Selberg trace formula and Jacobi forms

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1 Introduction

In this note we present a calculation of the traces of Hecke operators acting on the spaces of Jacobi forms via the general Selberg trace formula. We can represent those traces in a closed form with the use of some arithmetic quantities and the residues at poles of certain Selberg type zeta functions. The calculation of those traces has been done by Skoruppa-Zagier ([S-Z1, 2]) in some cases in a different manner. They have employed the Bergman kernel functions for the spaces of Jacobi forms and also some results of Shimura [Sh] concerning modular forms of half integral weight. Here we use the general Selberg trace formula due originally to Selberg [Se] and to Hejhal [He], Fischer [Fi]. For our calculation we exclusively follow Fischer [Fi].

In this short survey we exhibit only the results which is a generalization of our previous work [Ar] and we shall give a proof in another occasion in details.

2 Jacobi forms and Hecke operators

We use the symbol $e(\alpha)$ as an abbreviation of $\exp(2\pi i \alpha)$. Let $l$ be a positive integer. Let $G_Q^J$ be the Jacobi group defined over $\mathbb{Q}$:

$$G_Q^J = \{(g, (\lambda, \mu), \rho) \mid g \in \mathbb{Q}^l, \lambda, \mu \in \mathbb{Q}, \rho \in \mathrm{Sym}_l(\mathbb{Q})\},$$

where $\mathbb{Q}^l$ (resp. $\mathrm{Sym}_l(\mathbb{Q})$) denotes the space of rational column vectors (resp. rational symmetric matrices) of size $l$. The composition law of $G_Q^J$ is given by

$$g_1 g_2 = (M_1 M_2, (\lambda_1, \mu_1) M_2 + (\lambda_2, \mu_2), \rho_1 + \rho_2 - \mu_1 \lambda_1 + \mu^* \lambda^* + \lambda^* \mu_2 + \mu_2 \lambda^*)$$

$$g_j = (M_j, (\lambda_j, \mu_j), \rho_j) \in G_Q^J, j = 1, 2)$$

with $(\lambda^*, \mu^*) = (\lambda_1, \mu_1) M_2$. Denote by $G_{\mathbb{R}}^J$ the group of real points of $G_Q^J$. Denote by $\mathcal{D}$ the product of the upper half plane $\mathfrak{H}$ and $\mathbb{C}^l$, the space of complex column vectors of size $l$: $\mathcal{D} = \mathfrak{H} \times \mathbb{C}^l$. The Jacobi group $G_Q^J$ acts on $\mathcal{D}$ in the following manner:

$$g(\tau, z) = \left( M \tau, \frac{z + \lambda \tau + \mu}{J(M, \tau)} \right)$$
$(g = (M, (\lambda, \mu), \rho) \in G_{\bullet}^{J}, (\tau, z) \in \mathcal{D})$

where $J(M, \tau) = c\tau + d$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $S$ be a positive definite half integral symmetric matrix of size $l$. We define a factor of automorphy $J_{k,S}(g, (\tau, z))$ associated to $S$ and a half integer $k$ by

\[ J_{k,S}(g, (\tau, z)) = J(M, \tau)^k e\left(-\text{tr}(S\rho) - \tau S[\lambda] - 2S(\lambda, z) + \frac{c}{J(M, \tau)}S[z + \lambda\tau + \mu]\right) \]

where the branch of $J(M, \tau)^k = \exp(k \log J(M, \tau))$ is chosen so that $-\pi < \arg J(M, \tau) \leq \pi$.

Let $\Gamma$ be a congruence subgroup of $SL_2(\mathbb{Z})$ having the element $-1_2$ and $\Gamma^{J}$ the subgroup of $G_\bullet^{J}$ given by

$\Gamma^{J} = \{(M, (\lambda, \mu), \rho) | M \in \Gamma, \lambda, \mu \in \mathbb{Z}^l, \rho \in Sym_1(\mathbb{Z})\}$

where $\mathbb{Z}^l$ (resp. $Sym_1(\mathbb{Z})$) denotes the $\mathbb{Z}$-lattice consisting of integral column vectors (resp. integral symmetric matrices) in $\mathbb{Q}^l$ (resp. $Sym_1(\mathbb{Q})$). For any function $\phi : \mathcal{D} \rightarrow \mathbb{C}$ and $g = (M, (\lambda, \mu), \rho) \in G_{\bullet}^{J}$, we set

\[ (\phi|_{k,S}g)(\tau, z) = J_{k,S}(g, (\tau, z))^{-1}\phi(g(\tau, z)), \]

\[ (\phi|_{k,S}^{*}g)(\tau, z) = J_{0,S}(g, (\tau, z))^{-1}(J(M, \tau))^{-k+l}|J(M, \tau)|^{-1}\phi(g(\tau, z)). \]

In the definition of the latter $(\phi|_{k,S}^{*}g)$, we may assume that $k$ is an integer, since only such cases can occur in the discussion later on. If $k$ is an integer, then these operations satisfy

\[ \phi|_{k,S}g_1g_2 = \phi|_{k,S}g_1|_{k,S}g_2 \]

and

\[ \phi|_{k,S}^{*}g_1g_2 = \phi|_{k,S}^{*}g_1|_{k,S}^{*}g_2. \]

Note that $\tilde{\mathfrak{H}} \cup \{\infty\} \cup \mathbb{Q}$ is the total set of cusps of $\Gamma$. For each element $M$ of $\Gamma$, put $M \infty = \zeta$. Denote by $\Gamma_\zeta$ the stabilizer group of $\zeta$ in $\Gamma$: $\Gamma_\zeta = \{\sigma \in \Gamma | \sigma \zeta = \zeta\}$. There exists a unique positive integer $N$ such that the group $M^{-1}\Gamma_\zeta M$ of $SL_2(\mathbb{Z})$ is generated by $-1_2$ and \( \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \). Let $k$ be a positive integer. Now we define the space $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^{*}(\Gamma)$) of holomorphic (resp. skew-holomorphic) Jacobi forms of index $S$ and weight $k$ with respect to $\Gamma^{J}$. We define $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^{*}(\Gamma)$) to be the space consisting of all functions $\phi : \mathcal{D} \rightarrow \mathbb{C}$ which satisfy the following three conditions:
(i) $\phi(\tau, z)$ is holomorphic in $\tau$ and $z$

   (resp. $\phi(\tau, z)$ is a smooth function in $\tau$ and holomorphic in $z$)

(ii) $\phi(\tau, z)$ satisfies the identity

   $\phi|_{k, S} \gamma = \phi$  (resp. $\phi^*|_{k, S} \gamma = \phi$)  \quad \text{for} \quad \forall \gamma \in \Gamma^J$

(iii) The function $\phi|_{k, S} M$ (resp. $\phi^*|_{k, S} M$) for any $M \in SL_2(\mathbb{Z})$ has a Fourier Jacobi expansion of the form

   \[
   (\phi|_{k, S} M)(\tau, z) = \sum_{n \in \mathbb{Z}, r \in \mathbb{Z}^J} c(n, r) e \left( \frac{nr}{N} + 4nrS^{-1}r \right)
   \]

   \[
   \left( \text{resp.} \quad (\phi|_{k, S}^* M)(\tau, z) = \sum_{n \in \mathbb{Z}, r \in \mathbb{Z}^J} c(n, r) e \left( \frac{nr}{N} + \frac{i\eta}{2}(rS^{-1}r) + 4nrS^{-1}r \right) \right),
   \]

where $\eta = \text{Im} \tau$ and a positive integer $N$ is chosen for each $M$ in the above manner.

In the above (iii), $M \in SL_2(\mathbb{Z})$ is identified with the element $(M, (0,0), 0)$ in $G_{\mathbb{Q}}^J$.

Denote by $J^\text{cusp}_{k, S}(\Gamma)$ (resp. $J^\text{skew}_{k, S}(\Gamma)$) the subspace of cusp forms of $J_{k, S}(\Gamma)$ (resp. $J^*_k(\Gamma)$) consisting of all Jacobi forms $\phi \in J_{k, S}(\Gamma)$ (resp. all skew-holomorphic Jacobi forms $\phi \in J^*_k(\Gamma)$) whose Fourier coefficients $c(n, r)$ in the above (iii) equals zero if $4n - N^t r S^{-1} r = 0$.

Let $\Delta \subseteq G_{\mathbb{Q}}^J$ be a finite union of double cosets with respect to $\Gamma^J$: $\Delta = \bigcup_j \Gamma^J \sigma_j \Gamma^J$ ($\sigma_j \in G_{\mathbb{Q}}^J$). Following Skoruppa-Zagier [S-Z2], we define an operator $H_{k, S, \Gamma}(\Delta)$ (resp. $H^\text{skew}_{k, S, \Gamma}(\Delta)$) acting on $J_{k, S}(\Gamma)$ (resp. $J^*_k(\Gamma)$) by

\[
\phi|H_{k, S, \Gamma}(\Delta) = \sum_{\xi \in \Gamma^J \setminus \Delta} \phi|_{k, S} \xi
\]

\[
\left( \text{resp.} \quad \phi|H^\text{skew}_{k, S, \Gamma}(\Delta) = \sum_{\xi \in \Gamma^J \setminus \Delta} \phi^*|_{k, S} \phi \right),
\]

where the summation is taken over a complete set of representatives $\xi$ for the left $\Gamma^J$-cosets of $\Delta$. The operator $H_{k, S, \Gamma}(\Delta)$ (resp. $H^\text{skew}_{k, S, \Gamma}(\Delta)$) is well-defined and maps $J_{k, S}(\Gamma)$ (resp. $J^*_k(\Gamma)$) to $J_{k, S}(\Gamma)$ (resp. $J^*_k(\Gamma)$) and cusp forms to cusp forms (see Proposition 1.1 of [S-Z2]). For $L$-functions associated with common eigen Jacobi forms in this situation we refer the reader to Sugano [Su].
3 An operator acting on the space of theta series

Let $S$ be a positive definite half-integral symmetric matrix of size $l$ as before and $R_S$ denote the $\mathbb{Z}$-module $(2S)^{-1}/\mathbb{Z}^l$. Set

$$d = \det(2S) = \#(R_S).$$

We write, for simplicity,

$$S(u, v) = \iota_{uSv}$$

and

$$S[u] = e_{uSu}$$

for $u, v \in C^\iota$.

Denote by $V = C^d$ the $C$-vector space consisting of column vectors $(x_r)_{r \in R_S}$ $(x_r \in C)$. Let $<x, y>_S$ be the positive definite hermitian scalar product given by

$$<x, y>_S = \sum_{r \in R_S} x_r \overline{y_r} \quad (x = (x_r)_{r \in R_S}, y = (y_r)_{r \in R_S} \in V).$$

For each $r \in (2S)^{-1}/\mathbb{Z}^l$, we define a theta series $\theta_r(\tau, z)$ to be the sum

$$\sum_{q \in \mathbb{Z}^l} e(\tau S[q + r] + 2S(q + r, z)) \quad ((\tau, z) \in \mathcal{D}).$$

Since $\theta_{r+\mu}(\tau, z) = \theta_r(\tau, z)$ for any $\mu \in \mathbb{Z}^l$, one can define $\theta_r(\tau, z)$ for each $r \in R_S$. For each $\tau \in \mathfrak{H}$, let $\Theta_{S, \tau}$ denote the space of holomorphic functions $\theta : C^l \rightarrow C$ with the property

$$\theta(z + \lambda \tau + \mu) = e(-\tau S[\lambda] - 2S(\lambda, z))\theta(z).$$

It is known that $\{\theta_r(\tau, z)\}_{r \in R_S}$ forms a basis of the space $\Theta_{S, \tau}$. For each element $X = (\lambda, \mu) \in Q^l \times Q^l$, we denote by $[X]$ the element $(1_2, X, 0)$ of $G^l_Q$. We set

$$L = \mathbb{Z}^l \times \mathbb{Z}^l,$$

$$H_{\mathbb{Z}} = \{(1_2, X, \rho) \mid X \in L, \rho \in \text{Sym}_l(\mathbb{Z})\}.$$

Then, $H_{\mathbb{Z}}$ is a subgroup of $G^l_Q$. For each $\xi \in G^l_Q$, denote by $L_{\xi}$ the sublattice $\{X \in L \mid \xi[X]^{-1} \in H_{\mathbb{Z}}\}$ of $L$. Following Skoruppa-Zagier [S-Z2], we define an operator $U_S(\xi)$ acting on $\Theta_{S, \tau}$ as follows:

$$\theta|U_S(\xi) = \left( \sum_{X \in L_{\xi} \setminus L} \theta_{|/2_\xi} \right) \times \frac{1}{[L : L_{\xi}]}.$$

For this operator Skoruppa-Zagier (Proposition 4.1 of [S-Z2]) proved the following.
Theorem 1 (Skoruppa-Zagier) (i) For each $\theta \in \Theta_{S,\tau}$ and $\xi \in G_{Q}^{J}$, $\theta|U_{S}(\xi) \in \Theta_{S,\tau}$

(ii) We arrange $\theta_{r}, \theta_{r}|U_{S}(\xi), (r \in R_{S})$ as column vectors of $C^{d}$. Then there exists a matrix $U_{S}(\xi)$ of size $d$ (or a linear transformation of $V = C^{d}$) such that

\[(\theta_{r}|U_{S}(\xi))_{r \in R_{S}} = U_{S}(\xi)(\theta_{r})_{r \in R_{S}},\]

where $U_{S}(\xi)$ is independent of the choice of $\tau \in \mathfrak{H}$.

Remark. (1) For the matrix $U_{S}(\xi)$, we have used the same notation as for the operator $U_{S}(\xi)$ by abuse of notation.

(2) If $\xi = (M, 0, 0)$ and $M \in SL_{2}(Z)$, then the identity (2.1) is nothing but the theta transformation formula:

\[(\theta_{r}(M(\tau, z)))_{r \in R_{S}} = J_{l/2,S}(M, (\tau, z))U_{S}(M)(\theta_{r}(\tau, z))_{\epsilon R_{S}},\]

where $M(\tau, z) = (M\tau, \frac{z}{c\tau+d})$ and $U_{S}(M) = U_{S}((M, 0, 0))$ in this case is a unitary matrix with respect to the inner product $<, >_{S}$.

4 Where does $U_{S}(\xi)$ come from?

Let $k$ be a positive integer and put $\kappa = (k - l/2)/2$. We define a factor of automorphy $j_{M}(\tau)$ by

\[j_{M}(\tau) = \exp(2i\kappa \arg J(M, \tau)).\]

Denote by $\mathcal{M}_{S,k-l/2}(\Gamma)$ the space of all functions $f : \mathfrak{H} \rightarrow V$ satisfying the following conditions

(i) $\eta^{-\kappa} f(\tau)$ is holomorphic on $\mathfrak{H}$ and also finite at any cusps of $\Gamma$

(ii) $f(M\tau) = \overline{U_{S}(M)} j_{M}(\tau) f(\tau)$ for any $M \in \Gamma$.

Since each Jacobi form $\phi(\tau, z)$ of $J_{k,S}(\Gamma)$ is an element of $\Theta_{S,\tau}$ as a function of $z$, $\phi(\tau, z)$ has an expression as a linear combination of $\theta_{r}$'s:

\[\phi(\tau, z) = \sum_{r \in R_{S}} \eta^{-\kappa} f_{r}(\tau) \theta_{r}(\tau, z).\]

Then the collection $f(\tau) = (f_{r}(\tau))_{r \in R_{S}}$ is a modular form of $\mathcal{M}_{S,k-l/2}(\Gamma)$. It is well-known that $J_{k,S}(\Gamma)$ is isomorphic to $\mathcal{M}_{S,k-l/2}(\Gamma)$ as $C$-linear spaces via the correspondence $\iota : \phi \rightarrow f = (f_{r})_{r \in R_{S}}$. Let $\Delta \subseteq G_{Q}^{J}$ be a finite union of $\Gamma^{J}$-double cosets. Let $p : G_{Q}^{J} \rightarrow SL_{2}(Q)$ denote the natural projection map. For each $A$ of $p(\Delta)$ we put

\[V_{\Delta}(A) = \sum_{L \in H_{\Delta}} [L : L_{\xi}] U_{S}(\xi),\]
where the summation is over a complete set of representatives $\xi$ of the double cosets of $p^{-1}(A) \cap \Delta$ with respect to $H_{\mathbb{Z}}$ (this is a finite sum). Then this quantity $V_{\Delta}(A)$ is well-defined. If $\Delta = \Gamma^{J}$, then $V_{\Delta}(A)$ equals the linear operator $U_{S}(A) = U_{S}((A, 0, 0))$. The action of Hecke operators $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\underline{\Gamma})$ is transferred in terms of modular forms of $\mathcal{M}_{S,k-\frac{1}{2}}(\Gamma)$. There exists a linear operator $\widetilde{H}_{k,S,\Gamma}(\Delta)$ acting on $\mathcal{M}_{S,k-\frac{1}{2}}(\Gamma)$ such that $\iota \circ H_{k,S,\Gamma}(\Delta) = \widetilde{H}_{k,S,\Gamma}(\Delta) \circ \iota$. Then we easily have

$$(f|\widetilde{H}_{k,S,\Gamma}(\Delta))(\tau) = \sum_{A \in p(\Delta)} V_{\Delta}(A) j_{A}(\tau)^{-1} f(A \tau) \quad (f \in \mathcal{M}_{S,k-\frac{1}{2}}(\Gamma)),$$

where $A$ runs over a complete set of representatives of the left $\Gamma$-cosets of $p(\Delta)$ and the sum is well-defined.

In this manner the operator $U_{S}(\xi)$ is coming in our sight. It seems that $U_{S}(\xi)$ is a very attractive arithmetic object.

5 Selberg type zeta functions

For $M \in SL_{2}(\mathbb{Z})$, we write $U_{S}(M)$ instead of $U_{S}((M, 0, 0))$ in (2.1). We set

$$R_{S}^{0} = \{ r \in R_{S} | r \equiv -r \pmod{\mathbb{Z}} \}.$$

Since $U_{S}(-1_{2})$ has eigenvalues $\pm e^{-\pi i/2}$ (see (1.6) of [Ar]), it has the block decomposition

$$(4.1) \quad U_{S}(-1_{2}) = e^{-\pi i/2} Q \begin{pmatrix} 1_{d(+)} & 0 \\ 0 & -1_{d(-)} \end{pmatrix} Q^{-1},$$

where $Q$ is a certain unitary matrix of size $d$ and $d(+) = (d + d_{0})/2$ (resp. $d(-) = (d - d_{0})/2$). We easily have

$$V_{\Delta}(A) U_{S}(-1_{2}) = U_{S}(-1_{2}) V_{\Delta}(A) \quad \text{for any } A \in p(\Delta).$$

Therefore, $V_{\Delta}(A)$ has the block decomposition similar to (4.1):

$$(4.2) \quad V_{\Delta}(A) = Q \begin{pmatrix} V_{\Delta}^{+}(A) & 0 \\ 0 & V_{\Delta}^{-}(A) \end{pmatrix} Q^{-1}$$

with $V_{\Delta}^{+}(A)$ (resp. $V_{\Delta}^{-}(A)$) a matrix of size $d(+) \quad \text{resp.} \quad d(-))$. For $A \in SL_{2}(\mathbb{Q})$, let $Z_{\Gamma}(A)$ denote the centralizer of $A$ in $\Gamma$. Denote by $Hyp^{+}(\Delta)$ the set of hyperbolic elements $P$ of $p(\Delta)$ with $\text{tr}P > 2$ which do not fix any cusps of $\Gamma$. We set, for $\epsilon = \pm$,

$$\zeta_{\Delta,S,\epsilon}(s) = \sum_{P \in Hyp^{\epsilon}(\Delta) \Gamma} \text{tr}V_{\Delta}^{\epsilon}(P) \log N(P_{0}) \times \frac{N(P)^{-s}}{1 - N(P)^{-1}},$$

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where $Hyp^+(\Delta)/\Gamma$ denote a complete set of representatives of the $\Gamma$-conjugacy classes of elements of $Hyp^+(\Delta)$, and where, for each $P \in Hyp^+(\Delta)$, $P_0$ together with the element $-1_2$ is the generator of the centralizer $Z_\Gamma(P)$. It can be shown that $\zeta_{\Delta,S,e}(s)$ is absolutely convergent for $\Re(s) > 1$. If $\Delta = \Gamma^J$, then, $\zeta_{\Delta,S,e}(s)$ coincides with the logarithmic derivative of the Selberg zeta function associated with $\Gamma$, $S$:

$$\zeta_{\Delta,S,e}(s) = (Z_{\Gamma,S,e}/Z_{\Gamma,S,e})(s),$$

where $\varepsilon = \pm$ and

$$Z_{\Gamma,S,e}(s) = \prod_{\{P_0\}_\Gamma, \tr P_0 > 2} \prod_{n=0}^{\infty} \det(1_{d(e)} - U_S^e(P_0) N(P_0)^{-s-n}),$$

$P_0$ running over the $\Gamma$-conjugacy classes of primitive hyperbolic elements of $\Gamma$ with $\tr P_0 > 2$. Here, $U^\pm_S(A)$ ($A \in SL_2(\mathbb{Z})$) is defined similarly as in (4.2) from $U_S(A)$. For details concerning the Selberg zeta functions $Z_{\Gamma,S,e}(s)$ we refer to [Ar]. Via the theory of general Selberg trace formula the Selberg type zeta functions $\zeta_{\Delta,S,e}(s)$ can be analytically continued to a meromorphic function of $s$ in the whole complex plane. This analytic continuation is crucial to the calculation of the traces of Hecke operators.

### 6 Traces of Hecke operators

Let $\Delta$ be as before. Each elliptic element $R$ of $SL_2(\mathbb{R})$ is $SL_2(\mathbb{R})$-conjugate to some

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

with $0 < \theta < 2\pi$, where $\theta$ is uniquely determined by $R$. We often write $\theta(R)$ for this $\theta$. Denote by $Ell^+(\Delta)$ the set of all elliptic elements $R$ of $p(\Delta)$ with $0 < \theta(R) < \pi$. Denote by $Ell^+(\Delta)/\Gamma$ a complete set of representatives of the $\Gamma$-conjugacy classes of all elements of $Ell^+(\Delta)$. Let $\zeta_1, \zeta_2, \ldots, \zeta_h$ be a complete set of representatives of the $\Gamma$-equivalece classes of cusps of $\Gamma$. For each $j$ ($1 \leq j \leq h$), one can choose an element $A_j \in SL_2(\mathbb{R})$ such that $-1_2$ and $T_j := A_j^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A_j$ generate the stabilizer group $\Gamma_{\zeta_j}$ of the cusp $\zeta_j$. For each $j$ ($1 \leq j \leq h$), denote by $Hyp_j^+(\Delta)$ the set of all hyperbolic elements $P$ of $p(\Delta)$ with $\tr P > 2$ and $P\zeta_j = \zeta_j$. The set $Hyp_j^+(\Delta)$ is stable under the conjugation by any element of $\Gamma_{\zeta_j}$. Denote by $Hyp_j^+(\Delta)/\Gamma_{\zeta_j}$ a complete set of representatives of the $\Gamma_{\zeta_j}$-conjugacy classes of all elements of $Hyp_j^+(\Delta)$. Moreover for each $j$ ($1 \leq j \leq h$), we denote by $Par_j^+(\Delta)$ the set of all parabolic elements $P$ of $p(\Delta)$ satisfying the conditions $\tr P = 2$, $P\zeta_j = \zeta_j$ and $P \neq 1_2$. Let $N = 4 \det(2S) = 4d$ and $\Gamma(N)$ the principal congruence subgroup of $SL_2(\mathbb{Z})$ with level $N$. Set, for each $j$ ($1 \leq j \leq h$),

$$\Gamma_j^+ = \Gamma_{\zeta_j} \cap \Gamma(N).$$
Then the group $\Gamma_j^+$ is generated by $T_j^{n_j}$ with a positive integer $n_j$. This integer $n_j$ is uniquely determined. We call two elements $A, B$ of $Par_j^+(\Delta)$ $\Gamma_j^+$-equivalent, if there exists an element $M$ of $\Gamma_j^+$ with $B = MA$. Denote by $Par_j^+(\Delta)/\Gamma_j^+$ a complete set of representatives of the $\Gamma_j^+$-equivalence classes of all elements of $Par_j^+(\Delta)$. Each element $P$ of $Par_j^+(\Delta)$ has an expression

$$P = A_j^{-1} \begin{pmatrix} 1 & r(P) \\ 0 & 1 \end{pmatrix} A_j$$

with a uniquely determined rational number $r(P) \in \mathbb{Q}$, $r(P) \neq 0$. If $P$ is a representative of $Par_j^+(\Delta)/\Gamma_j^+$, $r(P)$ is uniquely determined modulo $n_j$.

Let $k$ be an integer and set

$$\kappa = (k - l/2)/2.$$ 

Denote by $e(k)$ the sign $+$ or $-$ according as $k$ is even or not. We set

$$C_{\Gamma, \Delta}(k) = \frac{1}{4\pi} vol(\Gamma \backslash \mathfrak{H}) tr(V_\Delta^{e(k)}(1_2))(2\kappa - 1)$$

$$+ \sum_{R \in Bl^{+} \star(\Delta)l^{r}} tr(V_\Delta^{e(k)}(R)) \times \frac{e^{-2i\theta(R) + i\theta(R)}}{\nu(e^{i\theta(R)} - e^{-i\theta(R)})}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \in Hy} tr(V_\Delta^{e(k)}(P)) \times \frac{N(P)^{-\kappa}}{1 - N(P)^{-1}}$$

$$- \frac{1}{2n_j} \sum_{j=1}^{h} \sum_{P \in Par_j^+(\Delta)/\Gamma_j^+} \frac{1}{2n_j} tr(V_\Delta^{e(k)}(P)) \times \begin{cases} 1 - i \cot \frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \text{ mod } n_j \\ 1 & \cdots r(P) \equiv 0 \text{ mod } n_j \end{cases}$$

and

$$C_{\Gamma, \Delta}^*(k) = \frac{1}{4\pi} vol(\Gamma \backslash \mathfrak{H}) tr(V_\Delta^{e(k)}(1_2))(-2\kappa - 1)$$

$$+ \sum_{R \in Bl^{+} \star(\Delta)l^{r}} tr(V_\Delta^{e(k)}(R)) \times \frac{e^{-2j\kappa \theta(R) - j\theta(R)}}{\nu(e^{j\theta(R)} - e^{-j\theta(R)})}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \epsilon Hy} tr(V_\Delta^{e(k)}(P)) \times \frac{N(P)^{\kappa}}{1 - N(P)^{-1}}$$

$$- \sum_{j=1}^{h} \sum_{P \epsilon Par_j^+(\Delta)/\Gamma_j^+} \frac{1}{2n_j} tr(V_\Delta^{e(k)}(P)) \times \begin{cases} 1 + i \cot \frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \text{ mod } n_j \\ 1 & \cdots r(P) \equiv 0 \text{ mod } n_j \end{cases},$$

where

$$vol(\Gamma \backslash \mathfrak{H}) = \int_{\Gamma \backslash \mathfrak{H}} \eta^{-2} d\xi d\eta \quad (\xi = \text{Re} \tau, \eta = \text{Im} \tau).$$
We denote by $\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma))$ the trace of the action of $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\Gamma)$ and so on. Let $\Theta_{S,\Gamma}$ denote the space of theta functions $\theta(\tau, z)$ satisfying the following conditions:

(i) $\theta(\tau, z)$, as a function of $z$, is an element of $\Theta_{S,\Gamma}$

(ii) $\theta|_{l/2, S} M = \theta$ for any $M \in \Gamma$.

Then, $\Theta_{S,\Gamma}$ is isomorphic to the space $J_{l/2, S}(\Gamma)$ of Jacobi forms of weight $l/2$ with respect to $\Gamma^J$. The Hecke operator $H_{l/2, S, \Gamma}(\Delta)$ operates on $\Theta_{S,\Gamma}$. We have the following theorem.

**Theorem 2** Asssume that $\Gamma$ is a congruence subgroup of $SL_2(\mathbb{Z})$ having the element $-1_2$. Let $k$ be an integer and $\Delta \subseteq G^J_\mathbb{Q}$ a finite union of $\Gamma^J$-double cosets.

(i) If $k > l/2 + 2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{\text{cusp}}(\Gamma)) = C_{\Gamma,\Delta}(k).$$

If $k < l/2 - 2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{*\text{cusp}}(\Gamma)) = C_{\Gamma,\Delta}^{*}(k).$$

(ii) Assume that $l$ is odd. Denote by $\epsilon$ the sign + or – according as $l$ is congruent to 1 or 3 modulo 4 (i.e., $\epsilon = \epsilon((l-1)/2)$.

If $k = (l+3)/2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{*\text{cusp}}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,\epsilon}(s) + C_{\Gamma,\Delta}(k).$$

If $k = (l+1)/2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,-\epsilon}(s).$$

If $k = (l-1)/2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{*\text{cusp}}(\Delta), J_{l-k,S}^{*}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,\epsilon}(s).$$

If $k = (l-3)/2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}(\Delta), J_{l-k,S}^{*\text{cusp}}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,-\epsilon}(s) + C_{\Gamma,\Delta}^{*}(k).$$

(iii) Assume that $l$ is even. Let $\epsilon = \epsilon(l/2)$.

If $k = l/2 + 2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{\text{cusp}}(\Gamma)) = \text{Res}_{s=1} \zeta_{\Delta,S,\epsilon}(s) + C_{\Gamma,\Delta}(k).$$

If $k = l/2 - 2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{*\text{cusp}}(\Delta), J_{l-k,S}^{*}(\Gamma)) = \text{Res}_{s=1} \zeta_{\Delta,S,\epsilon}(s) + C_{\Gamma,\Delta}^{*}(k).$$

If $k = l/2$, then,

$$\text{tr}(H_{l/2, S, \Gamma}(\Delta), \Theta_{S,\Gamma}) = \text{Res}_{s=1} \zeta_{\Delta,S,\epsilon}(s).$$
For the proof we use Fischer's resolvent trace formula [Fi] and the method of Skoruppa-Zagier [S-Z2]. We can deduce the following corollary from (ii), (iii) of the above theorem.

**Corollary 3** (i) Assume that $l$ is odd. Then,

$$
\text{tr}(H_{(l+3)/2, S, \Gamma}(\Delta), J_{(l+3)/2, S}^{\text{cusp}}(\Gamma)) = \text{tr}(H_{(l+1)/2, S, \Gamma}(\Delta), J_{(l+1)/2, S}^{*}(\Gamma)) + C_{\Gamma, \Delta}\left(\frac{l+3}{2}\right),
$$

$$
\text{tr}(H_{(l+3)/2, S, \Gamma}^{\text{skew}}(\Delta), J_{(l+3)/2, S}^{*}(\Gamma)) = \text{tr}(H_{(l+1)/2, S, \Gamma}^{\text{skew}}(\Delta), J_{(l+1)/2, S}^{*}(\Gamma)) + C_{\Gamma, \Delta}^{*}\left(\frac{l+3}{2}\right).
$$

(ii) Assume that $l$ is even. Then,

$$
\text{tr}(H_{l/2+2, S, \Gamma}(\Delta), J_{l/2+2, S}^{\text{cusp}}(\Gamma)) = \text{tr}(H_{l/2, S, \Gamma}(\Delta), \Theta_{S, \Gamma}) + C_{\Gamma, \Delta}\left(\frac{l}{2} + 2\right),
$$

$$
\text{tr}(H_{l/2+2, S, \Gamma}^{\text{skew}}(\Delta), J_{l/2+2, S}^{*}(\Gamma)) = \text{tr}(H_{l/2, S, \Gamma}^{\text{skew}}(\Delta), \Theta_{S, \Gamma}) + C_{\Gamma, \Delta}^{*}\left(\frac{l}{2} - 2\right).
$$

**Remark.** In the case of $l = 1$ the first identity of the above (i) has been already obtained by Skoruppa-Zagier [S-Z1]. The results in Theorem 2 and Corollary 3 are consistent with those of [S-Z1, 2].

**References**


[Su] Sugano, T., *Jacobi forms and the theta lifting*, preprint
