma\vartheta(x)}P(x, D)(e^{\gamma\vartheta(x)}u)\) for \(\zeta(x) \in B^\infty(\mathbb{R}^n)\).

(P-2) \(\forall z^0 \in S^*\mathbb{R}^n\) with \(z^0 \neq 0 \) and \(dp(z^0) = 0, \forall \theta^0 \in \Gamma(p(z^0, \cdot), \vartheta), \forall k > 0, \exists \delta > 0 \text{ s.t. } \) \(P(z,\xi;\gamma)\) for some \(z(x)\) which is equal to \((z - z^0) \cdot \vartheta^0 + k|x - z^0|^2\) near \(z^0\) (resp. \(P(z, \xi + iy\vartheta)\) satisfies \((E; z^0, \{at_+(z, \xi, z^0)\}_{a \geq \delta_0}, \infty)\) (resp. \((E; z^0, \{at_-(z, \xi, z^0)\}_{a \geq \delta_0}, \infty)\), where \(t_\pm(z, \xi, z^0) = \pm(z_1 - z_1^0) + |z - z^0|^2\)}
+ | \xi/ | | \xi | -\xi^0 |^2 in a conic nbd of \xi^0, \Gamma(p(z, D), \eta) denotes the component of \{\xi \in \mathbb{R}^n; p(z, \xi) \neq 0\} containing \eta, and \trans{P}(z, D) denotes the transposed of \begin{array}{c}
p(z, D)
\end{array}.

Assume that for each \begin{array}{c}
z^0 = (z^0, \xi^0) \in \mathbb{R}^{n-1} \mathbb{R}^n, \end{array} \begin{array}{c}
P(z, D) can be written as \begin{array}{c}
P(z, D) = P_1(z, D) \cdots P_m(z, D) + R(z, D)z^0 in C'((z^0), where C'((z^0), is a conic nbd of \begin{array}{c}
(0, z^0) \in \mathbb{R} \times (T^n \mathbb{R}^{n-1})0, z^0 = (z_1, \ldots, z_n), \nu = \nu(z^0), P^k(z, \xi, z^0) \in \mathbb{R}^{m0}
(\text{ the coef. of } \xi^{m0}=1, R(z, \xi, z^0) \in \mathbb{n, a(\xi) \in S_{1,0}^{m0} \leftrightarrow a(z, \xi) = \sum_{i=0}^{m} a_i(z, \xi, \xi') a_i(z, \xi, \xi') \in S_{1,0}^{m0} \text{ classical). Put } t^\pm(x, \xi', z^0) = \pm z_1 + z^2 + |z' - z^0|^2 + |\xi'/| \xi'| |\xi^0|^2 \text{ in a conic nbd of } (0, z^0).

(P3) \forall z^0 = (z^0, \xi^0) \in \mathbb{R}^{n-1}, 1 \leq \forall k \leq \nu(z^0), \exists P^k(z, \xi, z^0) \in \mathbb{R}^{m0} \text{ ( hyp. w.r.t. } \eta, \text{ the coef. of } \xi^{m0} = 1 \text{ s.t. } (1) P^k(x, \xi, z^0) = P^k(x, \xi, z^0) \text{ ( } z, \xi' \in C'((z^0), \xi^0) \text{ s.t. } \eta^0 \geq 1 \text{. (2) } P^k(x, \xi, z^0) = P^k(z^0) \text{ ( } z \notin \Omega(z^0) \subset \mathbb{R}^n); \text{ a nbd of } (0, z^0)), \text{ where } P^k(x, \xi, z^0) \text{ is pos. homo. for } \xi^0 \geq 1 \text{. (3) } \forall z^1 = (z^1, \xi^1) \in \mathbb{R}^{n-1}, \text{ and } dP^k(z^2) = 0, \exists \eta^0 \in \Omega(z^0), \text{ satisfying } (A) \text{ s.t. } \eta^0 \in \Omega(z^0), \text{ for some } \eta^0 \in \Gamma(z^0) \text{. }

(P4) K_{z^0} \cap \{z_1 \geq 0\} \text{ is bdd for every } z^0 \in \mathbb{R}^n.

**Theorem 1.** Assume that (P1)-(P4) are satisfied. Then \forall f \in D' ( resp. C^\infty) with supp \ f \subset \{z_1 \geq 0\}, \exists u \in D' \text{ ( resp. C^\infty) s.t. } \begin{array}{c}
P(z, D)u = f, \text{ supp } u \subset \{z_1 \geq 0\}.
\end{array}
Moreover, supp \ u \subset \{z \in \mathbb{R}^n; z \in K^+_y \text{ for some } y \in \text{ supp } f\}.

To prove Theorem 1 we first derive local a priori estimates ( Carleman type estimates) from microlocal a priori estimates. This proves uniqueness of the Cauchy problem. This argument can be also applicable to operators which are not necessarily hyperbolic. Next we derive hyperbolic estimates for \trans{P}(z, D) from (P3), which proves existence of solutions to the Cauchy problem. In deriving hyperbolic estimates the condition (2) of (P3) plays an important role. For the detail we refer to [7].

2. The Cauchy problem.

We shall assume that \begin{array}{c}
P(z, D) satisfies at least one of the conditions (A)\_{z^0}, (B)\_{z^0} \text{ and (C)}\_{z^0} \text{ for } z^0 \in \mathbb{R}^{n, \text{ with dp}(z^0) = 0, which will be defined below. Let } z^0 = (z^0, \xi^0) \in \mathbb{R}^n \text{ satisfy dp}(z^0) = 0. \text{ We say that } P(z, D) satisfies the condition (A)\_{z^0} \text{ if (A-1)}\_{z^0} \text{ and (A-2)}\_{z^0} \text{ are fulfilled.}
\end{array}

(A-1)\_{z^0} \exists \text{ conic nbd } \mathcal{C} \text{ of } z^0, \text{ conic nbd } \bar{\mathcal{C}} \text{ of } (y^0, \eta^0), \exists \text{ homo. canonical transf. } \chi: \bar{\mathcal{C}} \otimes \mathcal{C}, \exists \text{ symbols } q(\eta), e(\eta, \eta) \text{ s.t. } z^0 = \chi(y^0, \eta^0), q(\eta): \text{ pos. homo. of deg. } m^t \in \mathbb{N} \text{ and } p(\chi(y, \eta)) = e(\eta, q(\eta), e(\eta, \eta) \neq 0 \text{ for } (y, \eta) \in \bar{\mathcal{C}}.

Let \begin{array}{c}
F_1 \text{ and } F_2 \text{ be classical Fourier integral operators corresponding to } \chi \text{ and } \chi^{-1} \text{ which are elliptic at } (y^0, \eta^0) \text{ and } z^0, \text{ respectively. Under (A-1)}\_{z^0} \text{ we can write}
\begin{array}{c}
s(F_2 P(z, D)F_1)(y, \eta) = \bar{\omega}(y, \eta)(q(\eta) + s(y, \eta))
\end{array}

in a conic nbd $\tilde{C}_0$ of $(y^0, \eta^0)$ if $|\eta| \geq 1$, where $\tilde{e}(y, \eta)$ is a classical symbol, which is elliptic in $\tilde{C}_0$, and $s(y, \eta) \in S^m_{1,0}$ is a classical symbol.

(A-2)$_{s^0}$ \(\exists C_0 > 0, \exists C_\alpha > 0\) for \(\forall \alpha\) s.t. 

$$|s_{(a)}(y, \eta)| \leq C_\alpha |q(\eta; \tilde{\theta})| \quad \text{for} \quad (y, \eta) \in \tilde{C}_0 \text{ with } |\eta| \geq C_0,$$

where $\tilde{\theta} = d\eta_{s^0}(0, \theta), \chi^{-1}(x, \xi) = (y(x, \xi), \eta(x, \xi))$ and $q(\eta; \zeta) = \sum_{|\alpha| \leq m} (-i\zeta)\alpha q^{(\alpha)}(\eta) \times \alpha!$ for $\zeta \in \mathbb{R}^n$.

We note that $q(\eta)$ is microhyperbolic with respect to $\tilde{\theta}$ at $(y^0, \eta^0)$ (see [16]). By the definition of microhyperbolicity we may assume that $|q(\eta; s\tilde{\theta})| \geq cs^m$ for $(y, \eta) \in \tilde{C}_0$ and $0 < s \leq s_0$, where $c > 0$ and $s_0 > 0$. This implies that $\tilde{\theta} \neq 0$ (see, also, Lemma 3.2 in [15]). We say that $P(z, D)$ satisfies the condition (B)$_{s^0}$ if $(B-1)_{s^0} - (B-3)_{s^0}$ below are fulfilled.

(B-1)$_{s^0}$ \(\exists\) conic nbd $C$ of $z^0$, \(\exists\) conic nbd $\tilde{C}$ of $(y^0, \eta^0)$, \(\exists\) homo. canonical transf. $\chi : \tilde{C} \overset{\sim}{\rightarrow} C$, \(\exists\) real-valued funs $t_j(y) \in B^\infty(\mathbb{R}^n)$ ( \(1 \leq j \leq N\)), \(\exists\) real-valued symbols $\lambda(y, \eta')$, $\alpha(y, \eta')$, $e(y, \eta)$ and $\exists C > 0$ s.t. $z^0 = \chi(y^0, \eta^0), \eta^0 = (0, \cdots, 0, 1)$, $d\chi(y^0, \eta^0)(0, \theta), d\chi_{(y^0, \eta^0)}(0, \ell \tilde{\theta}) \in \Gamma(p_{z^0}, (0, \theta))$ ( \(1 \leq j \leq N\)), $\lambda(y, \eta'), \alpha(y, \eta')$: pos. homo. of deg. 1, 2 (resp.), $p(\chi(y, \eta)) = e(y, \eta)(\eta_1(y_1 - \lambda(y, \eta')) - \alpha(y, \eta'))$ and $e(y, \eta) \neq 0$ for $(y, \eta), \lambda(y^0, \eta^0') = \alpha(y^0, \eta^0') = 0$, and $\alpha(y, \eta') \geq 0$ and $T(y)\partial \alpha/\partial y_1(y, \eta') \leq C\alpha(y, \eta')$ for $(y, \eta') \in \tilde{C}'$, where $p_{z^0}(\delta z)$ ( \(\neq 0\) in $\delta z \in \mathbb{R}^{2n}$) is defined as $p(z^0 + s\delta z) = s^{\alpha}p_{z^0}(\delta z) + o(1)$ when $s \rightarrow 0$, $T(y) = \min_{1 \leq j \leq N} |t_j(y)|$ and $\tilde{C}' = \{(y, \eta'); (y, \eta) \in \tilde{C} \text{ for some } \eta_1\}$.

Let $F_1$ and $F_2$ be classical Fourier integral operators corresponding to $\chi$ and $\chi^{-1}$ which are elliptic at $(y^0, \eta^0)$ and $z^0$, respectively. Under (B-1)$_{s^0}$ we can write 

$$\sigma(F_2P(x, D)F_1)(y, \eta) = \tilde{e}(y, \eta)\{\eta_1(\eta_1 - \lambda(y, \eta')) - \alpha(y, \eta') + \beta(y, \eta)\}$$

in a conic nbd $\tilde{C}_0$ of $(y^0, \eta^0)$ if $|\eta| \geq 1$, where $\tilde{e}(y, \eta)$ is a classical symbol, which is elliptic in $\tilde{C}_0$, and $\beta(y, \eta) \in S^m_{1,0}$ is a classical symbol.

(B-2)$_{s^0}$ \(\exists\) classical symbol $A(y, \eta') \in S^m_{1,0}, \exists C > 0$ s.t. 

$$T(y)|\Re \beta(y, 0, \eta') - B(y, \eta')\lambda(y, \eta')| \leq C(\alpha(y, \eta')^{1/2} + 1) \quad \text{for} \quad (y, \eta') \in \tilde{C}_0.'$$

We say that the condition (B-3)$_{s^0}$ is satisfied if at least one of the following conditions (B-3-1)$_{s^0}$ and (B-3-2)$_{s^0}$ is satisfied:

(B-3-1)$_{s^0}$ \(\exists\) classical symbol $B(y, \eta') \in S^m_{1,0}, \exists C > 0$ s.t. 

$$T(y)|\Re \beta(y, 0, \eta') - B(y, \eta')\lambda(y, \eta')| \leq C(\alpha(y, \eta')^{1/2} + 1)$$

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for \((y, \eta') \in \tilde{C}'_0\).

\[ (B-3-2)_{z^0} \quad \Re P_{m-1}(z^0) < \text{Tr}^+ F_p, \]

where \(F_p\) denotes the Hamilton map (fundamental matrix) corresponding to the Hessian of \(p(z)/2\) at \(z = z^0\) and \(\text{Tr}^+ F_p = \sum \lambda_j\), the \(i\lambda_j\) are the eigenvalues of \(F_p\) with positive imaginary parts.

We say that \(P(z, D)\) satisfies the condition \((C)_{z^0}\) if the following conditions \((C-1)_{z^0}\) and \((C-2)_{z^0}\) are fulfilled:

\((C-1)_{z^0} \quad \exists \text{conic nbd } C \text{ of } z^0, \exists \text{conic nbd } \tilde{C} \text{ of } (y^0, \eta^0), \exists \text{homo. canonical transf. } \chi: \tilde{C} \sim C, \exists \text{real-valued funs } t_j(y) \in B^\infty(\mathbb{R}^n) \text{ (} 1 \leq j \leq N\text{), } \exists c > 0 \text{ s.t. } z^0 = \chi(y^0, \eta^0), \exists \text{conic } C > 0 \text{ s.t. } h_{m-1}(x, \xi) \geq c h_{m-2}(x, \xi) T(x, \xi)^2 \]

for \((x, \xi) \in C \text{ with } |\xi| = 1\), where \(h_k(x, \xi) \text{ (} 0 \leq k \leq m\text{)}\) are defined as \(|p(x, \xi - is\theta)|^2 = \sum_{j=0}^{m} e^{2j} h_{m-j}(x, \xi)\) for \((x, \xi) \in T^* \mathbb{R}^n\) and \(s \in \mathbb{R}, T(x, \xi) = \min_{1 \leq j \leq N} |t_j(y(x, \xi))|\) and \(\chi^{-1}(z, \xi) = (y(x, \xi), \eta(x, \xi))\).

\((C-2)_{z^0} \quad \exists C > 0 \text{ s.t. } |P_j(x, \xi)| \leq C h_{2j-m}(x, \xi)^{1/2} \]

for \((x, \xi) \in C \text{ with } |\xi| = 1 \text{ and } m/2 + 1 \leq j \leq m - 1\), where \(P_j(x, \xi) = \sum_{|\alpha| = j} a_\alpha(x) \xi^\alpha \text{ (} 0 \leq j \leq m - 1\text{)}\).

**Theorem 2.** Assume that \((P-1)\) and \((P-4)\) are satisfied and that at least one of the conditions \((A)_{z^0}, (B)_{z^0}\) and \((C)_{z^0}\) is satisfied if \(z^0 = (z^0, \xi^0) \in S^* \mathbb{R}^n, z_1^0 \geq 0\) and \(dp(z^0) = 0\). Then \(\forall f \in D' \text{ (resp. } C^\infty) \text{ with supp } f \subset \{z_1 \geq 0\}, \exists u \in D' \text{ (resp. } C^\infty) \text{ s.t. } P(z, D)u = f\), \(u \subset \{z_1 \geq 0\}\). Moreover, \(u \subset \{z \in \mathbb{R}^n; z \in K_1^+ \text{ for } \exists y \in \text{supp } f\}\).

Theorem 2 can be proved, applying Theorem 1 (or an improved version) (see Theorem 5.2 in [8]). If one finds other conditions to give microlocal a priori estimates at \(z^0 \in S^* \mathbb{R}^n\), one can add these conditions to \((A)_{z^0}, (B)_{z^0}\) and \((C)_{z^0}\) and improve Theorem 2. The condition \((A)_{z^0}\) is satisfied if the characteristic roots of \(p(z, \xi) = 0\) are involutive and the lower order terms of \(P(z, D)\) satisfy the so-called Levi conditions, which were studied by Zeman [17] (see, also, [1], [10]). The condition \((B)_{z^0}\) implies that \(p(z, \xi)\) has double characteristics at \(z^0\). If \(P(z, D)\) is effectively hyperbolic, then \(P(z, D)\) satisfies both the conditions \((B)_{z^0}\) and \((C)_{z^0}\). \(C^\infty\) well-posedness of the Cauchy problem for effectively hyperbolic operators was proved by Iwasaki [6] (see, also, [3], [5], [11], [12], [14]). The operators, which was treated by Ivrii [4], satisfy the condition \((B)_{z^0}\) with \(T(y) \equiv 1\) (see, also, [2]). We can treat both effectively hyperbolic cases and non effectively hyperbolic cases at the same time by introducing \(T(y)\) in \((B)_{z^0}\). The condition \((C)_{z^0}\) is closely related to the conditions given in [13]. Some results on propagation of singularities were essentially obtained in [15] if the operators are microhyperbolic at \(z^0\) and satisfy \((A)_{z^0}\). For microhyperbolic operators satisfying \((B)_{z^0}\) we obtained theorems on propagation of singularities in [13]. Applying the methods
developed in [7], we can also obtain theorems on propagation of singularities for microhyperbolic operators which satisfy at least one of the conditions \((A)_{x^0}, (B)_{x^0}\) and \((C)_{x^0}\). For the proof of Theorem 2 and the detail we refer to [9].

References