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<th>ON WELL-BEHAVED C$^\ast$-ALGEBRAS RELATED TO ORDERS</th>
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ON WELL-BEHAVED C*-ALGEBRAS RELATED TO ORDERS

Kazuyuki SAITO

One of the outstanding problems in the theory of AW*-algebras is the monotone completeness of any AW*-algebras. For AW*-algebras of Type I, the answer is known to be yes (see Kaplansky [3]) but, for general AW*-algebras, this question is still open, although an impressive attack on this problem was made by Christensen and Pedersen [1].

In this note, we should like to make a survey of the development of the problem of monotonicity of AW*-algebras, with an outline of their proofs. This is a joint work with John D.M. Wright [6].

Let us recall that a C*-algebra $A$ is an AW*-algebra if (1) each maximal abelian *-subalgebra of $A$ is generated by its projections and (2) each orthogonal family of projections $\{e_\alpha\}$ in $A$ has a supremum $\Sigma A^e_\alpha$ in Proj($A$) (the complete lattice of all projections in $A$).

A natural line of attack on this problem would be to use the second dual $A''$ or the weak closures of representatives of $A$ on some Hilbert spaces.

Unfortunately, this is too naive. In general, the structure of the complete lattice Proj($A$) is not consistent with that of Proj($A''$) or that of the weak closures, because of the lack of the weak or strong topologies. In fact, if so, $A$ would be a von Neumann algebra.
Let $B$ be an $AW^*$-algebra and let $C$ be a unital $C^*$-subalgebra of $B$. We say that $C$ is normal in $B$ if for every orthogonal family $\{ e_\alpha \}$ in $\text{Proj}(C)$ with the supremum $\Sigma C e_\alpha$ in $\text{Proj}(C)$, $\Sigma C e_\alpha = \Sigma B e_\alpha$.

Let $A$ be an $AW^*$-algebra. Then $A$ sits inside its regular completion $\hat{A}$ [2]. $\hat{A}$ is a monotone complete $C^*$-algebra (and so an $AW^*$-algebra) which is, in general, not a von Neumann algebra. We say that $A$ is normal if $A$ is normal in $\hat{A}$. So our first question is this:

Are all $AW^*$-algebras normal?

It has been known for ten years that finite $AW^*$-algebras are normal [7], [2] and [4]. So, when establishing normality, we may confine our attention to properly infinite $AW^*$-algebras.

Quite recently, we showed that mild restrictions on the centre of an $AW^*$-algebra are sufficient to force it to be normal. In particular, all $AW^*$-factors are normal.

Let $A$ be an $AW^*$-algebra whose centre is locally countably decomposable. Then $A$ must be normal.

Detailed proof will appear in the Journal of the London Mathematical Society (see [6]).

It is a pleasure to thank Professor John D.M. Wright and the staff of the Mathematics Department of the University
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Let $B$ be a $C^*$-algebra. A net of increasing projections in $\text{Proj}(B)$, $\{ e_j \}_{j \in J}$, with the supremum $\text{LUB}_{\text{Proj}(B)} e_j$ in $\text{Proj}(B)$ is said to be well-behaved if $\text{LUB}_{\text{Proj}(B)} e_j$ is the supremum of $\{ e_j \}_{j \in J}$ in the partially ordered set $B$. $B$ is said to be well-behaved if, every such net $\{ e_j \}_{j \in J}$ is well-behaved.

Let us begin with the following lemma which plays an important role in proving the theorem.

Lemma 1. ([4]) Let $A$ be an $AW^*$-algebra. Then the following three conditions are equivalent.

1. $A$ is normal;

2. $A$ is well-behaved;

3. for every increasing net $\{ e_j \}_{j \in J}$ in $\text{Proj}(A)$ with the supremum $e$ in $\text{Proj}(A)$, whenever $x$, in $A$, satisfies $e_j x e_j \geq 0$ for all $j$, then $xe e \geq 0$.
Remark. (2) and (3) are equivalent even when \( A \) is a general unital C*-algebra.

Outline of the proof (see [4]).

(1) \( \Leftrightarrow \) (2). Suppose that \( A \) is normal, then, by a result of Pedersen and Saitô, for every increasing net \( \{ e_j \}_{j \in J} \) in \( \text{Proj}(A) \),

\[
\text{LUB}_{\text{Proj}(A)} e_j = \text{LUB}_{\hat{\text{Proj}}(\hat{A})} e_j.
\]

Since \( \hat{A} \) is monotone complete and \( \{ e_j \}_{j \in J} \) is well-behaved in \( \hat{A} \), and so it is well-behaved in \( A \) as well.

Conversely, suppose that \( A \) is well-behaved. Then, for every orthogonal family \( \{ e_j \}_{j \in J} \) in \( \text{Proj}(A) \), the net \( \{ \sum_{j \in F} e_j \mid F \text{ a non-empty finite subset of } J \} \) is well-behaved. So it follows that

\[
\sum_{A} e_j = \text{LUB}_{\text{Proj}(A)} \{ \sum_{j \in F} e_j \mid F \}
= \text{LUB}_{\hat{\text{Proj}}(\hat{A})} \{ \sum_{j \in F} e_j \mid F \}
= \sum_{\hat{A}} e_j.
\]

(2) \( \Leftrightarrow \) (3). It is given that \( \{ e_j \}_{j \in J} \) is an increasing net in \( \text{Proj}(A) \) with the supremum \( \text{LUB}_{\text{Proj}(A)} e_j \) (= e say). Suppose that \( \{ e_j \}_{j \in J} \) satisfies (3).

The claim is that \( \text{LUB}_{\text{Proj}(A)} e_j = \text{LUB}_{A_h} e_j \). We have only to check that \( e_j \leq a \) for all \( j \) for some \( a \in A_h \) implies
\[ e \leq a. \quad \text{Suppose that such an } a \text{ is given as above, then, because } a \geq 0, \text{ it follows that} \]
\[ (a + 1/n)^{-1/2} e_j (a + 1/n)^{-1/2} \leq 1 \]

for each \( j \) and \( n \). Thus we get that
\[ \| (a + 1/n)^{-1/2} e_j \| \leq 1 \]

for each \( j \) and \( n \). This implies that
\[ e_j (e - e(a + 1/n)^{-1} e)e_j \geq 0 \]

for all \( j \) and \( n \). Since \( \{ e_j \}_{j \in J} \) satisfies (3), it follows that
\[ e(e - e(a + 1/n)^{-1} e)e \geq 0 \]

and \( \| e(a + 1/n)^{-1} e \| \leq 1 \) for all \( n \). Thus we conclude that
\[ (a + 1/n)^{-1/2} e(a + 1/n)^{-1/2} \leq 1 \]

for all \( n \). This implies that \( e \leq a + 1/n \) for all \( n \) and so \( e \leq a \) follows.

Conversely suppose that \( \{ e_j \}_{j \in J} \) is well-behaved.
It is given \( x \in A_h \) such that \( e_j x e_j \geq 0 \) for all \( j \). To prove the claim, we may assume that \( e = 1 \) (consider it in \( eAe \)) and \( \| x \| \leq 1 \). Since
\[ (1 + x)(1 - e_j)(1 + x) - (1 - x)(1 - e_j)(1 - x) \]
\[ = 2x(1 - e_j) + 2(1 - e_j)x, \]

\[ - 5 - \]
we see that
\[ e_j x e_j - x = (1 - e_j)x(1 - e_j) - (1 - e_j)x - x(1 - e_j) \]
\[ = (1/2)((1 - x)(1 - e_j)(1 - x) - (1 + x)(1 - e_j)(1 + x)) + (1 - e_j)x(1 - e_j) \]
\[ \leq (1/2)(1 - x)(1 - e_j)(1 - x) + 1 - e_j \]
because \( \|x\| \leq 1 \) and \( x = x^* \). Take \( y = (1/2)(|x| + x) \) and \( z = (1/2)(|x| - x) \). We see that \( x = y - z, y, z \) in \( A_h \), \( yz = 0 \) and \( z \) and \( y \) are non-negative. Moreover, \( y \) and \( z \) commute with \( x \). Hence, it follows that
\[ z e_j x e_j z - zz x \leq (1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z. \]
Since \( z e_j x e_j z \) is non-negative for all \( j \) and \( zz x = -z^3 \), we see that
\[ z^3 \leq (1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z \]
for all \( j \). Since \( \{ e_j \}_{j \in J} \) is well-behaved, this implies that
\[ (1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z = 0 \] in \( A_h \),
and so \( z^3 = 0 \), that is, \( z = 0 \). This completes the proof.
Theorem 1. **Finite AW*-algebras are normal.**

Let \( \{ e_j \}_{j \in J} \) be any increasing net in \( \text{Proj}(A) \) with the supremum \( e \) in \( \text{Proj}(A) \). We shall show that \( \{ e_j \}_{j \in J} \) satisfies (3). To do this, we may assume that \( e = 1 \).

Suppose that \( x \) in \( A \) satisfies that \( e_j x e_j \geq 0 \) for all \( j \).

If \( x = x^+ - x^- \) (\( x^+ x^- = 0 \), \( x^+ \geq 0 \) and \( x^- \geq 0 \)) and \( x^- \neq 0 \),

then there is a non-zero projection \( q \) in \( A \) and a positive number \( \epsilon \) such that \( x^- \geq \epsilon q \) and \( (1 - q)x^+ = x^+ \). Set \( f_j = e_j^\wedge q \)

and we have

\[
0 \leq f_j e_j x e_j f_j = f_j x f_j = f_j qx q f_j = - f_j x^- q f_j \\
\leq -\epsilon f_j q f_j \leq -\epsilon f_j
\]

for all \( j \) and so \( f_j = 0 \) for all \( j \), that is, \( e_j^\wedge q = 0 \) for all \( j \).

Note that

\[
q = q - e_j^\wedge q \vee e_j^\vee q - e_j \leq 1 - e_j
\]

for all \( j \) and \( A \) is finite, this implies that \( q = 0 \), because \( 1 - e_j \neq 0 \) in \( \text{Proj}(A) \). This is a contradiction. Thus \( x^- = 0 \), that is \( x \geq 0 \). This completes the proof.

Now we are in the position to discuss about the properly infinite case. Since, as you see, above proof depends on the finiteness assumption on \( A \), we need to seek another way to establish the normality for properly infinite case.

Before going into the discussions, we need some definitions.

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For a given index set $\beta$, a family $\{ x_j \in A_h \mid j \in \beta \}$ is said to be **order summable** if

$$\{ \sum_{j \in F} x_j \mid F \text{ a non-empty finite subset of } \beta \}$$

is bounded above in $A_h$. When such a family is order summable its **order sum** is defined to be the supremum of the set

$$\{ \| \sum_{j \in F} x_j \| \mid F \text{ a non-empty finite subset of } \beta \}.$$

For a given indexed set $\beta$, a family $\{ x_j \in A_h \mid j \in \beta \}$ is said to be **well-behaved** if the set

$$\{ \sum_{j \in F} x_j \mid F \text{ a non-empty finite subset of } \beta \}$$

has a supremum in $A_h$. When $\{ x_j \mid j \in \beta \}$ is a family of orthogonal projections, this definition of well-behaved is consistent with our earlier one.

Let $\beta$ be a given index set. The algebra $A$ is said to be **$\beta$-complete** if each order summable, $\beta$-indexed family of positive elements in $A$ is well-behaved.

It is clear that if $A$ is $\beta$-complete and if $\gamma$ is a set where $\# \gamma \leq \# \beta$, then $A$ is $\gamma$-complete. We note further that $A$ is $\alpha$-complete for a sufficiently large ordinal $\alpha$, then $A$ is monotone complete, but we omit the details.

The rest of the discussions, we shall suppose that $A$ is a properly infinite $AW^*$-algebra.

We suppose for the moment that, for some infinite ordinal $\Omega$, $A$ has a system of matrix units $\{ e_{ij} \}_{0 \leq i,j < \Omega}$ where
$e_{00} \sim l$ in $A$.

A transfinite sequence $\{ a_j \}_{j < \alpha}$ in $e_{00}Ae_{00}$ is said to be dilatable in $A$ if there exists an orthogonal family of projections $\{ p_j \}_{j < \alpha}$ in $(\sum_{j < \alpha} e_{j j})A(\sum_{j < \alpha} e_{j j})$ such that $e_{00}p_j e_{00} = x_j$ for each $j < \alpha$. Let $\alpha$ be an ordinal number. We call $e_{00}Ae_{00}$ $\alpha$-dilatable if, whenever $\{ x_j \}_{j < \alpha}$ is an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$, with order sum less than 1, then the transfinite sequence is dilatable in $A$.

The following lemma is a modification of an ingenious argument by Christensen and Pedersen [1].

**Lemma 3.** Let $\alpha$ be an ordinal with $\alpha \leq \Omega$. Let $e_{00}Ae_{00}$ be $\alpha$-dilatable in $A$. Then $A$ is $\alpha$-complete.

Since $e_{00}Ae_{00}$ is $\ast$-isomorphic to $A$, it suffices to show that $e_{00}Ae_{00}$ is $\alpha$-complete. The proof is rather long. We shall omit the details. See [6].

We shall need the following lemma which is proved in [1, Lemma 3].

**Lemma 4.** Let $e$ and $p$ be projections in a unital $C^\ast$-algebra $B$ such that $\|epe\| < 1$ and let $x$ be a positive element of $B$ such that $x + epe \leq e$. Let $\{ f_{ij} \}_{1 \leq i, j \leq 2}$ be matrix units.
for $M_2(C)$. Then there exists a projection $q$ in $B\otimes M_2(C)$ such that $q$ is orthogonal to $p \otimes f_{11}$ and

$$(e \otimes f_{11})q(e \otimes f_{11}) = x \otimes f_{11}.$$ 

Let $z = (1 - p)(1 - epe)^{-1}x(1 - epe)^{-1}(1 - p)$. Then $z \in A_h$ such that $(1 - p)z(1 - p) = z$, $eze = x$ and $0 \leq z \leq 1$. Let

$$q = \begin{bmatrix} z & (z - z^2)^{1/2} \\ (z - z^2)^{1/2} & 1 - z \end{bmatrix}$$

via $\{ f_{ij} \}_{1 \leq i, j \leq 2}$. Then $q$ satisfies all the requirements.

Lemma 5. Let $\alpha < \Omega$ and let $\alpha + 1$ be the successor ordinal of $\alpha$. Let $e_{00}Ae_{00}$ be $\alpha$-dilatable in $A$. Then $e_{00}Ae_{00}$ is also $(\alpha + 1)$-dilatable.

In fact, let $\{ x_\xi \}_{\xi < \alpha + 1}$ be an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$. Let its order sum be $c$, where $c < 1$. By hypothesis, there exists a family of orthogonal projections $\{ p_\xi \}_{\xi < \alpha}$ in $(\Sigma_{i < \alpha} e_{i1})A(\Sigma_{i < \alpha} e_{i1})$ such that $e_{00}p_\xi e_{00} = x_\xi$ for each $\xi < \alpha$. By Lemma 3, $A$ is $\alpha$-complete and so

$$\Sigma_{i < \alpha} p_i = \text{LUB}_A \{ \Sigma_{i \in F} p_i \mid F \text{ a non-empty finite subset of } \alpha \}.$$ 

Thus
\[ e_{00}(\Sigma_{i<\alpha}a^i p_{ii})e_{00} = \text{LUB}_A \{ \Sigma_{i \in F} x_i \mid F \text{ a finite subset of } \alpha \} \]

and so \( \| e_{00}(\Sigma_{i<\alpha}a^i p_{ii})e_{00} \| \leq c < 1 \). Also

\[ e_{00}(\Sigma_{i<\alpha}a^i p_{ii})e_{00} + x_\alpha \leq ce_{00} < e_{00}. \]

We observe that \( \Sigma_{i<\alpha}a^i e_{ii} \triangleright e_{\alpha} = \Sigma_{i \leq \alpha}a^i e_{ii} \). Let \( f_{11} = \Sigma_{i \leq \alpha}a^i e_{ii} \), let \( f_{22} = e_{\alpha} \) and let \( f_{12} \) be any partial isometry in \( (\Sigma_{i \leq \alpha}a^i e_{ii})A(\Sigma_{i \leq \alpha}a^i e_{ii}) \) such that \( f_{12}^* f_{12} = f_{11} \) and \( f_{12}^* f_{12} = f_{22} \).

Let \( f_{21} = f_{12}^* \). Then, by the above lemma, there is a projection \( p_\alpha \) in \( (\Sigma_{i \leq \alpha}a^i e_{ii})A(\Sigma_{i \leq \alpha}a^i e_{ii}) \), such that \( p_\alpha \) is orthogonal to \( \Sigma_{i<\alpha}a^i p_{ii} \) and \( x_\alpha = e_{00} p_\alpha e_{00} \). Hence \( \{ x_\xi \mid \xi \leq \alpha \} = \{ x_\xi \mid \xi < \alpha + 1 \} \) is dilatable.

**Lemma 6.** Let \( \alpha \) be an infinite ordinal such that \( \alpha \leq \Omega \). Let \( A \) be \( \xi \)-complete for each \( \xi < \alpha \). Then \( e_{00} a e_{00} \) is \( \alpha \)-dilatable in \( A \).

In fact, let \( \{ x_i \}_{i<\alpha} \) be an order summable transfinite sequence of positive elements of \( e_{00} a e_{00} \) with order sum \( c \), where \( c < 1 \). To obtain a contradiction, let us assume that this transfinite sequence is not dilatable. Then, there exists a smallest ordinal \( \beta \), such that \( \{ x_i \}_{i<\beta} \) is not dilatable, and \( \beta \leq \alpha \). (Note that \( \omega < \beta \) by the results of Christensen and Pedersen [1].)

Let \( \gamma \) be any non-zero ordinal strictly less than \( \beta \). From the definition of \( \beta \), \( \{ x_i \}_{i<\gamma} \) is dilatable. So, there
is a family \( \{ p_i \}_{i < \gamma} \) of orthogonal projections in
\[ \left( \Sigma_{i < \gamma} e_{ii} \right) A \left( \Sigma_{i < \gamma} e_{ii} \right) \] such that \( x_i = e_{00} p_i e_{00} \) for each \( i < \gamma \). Let \( M_\gamma \) be the set of all such families of orthogonal projections and let \( M = \cup \{ M_\gamma \mid 0 < \gamma < \beta \} \). Clearly \( M \neq \emptyset \). For \( \Gamma = \{ p_i \}_{i < \gamma_1} \in M \) and \( \Xi = \{ q_j \}_{j < \gamma_2} \in M \), we define \( \Gamma \leq \Xi \) to mean that \( \gamma_1 \leq \gamma_2 \) and, for all \( i < \gamma_1 \), \( p_i = q_i \). This partially orders \( M \) inductively. So, by Zorn's lemma, \( M \) has a maximal element \( \{ p_i \}_{i < \zeta} \). Then, by applying the argument of Lemma 5 to \( \{ x_\xi \}_{\xi < \zeta + 1} \), we find a projection \( p_\zeta \) in \( \left( \Sigma_{i \leq \xi} e_{ii} \right) A \left( \Sigma_{i \leq \xi} e_{ii} \right) \) such that \( p_\zeta \) is orthogonal to \( \Sigma_{i < \zeta} p_i \) and \( e_{00} p_\zeta e_{00} = x_\zeta \). Thus \( \{ p_i \}_{i < \zeta + 1} \) is in \( M \). This contradicts maximality. Hence the assumption that \( \{ x_i \}_{i < \alpha} \) was not dilatable must be false. Hence \( e_{00} A e_{00} \) is \( \alpha \)-dilatable in \( A \).

By using these lemmas, we have the following:

**Theorem 2.** Let \( A \) be a properly infinite AW*-algebra. Let \( \Omega \) be an infinite ordinal such that there exists an \( \Omega \)-indexed system of matrix units in \( A \), \( \{ e_{ij} \}_{0 \leq i, j < \Omega} \). Then \( A \) is \( \Omega \)-complete.

In fact, assume that \( A \) is not \( \Omega \)-complete. Then there is a first ordinal \( \beta, \beta \leq \Omega \), such that \( A \) is not \( \beta \)-complete. So, for \( \alpha < \beta \), \( A \) is \( \alpha \)-complete. So, by the above lemma,
\( e_{00}^* A e_{00} \) is \( \beta \)-dilatable in \( A \). Then, by Lemma 3, \( A \) is \( \beta \)-complete. This is a contradiction. So \( A \) must be \( \Omega \)-complete.

**Corollary 1([1]).** Let \( A \) be a properly infinite AW*-algebra. Then \( A \) is monotone \( \sigma \)-complete.

Since \( A \) has a countable system of matrix units, \( A \) is \( \omega \)-complete. So, \( A \) is monotone \( \sigma \)-complete.

Now we are in the position to discuss about normality in properly infinite AW*-algebras.

**Theorem 3.** Let \( A \) be a properly infinite AW*-algebra whose centre, \( Z \), is locally countably decomposable. Then \( A \) is normal.

**Outline of the proof.** (See [6].)

Let \( A \) be an infinite AW*-factor. Let \( \Pi \) be an infinite set of orthogonal projections in \( A \). Let \( \aleph \) be the cardinality of \( \Pi \). Then \( A \) is \( \aleph \)-complete. Since \( A \) is monotone \( \sigma \)-complete, there is nothing further to prove if \( \Pi \) is countable. So let us suppose \( \Pi \) to be uncountably infinite.

We may decompose \( \Pi \) into a family of disjoint set \( \{ \Pi_\lambda \mid \lambda \in \Lambda \} \), where each \( \Pi_\lambda \) is of the same cardinality as \( \Pi \),
and where \( \#\Pi = \#\Lambda \). Let \( p_\lambda = \Sigma \Pi_\lambda \). Then \( \{ p_\lambda \mid \lambda \in \Lambda \} \) is an orthogonal family of non-zero projections. We shall show that each \( p_i \) is infinite. Suppose that \( p_i \) is finite for some \( i \). Then \( p_i A p_i \) is a finite AW*-factor and so it is \( \sigma \)-finite. Since \( \Pi_i \) is uncountable. This is a contradiction. So all \( p_i \) are infinite. Since \( A \) is infinite, there exists a minimal infinite projection \( e_0 \) in \( A \) such that \( e_0 \not\preceq p_i \) for all \( i \in \Lambda \). So there is a set \( \Pi' \) of mutually orthogonal family of projections \( \{ e_\lambda \mid \lambda \in \Lambda \} \) such that \( e_\lambda \sim e_0 \) for all \( \lambda \in \Lambda \). By Zorn's lemma, \( \Pi' \) can be extended to a maximal collection \( \Gamma \) of mutually orthogonal infinite projections, each of which is equivalent to \( e_0 \). Clearly \( \#\Gamma \geq \aleph_0 \). By a general theory of AW*-algebras, one can find a \( \#\Gamma \)-homogeneous partition of \( 1 \) in \( A \). Hence we can construct \( \#\Gamma \)-system of matrix units \( \{ e_{ij} \mid ij \in \Gamma \} \) in \( A \) such that \( e_{ii} \sim 1 \) for all \( i \). Since \( \#\Lambda \leq \#\Gamma \), \( \Lambda \) must be \( \#\Lambda \)-complete, and so \( \Pi \) is well-behaved. For the general case, see [6].

References


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