

On Computation of Calibers of Real Quadratic Fields

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1. The fields of the caliber 2

Let F be a real quadratic field. Each number $x \in F - \mathbb{Q}$ can be expressed in a unique way as continuous fraction:

$$x = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots}}} \quad (a_i \in \mathbb{Z}, a_i \geq 1 \text{ if } i \geq 2).$$

It is well-known that the sequence $\{a_1, a_2, \dots\}$ becomes periodic.

Let m be the period of x . There exists some positive integer i_0 such that $a_{i+m} = a_i$ for all $i \geq i_0$. An element x of F is called *reduced*

if $x > 1, 0 > x' > -1$. As is well-known x is reduced if and only if the

continuous fractional expansion is purely periodic, i.e. $a_{i+m} = a_i$ for

all $i \geq 1$. Let A be an ideal class of F (the usual ideal class). One

can choose a representative ideal \mathfrak{a} of A such that $\mathfrak{a} = \mathbb{Z} + \mathbb{Z}x$ with a

reduced number x . The period m of x depends only on the class A . We

write $\kappa(A)$ for m and call $\kappa(A)$ the *caliber* of A . We set

$$(1) \quad \kappa(F) = \sum_A \kappa(A),$$

where A runs over all ideal classes of F . Let D be the discriminant

of F . Then we write $\kappa(D)$ instead of $\kappa(F)$ also. The positive integer

$\kappa(F)$ is called the *caliber* of F . Moreover we define a subset DI of F by putting

$$(2) \quad DI = \{\theta \in F \mid \theta > 1, 0 > \theta' > -1 \text{ and } \mathbb{Z} + \mathbb{Z}\theta \text{ is an ideal of } F\}.$$

It is easy to prove that DI is a finite set and

$$\kappa(D) = \#(DI),$$

where $\#(DI)$ denotes the cardinality of the set DI . It is known by [4] that the number of real quadratic fields with a given caliber is finite. Each element θ of the set DI satisfies the quadratic equation

$$a\theta^2 - b\theta + c = 0 \quad (a, b, c \in \mathbb{Z}, a > 0)$$

with the condition $b^2 - 4ac = D$, and we get

$$(3) \quad \theta = \frac{b + \sqrt{D}}{2a},$$

Since $\theta \in DI$, the pair (a, b) satisfies the following condition

$$(4) \quad \begin{cases} b > 2a - \sqrt{D}, \sqrt{D} > b > -2a + \sqrt{D}, \\ b^2 \equiv D \pmod{4a}. \end{cases}$$

We set

$$(5) \quad S(D) = \{(a, b) \in \mathbb{Z} \mid b > 2a - \sqrt{D}, \sqrt{D} > b > -2a + \sqrt{D}, b^2 \equiv D \pmod{4a}\}.$$

Thus the caliber $\kappa(D)$ is the number of the pairs (a, b) of integers satisfying the condition (4), i.e. $\kappa(D) = \#S(D)$.

Let $h(D)$ be the *ideal class number* of the quadratic field $F = \mathbb{Q}(\sqrt{D})$ with discriminant D . Then we can prove

Proposition 1.

If $\kappa(D) \leq 3$, then $h(D) \leq 1$.

proof) By the definition (1) of $\kappa(D)$ and the fact $\mathfrak{m}(A) \geq 1$, it is clear that $h(D) \leq 3$. Suppose that $h(D) = 2$ (or 3). Let A and B (and C) be the unequivalent ideal classes of F . It is well known that $\mathfrak{m}(A) \equiv \mathfrak{m}(B) (\equiv \mathfrak{m}(C)) \pmod{2}$. Since $\kappa(D) = \mathfrak{m}(A) + \mathfrak{m}(B) + \mathfrak{m}(C)$, we can show a contradiction.

Proposition 2.

Let $F = \mathbb{Q}(\sqrt{D})$, the quadratic field with discriminant D and $\kappa(D) = 2$.

If $D \equiv 1 \pmod{4}$, then D is a product of two primes.

If $D \equiv 4 \pmod{8}$, then $D/4$ is a prime. If $D \equiv 0 \pmod{8}$, then $D/8$ is a prime.

proof)

From $\kappa(D) = 2$ and proposition 1, we have $h(D) = 1$, so that the caliber of the principal ideal class of F equals to 2. Therefore there exists a number $\alpha \in DI$ whose period is 2, and every number of F has even period. Hence the norm of the fundamental unit of F is +1. Therefore the class number $h^*(D)$ of F in the narrow sense is equal to $2h(D)$, i.e. $h^*(D) = 2^{2-1}$. Therefore number of different prime divisors of D is less than 2. From this we can show the assertions of the proposition.

Proposition 3.

Let the notation be as above. Then the following facts hold.

i) There exists a number $\alpha \in DI$ such that $\alpha = \frac{b + \sqrt{D}}{2}$ ($b \in \mathbb{Z}$) and b satisfies $b > 2 - \sqrt{D}$, $\sqrt{D} > b > -2 + \sqrt{D}$, $b^2 \equiv D \pmod{4}$ (see (2), (3), (4)).

ii) If $D > 16$ and $D \equiv 0$ or 1 or $4 \pmod{8}$, then there exists a number

$\alpha \in \mathbb{I}$ such that $\alpha = \frac{b + \sqrt{D}}{2 \cdot 2}$ ($b \in \mathbb{Z}$) and b satisfies $b > 4 - \sqrt{D}$,

$\sqrt{D} > b > -4 + \sqrt{D}$, $b^2 \equiv D \pmod{8}$ (see (2), (3), (4)).

iii) If $D > 36$ and $D \equiv 0$ or $4 \pmod{12}$, then there exists a number $\alpha \in \mathbb{I}$

such that $\alpha = \frac{b + \sqrt{D}}{2 \cdot 3}$ ($b \in \mathbb{Z}$) and b satisfies $b > 6 - \sqrt{D}$, $\sqrt{D} > b > -6 + \sqrt{D}$,

$b^2 \equiv D \pmod{12}$ (see (2), (3), (4)).

proof)

We prove only the case iii). Other cases can be proved by the same

method. In the case iii), we prove that there exists an integer b

satisfying the condition $b > 6 - \sqrt{D}$, $\sqrt{D} > b > -6 + \sqrt{D}$, $b^2 \equiv D \pmod{12}$, i.e.

$$\sqrt{D} > b > \sqrt{D} - 6, \quad b^2 \equiv D \pmod{12},$$

because $D > 36$. Suppose $D \equiv 0 \pmod{12}$. Then the condition $b^2 \equiv D$

$\pmod{12}$ is equivalent to $b \equiv 0 \pmod{6}$. It is easy to see that there

exists an integer b such that $\sqrt{D} > b > \sqrt{D} - 6$, and $b \equiv 0 \pmod{6}$. For

the case $D \equiv 4 \pmod{12}$, the condition $b^2 \equiv D \pmod{12}$ becomes $b \equiv 2$ or 4

$\pmod{6}$. We can find an integer b such that $\sqrt{D} > b > \sqrt{D} - 6$, $b \equiv 2$ or 4

$\pmod{6}$. Therefore the case iii) is completely proved.

Now we can prove

Theorem 1.

Let F be $\mathbb{Q}(\sqrt{D})$, the quadratic field with the discriminant D and let

the caliber $\kappa(D)$ of F be 2. If $D > 16$ and $D \equiv 1 \pmod{4}$, then there

exist two positive integer n, k satisfying $D = n^2 k^2 + 4k$ and $k \geq 3$. If

$D > 36$ and $D \equiv 0 \pmod{4}$, then $D = 36n^2 + 8$ for some positive integer n .

proof)

From proposition 3-i), we can find a number $\alpha = \frac{b + \sqrt{D}}{2} \in DI$ ($b \in \mathbb{Z}$).

Since $\kappa(D) = 2$, we get $h(D) = 1$ by proposition 1, so that the period of α is 2. Put $\alpha = \{l, n\}$, where $l, n \in \mathbb{Z}$, $l, n > 0$ and $l \neq n$. Then α satisfies the quadratic equation $n^2\alpha^2 - l\alpha - l = 0$. Hence

$\alpha = \frac{l + \sqrt{l^2 + 4(l/n)}}{2}$. Compare $\alpha = \frac{b + \sqrt{D}}{2}$ and the above expression of

Then we get $b = l$ and $D = l^2 + 4(l/n)$. It follows from $\alpha \in DI$ that $(b^2 - D)/4 \in \mathbb{Z}$, i.e. $l/n \in \mathbb{Z}$. Put $l = nk$ with a positive integer k . Since $n, l > 0$ and $n \neq l$, we get $k \geq 2$ and $D = n^2k^2 + 4k$. If $D \equiv 1 \pmod{4}$ then k is odd, so that $k \geq 3$. In the case $D \equiv 0 \pmod{4}$, suppose that $k \geq 3$ and put $\gamma = (nk + \sqrt{D})/(2k)$, a root of the equation $k\gamma^2 - nk\gamma - 1 = 0$. Then γ is an element of DI and $\gamma \neq \alpha$. From proposition 3-ii), there exists a number $\beta = (c + \sqrt{D})/4 \in DI$ ($c \in \mathbb{Z}$). It follows from $\alpha \neq \beta, \beta \neq \gamma$ and $\gamma \neq \alpha$ that $\kappa(D) = \#(DI) \geq 3$. This is a contradiction. Therefore we get $k = 2$ and $D = 4n^2 + 8$ for an integer n . Furthermore, from proposition 3-iii) and $\#(DI) = \kappa(D) = 2$, it is necessary that $D \equiv 8 \pmod{12}$. So that $D = 36n^2 + 8$ for an integer n .

From proposition 1, 2 and theorem 1, we get immediately

Proposition 4.

If $\kappa(D) = 2$, then $h(D) = 1$. Moreover we have the following facts.

i) If $D \equiv 1 \pmod{4}$ and $D > 16$, then there exist two integer n, k such

that $D = n^2 k^2 + 4k$ and that $k, n^2 k + 4$ are primes.

ii) If $D \equiv 0 \pmod{4}$ and $D > 36$, then there exists an integer n such that $D = 36n^2 + 8$. If n is odd, then $9n^2 + 2$ is a prime. If n is even, then $(9n^2/2) + 1$ is a prime.

Proposition 5.

i) If $D = n^2 k^2 + 4k \equiv 1 \pmod{4}$, then $\varepsilon = (nk + \sqrt{D})^2 / (4k)$ is a unit of $\mathbb{Q}(\sqrt{D})$.

ii) If $D = 36n^2 + 8$ then $\varepsilon = (6n + \sqrt{D})^2 / 8$ is a unit of $\mathbb{Q}(\sqrt{D})$.

The following theorem is proved in [7].

Theorem 2 (Tatsuzawa).

Let $\frac{1}{2} > t > 0$ and $N > \max(e^{(1/t)}, e^{11.2})$. Then

$$(6) \quad L(1, \chi) > 0.655 \frac{t}{N^t}$$

with one possible exception χ , where χ is a non-principal primitive real character of modulus N and $L(s, \chi)$ is the L-function with character χ .

Choose χ_D , the Kronecker character belonging to the quadratic field $\mathbb{Q}(\sqrt{D})$ as χ , $\frac{1}{15}$ as t . Then $N = D$ and we have the inequality

$$(7) \quad h(D) > \frac{0.655 D^{(13/30)}}{30 \log(\varepsilon_0)} \quad (\text{for } D > e^{15}),$$

where ε_0 is the fundamental unit of $\mathbb{Q}(\sqrt{D})$ (see [3]). If $\kappa(D) = 2$, then it follows from theorem 1 and proposition 5 that

$$\log(\mathfrak{S}_0) \leq \log(\mathfrak{S}) < \begin{cases} \log(D) - \log(3) & \text{if } D \equiv 1 \pmod{4} \text{ and } D > 16 \\ \log(D) - \log(2) & \text{if } D \equiv 0 \pmod{4} \text{ and } D > 36. \end{cases}$$

Therefore for $F = \mathbb{Q}(\sqrt{D})$ with $\kappa(D) = 2$, we have

$$h(D) > \frac{0.655D^{(13/30)}}{30(\log(D) - \log(2))} \quad (\text{for } D > e^{15})$$

with one possible exception D . By proposition 1, we can prove

Proposition 6.

Let F be the real quadratic field $\mathbb{Q}(\sqrt{D})$ with discriminant D . If $\kappa(D) = 2$ and $D > e^{14}$, then D satisfies the inequality

$$1 > \frac{0.655D^{(13/30)}}{30(\log(D) - \log(2))}$$

with one possible exception D .

Note that the above inequality does not hold for $D > e^{15}$ ($= 3269017.23\dots$). So if $\kappa(D) = 2$, then $D < 3269018$ with one possible exception. If $\kappa(D) = 2$, then D satisfies the conditions represented in proposition 4. For these D , by aid of a computer, we can easily calculate $h(D)$ and $\kappa(D)$. Hence we get

Theorem 3.

Let F a the real quadratic field with the discriminant D . If the caliber $\kappa(D)$ of $F = 2$, then $D = 4 \cdot 3, 4 \cdot 6, 4 \cdot 11, 21, 4 \cdot 38, 77, 4 \cdot 83, 93, 4 \cdot 227, 237, 437, 453, 1133, 1253$ or one more possible $D_0 (> 1253)$.

2. The fields with caliber 3

Now we consider quadratic fields with caliber 3. Let F be a

quadratic field $\mathbb{Q}(\sqrt{D})$ of the discriminant D .

Proposition 7.

If the caliber $\kappa(D)$ of $F=\mathbb{Q}(\sqrt{D})$ is equal to 3, then the discriminant D is a prime and $D \equiv 1 \pmod{4}$.

proof)

By the condition $\kappa(D)=3$, we get $h(D)=1$ (see Proposition 1). So that there exists some number α in DI whose period is 3 by using the definition (1) of $\kappa(D)$. Hence the norm of the fundamental unit of F is -1 . By the same way as the proof of proposition 2, the number of different primes of D is 1, i.e. D is a prime. Since the discriminant D is a prime, we get $D \equiv 1 \pmod{4}$.

For a prime D which satisfies $D \equiv 1 \pmod{4}$ and a prime p , we define a symbol $\left(\frac{D}{p}\right)^*$ as follows:

if $p > 2$, then $\left(\frac{D}{p}\right)^*$ is usual Legendre symbol $\left(\frac{D}{p}\right)$,

if $p=2$, then $\left(\frac{D}{p}\right)^* = \begin{cases} 1 & D \equiv 1 \pmod{8}, \\ -1 & D \equiv 5 \pmod{8}. \end{cases}$

We denote

$$P(D) = \{p: \text{primes} \mid p < \frac{\sqrt{D}}{2}, \left(\frac{D}{p}\right)^* = 1\},$$

we put

$$\pi(D) = \#(P(D)).$$

Proposition 8.

If the discriminant D of $F=\mathbb{Q}(\sqrt{D})$ is a prime satisfying $D \equiv 1 \pmod{4}$, then $\kappa(D) \geq 1 + 2\pi(D)$.

proof)

By proposition 3-i) there exists a pair $(1, b)$ which satisfies the condition (4). If a prime p is an element of $P(D)$, then there exist two b 's such that each pair (p, b) is elements in $S(D)$ (see (5)). Hence we obtain $\kappa(D) \geq 1 + 2\pi(D)$.

Theorem 4.

Let D be the discriminant of $F = \mathbb{Q}(\sqrt{D})$ and the caliber $\kappa(D)$ be equal to 3. Put $D = 4d + 1$ (see Proposition 7). Then d is written by p , p^2 , p^3 , pq or p^2q for odd primes p and q ($p \neq q$).

proof)

Assume $d = p_1 p_2 p_3 \ell$, where p_1, p_2 and p_3 ($p_1 < p_2 < p_3$) are primes and ℓ is a positive integer. Then it is easy to see that p_1 and p_2 are elements of $P(D)$. So that $\kappa(D) \geq 1 + 2\pi(D) \geq 5$. Assume $d = p_1^{e_1} p_2^{e_2} \ell$, where p_1 and p_2 are primes, e_1, e_2 and ℓ are positive integer such that $e_1, e_2 \geq 2$. Then p_1 and p_2 are elements of $P(D)$, i.e. $\kappa(D) \geq 5$. Assume $D = p^e \ell$, where p is a prime, e and ℓ are positive integers such that $e \geq 4$. Then the fact $p \in P(D)$ holds. Furthermore there exists two elements (p^2, b) in $S(D)$. Hence we get the fact $\kappa(D) \geq 5$ for $D = 4p^e \ell + 1$ ($e \geq 1$). Assume that d is even. If we put $d = 2f$ ($0 < f \in \mathbb{Z}$), then $d = 8f + 1$. So that we get $D \equiv 1 \pmod{8}$ and $D \equiv 1$ or $9 \pmod{16}$. For $D \equiv 1 \pmod{8}$, we can see that there exist two pairs $(2, b)$ which satisfy (4) by proposition 3-ii). Further it is easy to show that for $D \equiv 1$ or $9 \pmod{16}$, there exist two pairs $(4, b)$ which are elements in $S(D)$.

Thus we have obtained $\kappa(D) \geq 5$ for even d and proved the assertion of the proposition.

Proposition 9.

If $\kappa(D) \geq 3$, then there exists a number $\alpha = \frac{b + \sqrt{D}}{2} \in DI$ ($b \in \mathbb{Z}$) (Proposition 3-i)). Moreover $\alpha = \{l, n, n\}$, where l, n are positive integers and $n^2 + 1$ divides $ln + 1$.

proof) It follows from $\kappa(D) = 3$ and $h(D) = 1$ (see Proposition 1) that $\alpha = \{l, m, n\}$, where l, m and n are positive integers excluding the case $l = m = n$. Therefore

$$\alpha = l + \frac{1}{m + \frac{1}{n + \frac{1}{\alpha}}}, \text{ i.e.}$$

α satisfies the equation

$$(mn+1)\alpha^2 + (m-l-n-lmn)\alpha - (lm+1) = 0.$$

We obtain

$$\alpha = \frac{l + \frac{n-m}{mn+1} \pm \sqrt{\left(1 + \frac{n-m}{mn+1}\right)^2 + 4\left(\frac{lm+1}{n^2+1}\right)}}{2}$$

Compare above representation of α and $\alpha = \frac{b + \sqrt{D}}{2}$. Then we get $m = n$, $l \neq m$, $n^2 + 1 \mid lm + 1$. Therefore we obtain

$$\alpha = \frac{l + \sqrt{l^2 + \frac{ln+1}{n^2+1}}}{2} \text{ and } \alpha = \{l, n, n\} \text{ (} l \neq n \text{)}.$$

By the equality

$$l + \frac{1}{n + \frac{1}{n + \frac{1}{\alpha}}} = \frac{\{l(n^2+1)+n\}\alpha + (ln+1)}{(n^2+1)\alpha + n}$$

and Theorem 3.9 in [6, p.215], we can prove immediately

Proposition 10.

Let the notation be as above. Then $(n^2+1)\alpha+n$ is a unit of $\mathbb{Q}(\sqrt{D})$.

Put $\mathfrak{E}=(n^2+1)\alpha+n$. Then it follows from $D=l^2+4(ln+1)/(n^2+1)$ and $\alpha=(1+\sqrt{D})/2$ that $\log \mathfrak{E} < \frac{3}{2} \log D$. In the unequation (6), put $x=x_D$ and $t=\frac{1}{16}$. Then we get

$$h(D) > \frac{0.655D^{7/16}}{32 \log \mathfrak{E}_0}$$

So that we can prove, by proposition 1,

Proposition 11.

Let F be the real quadartic field $\mathbb{Q}(\sqrt{D})$ with the discriminant D . If

$\kappa(D)=3$ and $D > e^{16}$, then D satisfies the inequality

$$1 > \frac{0.655D^{\frac{3}{7}}}{48 \log D}$$

with one possible exception D .

It is easy to show that the above inequality does not hold for $D > e^{16.18}$ ($=10638605.805\dots$). So that if $\kappa(D)=3$, then $D < 10638617$ and D satisfies the condition represented in Theorem 4. Furthermore by proposition 8, and $\kappa(D)=3$, we get $\#P(D)=0$ or 1. By aid of a computer, we get

Theorem 5.

Let F be the real quadratic field with discriminant D . If the

caliber $\kappa(D)$ of F is equals to 3, then $D=17, 37, 61, 101, 197, 317, 461, 557, 677, 773, 1877$ or one more possible $D_0 (>1877)$. Cororally (of Theorem 3 and Theorem 5)

Let the notation be as above. Then at least two of the following assertions hold.

i) $\kappa(D)=1$ if and only if $D=4\cdot 2, 5, 13, 29, 53, 173, 293$.

ii) $\kappa(D)=2$ if and only if $D=4\cdot 3, 4\cdot 6, 4\cdot 11, 21, 4\cdot 38, 77, 4\cdot 83, 93, 4\cdot 227, 237, 437, 453, 1133, 1253$.

iii) $\kappa(D)=3$ if and only if $D=17, 37, 61, 101, 197, 317, 461, 57, 677, 773, 1877$

proof)

If $\kappa(D)=1$, then it follows from § 3 in [4] and theorem 2 in [3] that $D=4\cdot 2, 5, 13, 29, 53, 173, 293$ or one more possible $D_0 (>293)$. D_0 is same as in Theorem 3 and Theorem 5 if it exists.

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```

LISP FLUID(' (DD));
LISP FLUID(' (FF));
LISP FLUID(' (EE));
LISP FLUID(' (LSTAB));
LISP FLUID(' (EE2));

```

```

SYMBOLIC PROCEDURE ISQRT(I);
BEGIN
  IF I<0 THEN RETURN NIL;
  IF I=0 THEN RETURN 0;
  IF I<4 THEN RETURN 1;
  RETURN ISQRT1(QUOTIENT(I, 2), 1);
END;

```

```

SYMBOLIC PROCEDURE ISQRT1(J, I);
BEGIN
  INTEGER K;
  K:=QUOTIENT(I, J);
  IF K<J THEN RETURN ISQRT1(QUOTIENT(J+K, 2), I);
  RETURN J;
END;

```

```

SYMBOLIC PROCEDURE ABLSTO(DD, EE, EE2, BB);
BEGIN
  INTEGER Y, A, I, B;
  SCALAR LL, LSTAB;
  I:=1;
  B:=BB;
  LSTAB:=NIL;

F1:
  Y:=(DD-B**2)/4;

  IF NOT (Y>0) THEN <<FF:=1;RETURN LSTAB>>;
  A:=QUOTIENT(EE-B, 2);

F2:
  A:=A+1;
  IF A>(EE+B)/2 THEN <<B:=B+2;GOTO F1>>;
  IF REMAINDER(Y, A) NEQ 0 THEN GOTO F2;
  LL:=CONS(A, B);
  LSTAB:= CONS(LL, LSTAB);
  I:=I+1;
  GOTO F2;

END;

```

```

SYMBOLIC PROCEDURE EQUIVO(LL, EE, EE2);
BEGIN
  SCALAR N, R, H, AB, A, B, LL1, ABO;
  H:=0;

F1:
  IF NULL(LL) THEN RETURN (-1)**N*H;
  AB:=CAR(LL);
  LL:=CDR(LL);

F2:

```

```
H:=H+1;
N:=0;
A:=CAR(AB);
B:=CDR(AB);
```

```
F3:
```

```
N:=N+1;
A:=(DD-B**2)/(4*A);
R:=REMAINDER(EE+B, 2*A);
B:=EE-R;
IF A>(EE+B)/2 THEN GOTO F3;
```

```
ABO:= A . B;
```

```
LL1:=TAKEOUT(LL, ABO);
```

```
IF LL1=LL THEN GOTO F1 ELSE <<LL:=LL1; GOTO F3>>;
```

```
END;
```

```
SYMBOLIC PROCEDURE TAKEOUT(L, A);
```

```
BEGIN
```

```
SCALAR LT, B;
```

```
LOOP:
```

```
IF (MEMBER(A, L) = NIL) THEN RETURN REVERSE(APPEND(REVERSE(L), LT));
```

```
B:=CAR(L);
```

```
L:=CDR(L);
```

```
IF A NEQ B THEN LT:=CONS(B, LT);
```

```
GO TO LOOP;
```

```
END;
```

```
SYMBOLIC PROCEDURE SQFREE(MK);
```

```
BEGIN
```

```
integer i, i2;
```

```
IF REMAINDER(MK, 4) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 9) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 25) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 49) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 121) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 169) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 289) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 361) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 529) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 841) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 961) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 1369) = 0 THEN RETURN NIL;
```

```
IF REMAINDER(MK, 1681) = 0 THEN RETURN NIL;
```

```
i:=47;
```

```
loop:
```

```
i2:=i*i;
```

```
if i2>mk then return t;
```

```
if gcd(mk, i2)=i2 then return nil;
```

```
i:=i+2;
```

```
goto loop;
```

```
END;
```

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SYMBOLIC PROCEDURE CALB(DIS, PF, XX);

BEGIN

INTEGER B, DD, DDO, EE, EE2;

SCALAR LL, PF;

DDO:=REMAINDER(DIS, 4);

IF DDO=0 THEN RETURN NIL;

IF DDO=1 THEN <<DD:=DIS;B:=1>> ELSE <<DD:=4*DIS;B:=2>>;

EE:=ISQRT(DD);

EE2:=EE/2;

LL:=ABLSTO(DD, EE, EE2, B);

IF XX=T THEN WRITE("h=", EQUIVO(LL, EE, EE2), " ");

IF PF=T THEN RETURN <<WRITE(LL);TERPRI();LENGTH(LL)>>;

RETURN <<write("caliber=");LENGTH(LL)>>;

END;

SYMBOLIC PROCEDURE CICALB(M, N, CLP);

BEGIN

INTEGER KK;

KK:=M;

LOOP:

IF KK>N THEN RETURN;

IF NOT SQFREE(KK) THEN <<KK:=KK+1;GOTO LOOP>>;

IF CLP=NIL THEN <<WRITE("d=", KK, " ", CALB(KK, NIL, NIL));TERPRI();>>

ELSE <<WRITE("d=", KK, " ", CALB(KK, NIL, T));TERPRI();>>;

KK:=KK+1;

GOTO LOOP;

END;

SYMBOLIC OPERATOR CALB;

SYMBOLIC OPERATOR CICALB;

CICALB(10, 100, T);

CICALB(10, 100, nil);

END;