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<th>Title</th>
<th>Selberg trace formula and Jacobi forms</th>
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Kyoto University
Selberg trace formula and Jacobi forms

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1 Introduction

In this note we present a calculation of the traces of Hecke operators acting on the spaces of Jacobi forms via the general Selberg trace formula. We can represent those traces in a closed form with the use of some arithmetic quantities and the residues at poles of certain Selberg type zeta functions. The calculation of those traces has been done by Skoruppa-Zagier ([S-Z1, 2]) in some cases in a different manner. They have employed the Bergman kernel functions for the spaces of Jacobi forms and also some results of Shimura [Sh] concerning modular forms of half integral weight. Here we use the general Selberg trace formula due originally to Selberg [Se] and to Hejhal [He], Fischer [Fi]. For our calculation we exclusively follow Fischer [Fi].

In this short survey we exhibit only the results which is a generalization of our previous work [Ar] and we shall give a proof in another occasion in details.

2 Jacobi forms and Hecke operators

We use the symbol $e(\alpha)$ as an abbreviation of $\exp(2\pi i \alpha)$. Let $l$ be a positive integer. Let $G_{Q}^{J}$ be the Jacobi group defined over $Q$:

$$G_{Q}^{J} = \{ (g, (\lambda, \mu), \rho) | g \in Q^{l}, \lambda, \mu \in Q^{l}, \rho \in Sym_{l}(Q) \},$$

where $Q^{l}$ (resp. $Sym_{l}(Q)$) denotes the space of rational column vectors (resp. rational symmetric matrices) of size $l$. The composition law of $G_{Q}^{J}$ is given by

$$g_{1}g_{2} = (M_{1}M_{2}, (\lambda_{1}, \mu_{1})M_{2} + (\lambda_{2}, \mu_{2}), \rho_{1} + \rho_{2} - \mu_{1}^{t}\lambda_{1} + \lambda^{*t}\mu_{2} + \mu_{2}^{t}\lambda^{*})$$

$$g_{j} = (M_{j}, (\lambda_{j}, \mu_{j}), \rho_{j}) \in G_{Q}^{J}, j = 1, 2)$$

with $(\lambda^{*}, \mu^{*}) = (\lambda_{1}, \mu_{1})M_{2}$. Denote by $G_{R}^{J}$ the group of real points of $G_{Q}^{J}$. Denote by $D$ the product of the upper half plane $\mathfrak{H}$ and $C^{l}$, the space of complex column vectors of size $l$: $D = \mathfrak{H} \times C^{l}$. The Jacobi group $G_{R}^{J}$ acts on $D$ in the following manner:

$$g(\tau, z) = \left( M\tau, \frac{z + \lambda\tau + \mu}{J(M, \tau)} \right)$$
\[(g = (M, (\lambda, \mu), \rho) \in G_{\bullet}^{J}, (\tau, z) \in \mathcal{D}),\]

where \(J(M, \tau) = c\tau + d\) for \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\).

Let \(S\) be a positive definite half integral symmetric matrix of size \(l\). We define a factor of automorphy \(J_{k,S}(g, (\tau, z))\) associated to \(S\) and a half integer \(k\) by

\[
J_{k,S}(g, (\tau, z)) = J(M, \tau)^{k}e\left(-\text{tr}(S\rho) - \tau S[\lambda] - 2S(\lambda, z) + \frac{c}{J(M, \tau)}S[z + \lambda\tau + \mu]\right)
\]

\((g = (M, (\lambda, \mu), \rho) \in G_{\bullet}^{J}, M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\tau, z) \in \mathcal{D})\),

where the branch of \(J(M, \tau)^{k} = \exp(k\log J(M, \tau))\) is chosen so that \(-\pi < \arg J(M, \tau) \leq \pi\).

Let \(\Gamma\) be a congruence subgroup of \(SL_{2}(\mathbb{Z})\) having the element \(-1_{2}\) and \(\Gamma^{J}\) the subgroup of \(G_{\bullet}^{J}\) given by

\[
\Gamma^{J} = \{(M, (\lambda, \mu), \rho) | M \in \Gamma, \lambda, \mu \in \mathbb{Z}^{l}, \rho \in \text{Sym}_{1}(\mathbb{Z})\},
\]

where \(\mathbb{Z}^{l}\) (resp. \(\text{Sym}_{1}(\mathbb{Z})\)) denotes the \(\mathbb{Z}\)-lattice consisting of integral column vectors (resp. integral symmetric matrices) in \(\mathbb{Q}^{l}\) (resp. \(\text{Sym}_{l}(\mathbb{Q})\)). For any function \(\phi : \mathcal{D} \rightarrow \mathbb{C}\) and \(g = (M, (\lambda, \mu), \rho) \in G_{\bullet}^{J}\), we set

\[
(\phi|_{k,S}g)(\tau, z) = J_{k,S}(g, (\tau, z))^{-1}\phi(g(\tau, z)),
\]

\[
(\phi|_{k,S}^{*}g)(\tau, z) = J_{0,S}(g, (\tau, z))^{-1}(J(M, \tau))^{-k+l}|J(M, \tau)|^{-1}\phi(g(\tau, z)).
\]

In the definition of the latter \((\phi|_{k,S}^{*}g)\), we may assume that \(k\) is an integer, since only such cases can occur in the discussion later on. If \(k\) is an integer, then these operations satisfy

\[
\phi|_{k,S}g_{1}g_{2} = \phi|_{k,S}g_{1}|_{k,S}g_{2}
\]

and

\[
\phi|_{k,S}^{*}g_{1}g_{2} = \phi|_{k,S}^{*}g_{1}|_{k,S}^{*}g_{2}.
\]

Note that \(\mathbb{H} \cup \{\infty\} \cup \mathbb{Q}\) is the total set of cusps of \(\Gamma\). For each element \(M\) of \(\Gamma\), put \(M\infty = \zeta\). Denote by \(\Gamma_{\zeta}\) the stabilizer group of \(\zeta\) in \(\Gamma\): \(\Gamma_{\zeta} = \{\sigma \in \Gamma | \sigma\zeta = \zeta\}\). There exists a unique positive integer \(N\) such that the group \(M^{-1}\Gamma_{\zeta}M\) of \(SL_{2}(\mathbb{Z})\) is generated by \(-1_{2}\) and \(\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}\). Let \(k\) be a positive integer. Now we define the space \(J_{k,S}(\Gamma)\) (resp. \(J_{k,S}^{*}(\Gamma)\)) of holomorphic (resp. skew-holomorphic) Jacobi forms of index \(S\) and weight \(k\) with respect to \(\Gamma^{J}\). We define \(J_{k,S}(\Gamma)\) (resp. \(J_{k,S}^{*}(\Gamma)\)) to be the space consisting of all functions \(\phi : \mathcal{D} \rightarrow \mathbb{C}\) which satisfy the following three conditions:

\[\text{...}\]
(i) $\phi(\tau, z)$ is holomorphic in $\tau$ and $z$

(resp. $\phi(\tau, z)$ is a smooth function in $\tau$ and holomorphic in $z$)

(ii) $\phi(\tau, z)$ satisfies the identity

$$\phi|_{k,S}\gamma = \phi \quad \text{(resp.} \quad \phi|^{*}_{k,S}\gamma = \phi \text{)} \quad \forall \gamma \in \Gamma^J$$

(iii) The function $\phi|_{k,S}M$ (resp. $\phi|^{*}_{k,S}M$) for any $M \in SL_2(\mathbb{Z})$ has a Fourier Jacobi expansion of the form

$$\left(\phi|_{k,S}M\right)(\tau, z) = \sum_{n \in \mathbb{Z}, r \in \mathbb{Z}^J, 4n - N^t r S^{-1} r \geq 0} c(n, r) e \left( \frac{nr}{N} + t rz \right)$$

(resp.

$$\left(\phi|^{*}_{k,S}M\right)(\tau, z) = \sum_{n \in \mathbb{Z}, r \in \mathbb{Z}^J, 4n - N^t r S^{-1} r \leq 0} c(n, r) e \left( \frac{nr}{N} + \frac{i\eta}{2}(t S^{-1} r) + t rz \right)$$

where $\eta = \text{Im}\tau$ and a positive integer $N$ is chosen for each $M$ in the above manner.

In the above (iii), $M \in SL_2(\mathbb{Z})$ is identified with the element $(M, (0,0), 0)$ in $G_{Q}^{J}$.

Denote by $J_{k,S}^{cu,p}(\Gamma)$ (resp. $J_{k,S}^{*cu,p}(\Gamma)$) the subspace of cusp forms of $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^{*}(\Gamma)$) consisting of all Jacobi forms $\phi \in J_{k,S}(\Gamma)$ (resp. all skew-holomorphic Jacobi forms $\phi \in J_{k,S}^{*}(\Gamma)$) whose Fourier coefficients $c(n, r)$ in the above (iii) equals zero if $4n - N^t r S^{-1} r = 0$.

Let $\Delta \subseteq G_{Q}^{J}$ be a finite union of double cosets with respect to $\Gamma^J$: $\Delta = \sum_{j} \Gamma^J \sigma_j \Gamma^J$ ($\sigma_j \in G_{Q}^{J}$). Following Skoruppa-Zagier [S-Z2], we define an operator $H_{k,S,\Gamma}(\Delta)$ (resp. $H_{k,S,\Gamma}^{skew}(\Delta)$) acting on $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^{*}(\Gamma)$) by

$$\phi|H_{k,S,\Gamma}(\Delta) = \sum_{\xi \in \Gamma^{J}\backslash \Delta} \phi|_{k,S}\xi$$

(resp. $\phi|H_{k,S,\Gamma}^{skew}(\Delta) = \sum_{\xi \in \Gamma^{J}\backslash \Delta} \phi|^{*}_{k,S}\phi$)

where the summation is taken over a complete set of representatives $\xi$ for the left $\Gamma^J$-cosets of $\Delta$. The operator $H_{k,S,\Gamma}(\Delta)$ (resp. $H_{k,S,\Gamma}^{skew}(\Delta)$) is well-defined and maps $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^{*}(\Gamma)$) to $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^{*}(\Gamma)$) and cusp forms to cusp forms (see Proposition 1.1 of [S-Z2]). For $L$-functions associated with common eigen Jacobi forms in this situation we refer the reader to Sugano [Su].
3 An operator acting on the space of theta series

Let $S$ be a positive definite half-integral symmetric matrix of size $l$ as before and $R_S$ denote the $\mathbb{Z}$-module $(2S)^{-1}\mathbb{Z}^l/\mathbb{Z}^l$. Set

$$d = \det(2S) = \#(R_S).$$

We write, for simplicity,

$$S(u, v) = \iota_{uSv}$$

for $u, v \in \mathbb{C}^l$.

Denote by $V = \mathbb{C}^d$ the $\mathbb{C}$-vector space consisting of column vectors $(x_r)_{r \in R_S}$ ($x_r \in \mathbb{C}$). Let $<x, y>_S$ be the positive definite hermitian scalar product given by

$$<x, y>_S = \sum_{r \in R_S} x_r\overline{y_r} \quad (x = (x_r)_{r \in R_S}, y = (y_r)_{r \in R_S} \in V).$$

For each $r \in (2S)^{-1}\mathbb{Z}^l$, we define a theta series $\theta_r(\tau, z)$ to be the sum

$$\sum_{q \in \mathbb{Z}^l} e(\tau S[q + r] + 2S(q + r, z)).$$

Since $\theta_{r+\mu}(\tau, z) = \theta_r(\tau, z)$ for any $\mu \in \mathbb{Z}^l$, one can define $\theta_r(\tau, z)$ for each $r \in R_S$. For each $\tau \in \mathfrak{H}$, let $\Theta_{S, \tau}$ denote the space of holomorphic functions $\theta : \mathbb{C}^l \to \mathbb{C}$ with the property

$$\theta(z + \lambda \tau + \mu) = e(-\tau S[\lambda] - 2S(\lambda, z))\theta(z).$$

It is known that $\{\theta_r(\tau, z)\}_{r \in R_S}$ forms a basis of the space $\Theta_{S, \tau}$. For each element $X = (\lambda, \mu) \in \mathbb{Q}^l \times \mathbb{Q}^l$, we denote by $[X]$ the element $(1_2, X, 0)$ of $G_\mathbb{Q}^l$. We set

$$L = \mathbb{Z}^l \times \mathbb{Z}^l,$$

$$H_\mathbb{Z} = \{(1_2, X, \rho) \mid X \in L, \rho \in \text{Sym}_l(\mathbb{Z})\}.$$

Then, $H_\mathbb{Z}$ is a subgroup of $G_\mathbb{Q}^l$. For each $\xi \in G_\mathbb{Q}^l$, denote by $L_\xi$ the sublattice $\{X \in L \mid \xi[X]\xi^{-1} \in H_\mathbb{Z}\}$ of $L$. Following Skoruppa-Zagier [S-Z2], we define an operator $U_S(\xi)$ acting on $\Theta_{S, \tau}$ as follows:

$$\theta|U_S(\xi) = \left( \sum_{X \in L_\xi \setminus L} \theta|_{1/2, S}\xi[X] \right) \times \frac{1}{[L : L_\xi]}.$$

For this operator Skoruppa-Zagier (Proposition 4.1 of [S-Z2]) proved the following.
Theorem 1 (Skoruppa-Zagier) (i) For each \( \theta \in \Theta_{S,r} \) and \( \xi \in G_{Q}^{J} \), \( \theta|U_{S}(\xi) \in \Theta_{S,r} \).

(ii) We arrange \( \theta_{r}, \theta_{r}|U_{S}(\xi), \) for \( r \in R_{S} \) as column vectors of \( C^{d} \). Then there exists a matrix \( U_{S}(\xi) \) of size \( d \) (or a linear transformation of \( V = C^{d} \)) such that

\[
(\theta_{r}|U_{S}(\xi))_{r \in R_{S}} = U_{S}(\xi)(\theta_{r})_{r \in R_{S}},
\]

where \( U_{S}(\xi) \) is independent of the choice of \( r \in R_{S} \).

Remark. (1) For the matrix \( U_{S}(\xi) \), we have used the same notation as for the operator \( U_{S}(\xi) \) by abuse of notation.

(2) If \( \xi = (M, 0, 0) \) and \( M \in SL_{2}(\mathbb{Z}) \), then the identity (2.1) is nothing but the theta transformation formula:

\[
(\theta_{r}(M(\tau, z)))_{r \in R_{S}} = J_{l/2,S}(M, (\tau, z))U_{S}(M)(\theta_{r}(\tau, z))_{r \in R_{S}},
\]

where \( M(\tau, z) = (M\tau, \frac{z}{c\tau+d}) \) and \( U_{S}(M) = U_{S}((M, 0, 0)) \) in this case is a unitary matrix with respect to the inner product \( <, >_{S} \).

4 Where does \( U_{S}(\xi) \) come from?

Let \( k \) be a positive integer and put \( \kappa = (k-l/2)/2 \). We define a factor of automorphy \( j_{M}(\tau) \) by

\[
j_{M}(\tau) = \exp(2i\kappa \arg J(M, \tau)).
\]

Denote by \( \mathcal{M}_{S,k-l/2}(\Gamma) \) the space of all functions \( f : \mathfrak{H} \rightarrow V \) satisfying the following conditions

(i) \( \eta^{-\kappa}f(\tau) \) is holomorphic on \( \mathfrak{H} \) and also finite at any cusps of \( \Gamma \)

(ii) \( f(M\tau) = \overline{U_{S}(M)}j_{M}(\tau)f(\tau) \) for any \( M \in \Gamma \).

Since each Jacobi form \( \phi(\tau, z) \) of \( J_{k,S}(\Gamma) \) is an element of \( \Theta_{S,r} \) as a function of \( z \), \( \phi(\tau, z) \) has an expression as a linear combination of \( \theta_{r} \)s:

\[
\phi(\tau, z) = \sum_{r \in R_{S}} \eta^{-\kappa}f_{r}(\tau)\theta_{r}(\tau, z).
\]

Then the collection \( f(\tau) = (f_{r}(\tau))_{r \in R_{S}} \) is a modular form of \( \mathcal{M}_{S,k-l/2}(\Gamma) \). It is well-known that \( J_{k,S}(\Gamma) \) is isomorphic to \( \mathcal{M}_{S,k-l/2}(\Gamma) \) as \( \mathbb{C} \)-linear spaces via the correspondence \( \iota : \phi \rightarrow f = (f_{r})_{r \in R_{S}} \). Let \( \Delta \subseteq G_{Q}^{J} \) be a finite union of \( \Gamma^{J} \)-double cosets. Let \( p : G_{Q}^{J} \rightarrow SL_{2}(\mathbb{Q}) \) denote the natural projection map. For each \( A \) of \( p(\Delta) \) we put

\[
V_{\Delta}(A) = \sum_{[L : L_{\xi}]U_{S}(\xi),}
\]
where the summation is over a complete set of representatives $\xi$ of the double cosets of $p^{-1}(A) \cap \Delta$ with respect to $H_{X}$ (this is a finite sum). Then this quantity $V_{\Delta}(A)$ is well-defined. If $\Delta = \Gamma^J$, then $V_{\Delta}(A)$ equals the linear operator $U_{S}(A) = U_{S}((A, 0.0))$. The action of Hecke operators $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\underline{\Gamma})$ is transferred in terms of modular forms of $M_{S,k-\frac{1}{2}}(\Gamma)$. There exists a linear operator $\overline{H}_{k,S,\Gamma}(\Delta)$ acting on $M_{S,k-\frac{1}{2}}(\Gamma)$ such that $\iota \circ H_{k,S,\Gamma}(\Delta) = \overline{H}_{k,S,\Gamma}(\Delta) \circ \iota$. Then we easily have

$$ (f_{| \overline{H}_{k,S,\Gamma}(\Delta)})(\tau) = \sum_{A \in \Gamma \Psi(\Delta)} iV_{\Delta}(A)J_{A}(\tau)^{-1}f(A\tau) \quad (f \in M_{S,k-\frac{1}{2}}(\Gamma)), $$

where $A$ runs over a complete set of representatives of the left $\Gamma$-cosets of $p(\Delta)$ and the sum is well-defined.

In this manner the operator $U_{S}(\xi)$ is coming in our sight. It seems that $U_{S}(\xi)$ is a very attractive arithmetic object.

5 Selberg type zeta functions

For $M \in SL_{2}(\mathbb{Z})$, we write $U_{S}(M)$ instead of $U_{S}((M, 0, 0))$ in (2.1). We set

$$ R_{S}^{0} = \{ r \in R_{S} \mid r \equiv -r \pmod{\mathbb{Z}'} \}. $$

Since $U_{S}(-1_{2})$ has eigen values $\pm e^{-\pi i/2}$ (see (1.6) of [Ar]), it has the block decomposition

$$ U_{S}(-1_{2}) = e^{-\pi i/2}Q \begin{pmatrix} 1_{d(+)} & 0 \\ 0 & -1_{d(-)} \end{pmatrix} Q^{-1}, $$

where $Q$ is a certain unitary matrix of size $d$ and $d(+) = (d + d_{0})/2$ (resp. $d(-) = (d - d_{0})/2$). We easily have

$$ V_{\Delta}(A)U_{S}(-1_{2}) = U_{S}(-1_{2})V_{\Delta}(A) \quad \text{for any } A \in p(\Delta). $$

Therefore, $V_{\Delta}(A)$ has the block decomposition similar to (4.1):

$$ V_{\Delta}(A) = Q \begin{pmatrix} V_{\Delta}^{+}(A) & 0 \\ 0 & V_{\Delta}^{-}(A) \end{pmatrix} Q^{-1} $$

with $V_{\Delta}^{+}(A)$ (resp. $V_{\Delta}^{-}(A)$) a matrix of size $d(\pm)$ (resp. $d(-)$). For $A \in SL_{2}(\mathbb{Q})$, let $Z_{\Gamma}(A)$ denote the centralizer of $A$ in $\Gamma$. Denote by $Hyp^{+}(\Delta)$ the set of hyperbolic elements $P$ of $p(\Delta)$ with $\text{tr}P > 2$ which do not fix any cusps of $\Gamma$. We set, for $\epsilon = \pm$,

$$ \zeta_{\Delta, S, \epsilon}(s) = \sum_{P \in Hyp^{+}(\Delta) \Gamma} \text{tr}V_{\Delta}^{\epsilon}(P) \log N(P_{0}) \times \frac{N(P)^{-s}}{1 - N(P)^{-1}}, $$

6
where \( H^+_{\Delta}(\Delta)/\Gamma \) denote a complete set of representatives of the \( \Gamma \)-conjugacy classes of elements of \( H^+_{\Delta}(\Delta) \), and where, for each \( P \in H^+_{\Delta}(\Delta) \), \( P_0 \) together with the element \(-1_2\) is the generator of the centralizer \( Z_{\Gamma}(P) \). It can be shown that \( \zeta_{\Delta,S,e}(s) \) is absolutely convergent for \( \text{Re}(s) > 1 \). If \( \Delta = \Gamma' \), then \( \zeta_{\Delta,S,e}(s) \) coincides with the logarithmic derivative of the Selberg zeta function associated with \( \Gamma', S \):

\[
\zeta_{\Delta,S,e}(s) = (Z_{\Gamma,S,e}'(s)/Z_{\Gamma,S,e}^e)(s),
\]

where \( \varepsilon = \pm \) and

\[
Z_{\Gamma,S,e}(s) = \prod_{P \in \pi_{\Gamma,S}} \prod_{\text{tr}P > 2} \prod_{n=0}^{\infty} \det(1_{2 \times 2} - U_2(P)N(P)^{-s-n}),
\]

\( P_0 \) running over the \( \Gamma \)-conjugacy classes of primitive hyperbolic elements of \( \Gamma \) with \( \text{tr}P > 2 \). Here, \( U_\Delta^e(A) \ (A \in SL_2(\mathbb{Z})) \) is defined similarly as in (4.2) from \( U_\Delta(A) \). For details concerning the Selberg zeta functions \( Z_{\Gamma,S,e}(s) \) we refer to [Ar]. Via the theory of general Selberg trace formula the Selberg type zeta functions \( \zeta_{\Delta,S,e}(s) \) can be analytically continued to a meromorphic function of \( s \) in the whole complex plane. This analytic continuation is crucial to the calculation of the traces of Hecke operators.

6 Traces of Hecke operators

Let \( \Delta \) be as before. Each elliptic element \( R \) of \( SL_2(\mathbb{R}) \) is \( SL_2(\mathbb{R}) \)-conjugate to some

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

with \( 0 < \theta < 2\pi \), where \( \theta \) is uniquely determined by \( R \). We often write \( \theta(R) \) for this \( \theta \). Denote by \( \text{Ell}^+(\Delta) \) the set of all elliptic elements \( R \) of \( p(\Delta) \) with \( 0 < \theta(R) < \pi \). Denote by \( \text{Ell}^+(\Delta)/\Gamma \) a complete set of representatives of the \( \Gamma \)-conjugacy classes of all elements of \( \text{Ell}^+(\Delta) \). Let \( \zeta_1, \zeta_2, \ldots, \zeta_h \) be a complete set of representatives of the \( \Gamma \)-equivalence classes of cusps of \( \Gamma \). For each \( j (1 \leq j \leq h) \), one can choose an element

\[
A_j \in SL_2(\mathbb{R})
\]

such that \(-1_2 \) and \( T_j := A_j^{-1} \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} A_j \) generate the stabilizer group \( \Gamma_{\zeta_j} \) of the cusp \( \zeta_j \). For each \( j (1 \leq j \leq h) \), denote by \( H^+_{\zeta_j}(\Delta) \) the set of all hyperbolic elements \( P \) of \( p(\Delta) \) with \( \text{tr}P > 2 \) and \( P \zeta_j = \zeta_j \). The set \( H^+_{\zeta_j}(\Delta) \) is stable under the conjugation by any element of \( \Gamma_{\zeta_j} \). Denote by \( H^+_{\zeta_j}(\Delta)/\Gamma_{\zeta_j} \) a complete set of representatives of the \( \Gamma_{\zeta_j} \)-conjugacy classes of all elements of \( H^+_{\zeta_j}(\Delta) \). Moreover for each \( j (1 \leq j \leq h) \), we denote by \( Par^+_{\zeta_j}(\Delta) \) the set of all parabolic elements \( P \) of \( p(\Delta) \) satisfying the conditions \( \text{tr}P = 2 \), \( P \zeta_j = \zeta_j \) and \( P \neq 1_2 \). Let \( N = 4 \det(2S) = 4d \) and \( \Gamma(N) \) the principal congruence subgroup of \( SL_2(\mathbb{Z}) \) with level \( N \). Set, for each \( j (1 \leq j \leq h) \),

\[
\Gamma_j^+ = \Gamma_{\zeta_j} \cap \Gamma(N).
\]
Then the group $\Gamma_j^+$ is generated by $T_j^{n_j}$ with a positive integer $n_j$. This integer $n_j$ is uniquely determined. We call two elements $A$, $B$ of $Par_j^+(\Delta)$ $\Gamma_j^+$-equivalent, if there exists an element $M$ of $\Gamma_j^+$ with $B = MA$. Denote by $Par_j^+(\Delta)/\Gamma_j^+$ a complete set of representatives of the $\Gamma_j^+$-equivalence classes of all elements of $Par_j^+(\Delta)$. Each element $P$ of $Par_j^+(\Delta)$ has an expression

$$P = A_j^{-1} \begin{pmatrix} 1 & r(P) \\ 0 & 1 \end{pmatrix} A_j$$

with a uniquely determined rational number $r(P) \in \mathbb{Q}$, $r(P) \neq 0$. If $P$ is a representative of $Par_j^+(\Delta)/\Gamma_j^+$, $r(P)$ is uniquely determined modulo $n_j$.

Let $k$ be an integer and set

$$\kappa = (k - l/2)/2.$$ 

Denote by $e(k)$ the sign $+$ or $-$ according as $k$ is even or not. We set

$$C_{\Gamma, \Delta}(k) = \frac{1}{4\pi} \text{vol}(\Gamma \backslash \mathfrak{H}) \text{tr}(V_{\Delta}^{e(k)}(1_{2}))(2\kappa - 1)$$

$$+ \sum_{R \in \text{Ell}^+(\Delta) \Gamma_j^+} \text{tr}(V_{\Delta}^{e(k)}(R)) \times \frac{e^{-2\text{i}\theta(R) + \text{i}\theta(R)}}{\nu(e^{\text{i}\theta(R)} - e^{-\text{i}\theta(R)})}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \in \text{Hy}^\prime} \text{tr}(V_{\Delta}^{e(k)}(P)) \times \frac{N(P)^{-\kappa}}{1 - N(P)^{-1}}$$

$$- \frac{1}{2n_j} \sum_{j=1}^{h} \sum_{P \in \text{Par}_j^+(\Delta)/\Gamma_j^+} \text{tr}(V_{\Delta}^{e(k)}(P)) \times \begin{cases} 1 - \text{i cot} \frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \text{ mod } n_j \\ 1 & \cdots r(P) \equiv 0 \text{ mod } n_j \end{cases}$$

and

$$C_{\Gamma, \Delta}^*(k) = \frac{1}{4\pi} \text{vol}(\Gamma \backslash \mathfrak{H}) \text{tr}(V_{\Delta}^{e(k)}(1_{2}))(-2\kappa - 1)$$

$$+ \sum_{R \in \text{Ell}^+(\Delta) \Gamma_j^+} \text{tr}(V_{\Delta}^{e(k)}(R)) \times \frac{e^{-2\text{i}\theta(R) - \text{i}\theta(R)}}{\nu(e^{\text{i}\theta(R)} - e^{-\text{i}\theta(R)})}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \in \text{Hy}^\prime} \text{tr}(V_{\Delta}^{e(k)}(P)) \times \frac{N(P)^{\kappa}}{1 - N(P)^{-1}}$$

$$- \frac{1}{2n_j} \sum_{j=1}^{h} \sum_{P \in \text{Par}_j^+(\Delta)/\Gamma_j^+} \text{tr}(V_{\Delta}^{e(k)}(P)) \times \begin{cases} 1 + \text{i cot} \frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \text{ mod } n_j \\ 1 & \cdots r(P) \equiv 0 \text{ mod } n_j \end{cases}$$

where

$$\text{vol}(\Gamma \backslash \mathfrak{H}) = \int_{\mathfrak{H}} \eta^{-2} d\xi d\eta \quad (\xi =\text{Re} \tau, \eta = \text{Im} \tau).$$
We denote by $\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma))$ the trace of the action of $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\Gamma)$ and so on. Let $\Theta_{S,\Gamma}$ denote the space of theta functions $\theta(\tau, z)$ satisfying the following conditions:

(i) $\theta(\tau, z)$, as a function of $z$, is an element of $\Theta_{S,\Gamma}$

(ii) $\theta|_{l/2, S} M = \theta$ for any $M \in \Gamma$.

Then, $\Theta_{S,\Gamma}$ is isomorphic to the space $J_{l/2, S}(\Gamma)$ of Jacobi forms of weight $l/2$ with respect to $\Gamma$. The Hecke operator $H_{l/2, S, \Gamma}(\Delta)$ operates on $\Theta_{S,\Gamma}$. We have the following theorem.

**Theorem 2** Assume that $\Gamma$ is a congruence subgroup of $SL_2(\mathbb{Z})$ having the element $-1_2$. Let $k$ be an integer and $\Delta \subseteq G^J$ a finite union of $\Gamma^J$-double cosets.

(i) If $k > l/2 + 2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{cusp}(\Gamma)) = C_{\Gamma, \Delta}(k).$$

If $k < l/2 - 2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{cusp}(\Gamma)) = C_{\Gamma, \Delta}^{*}(k).$$

(ii) Assume that $l$ is odd. Denote by $\epsilon$ the sign $+$ or $-$ according as $l$ is congruent to 1 or 3 modulo 4 (i.e., $\epsilon = \epsilon((l-1)/2)$.

If $k = (l+3)/2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{cusp}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,e}(s) + C_{\Gamma,\Delta}(k).$$

If $k = (l+1)/2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,-e}(s).$$

If $k = (l-1)/2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{cusp}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,e}(s).$$

If $k = (l-3)/2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{cusp}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,-e}(s) + C_{\Gamma,\Delta}^{*}(k).$$

(iii) Assume that $l$ is even. Let $\epsilon = \epsilon(l/2)$.

If $k = l/2 + 2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{cusp}(\Gamma)) = \text{Res}_{s=1} \zeta_{\Delta,S,e}(s) + C_{\Gamma,\Delta}(k).$$

If $k = l/2 - 2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{cusp}(\Gamma)) = \text{Res}_{s=1} \zeta_{\Delta,S,e}(s) + C_{\Gamma,\Delta}^{*}(k).$$

If $k = l/2$, then,

$$\text{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) = \text{Res}_{s=1} \zeta_{\Delta,S,e}(s).$$
For the proof we use Fischer's resolvent trace formula [Fi] and the method of Skoruppa-Zagier [S-Z2]. We can deduce the following corollary from (ii), (iii) of the above theorem.

**Corollary .3** (i) Assume that $l$ is odd. Then,

$$
\text{tr}(H_{(l+3)/2,S,\Gamma}(\Delta), J_{(l+3)/2,S}^{\text{cusp}}(\Gamma)) = \text{tr}(H_{(l+1)/2,S,\Gamma}(\Delta), J_{(l+1)/2,S}^{*}(\Gamma)) + C_{\Gamma,\Delta}\left(\frac{l+3}{2}\right),
$$

$$
\text{tr}(H_{(l+3)/2,S,\Gamma}^{\text{skew}}(\Delta), J_{(l+3)/2,S}^{*}(\Gamma)) = \text{tr}(H_{(l+1)/2,S,\Gamma}(\Delta), J_{(l+1)/2,S}^{*}(\Gamma)) + C_{\Gamma,\Delta}^{*}\left(\frac{l-3}{2}\right).
$$

(ii) Assume that $l$ is even. Then,

$$
\text{tr}(H_{l/2+2,S,\Gamma}(\Delta), J_{l/2+2,S}^{\text{cusp}}(\Gamma)) = \text{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) + C_{\Gamma,\Delta}\left(\frac{l}{2} + 2\right),
$$

$$
\text{tr}(H_{l/2+2,S,\Gamma}^{\text{skew}}(\Delta), J_{l/2+2,S}^{*}(\Gamma)) = \text{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) + C_{\Gamma,\Delta}^{*}\left(\frac{l}{2} - 2\right).
$$

**Remark.** In the case of $l = 1$ the first identity of the above (i) has been already obtained by Skoruppa-Zagier[S-Z1]. The results in Theorem 2 and Corollary 3 are consistent with those of [S-Z1, 2].

**References**


[Su] Sugano, T., *Jacobi forms and the theta lifting*, preprint
