A Class of Logic Functions
Expressible by Polynomial-Size Binary Decision Diagrams

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Abstract

In this paper we discuss properties of logic functions expressible by BDD's of feasible size. We define a class of logic functions expressible by BDD's whose size (number of nodes) are bounded by a polynomial of the number of input variables. We derive some properties of this class through the discussion on the relation between polynomial-size BDD's and Turing machines. We also focus on the relation between polynomial-size BDD's and combinational circuits and show that polynomial size BDD's can be synthesized into $O(n^{2})$ depth combinational circuits.

1 Introduction

A binary decision diagram (BDD) [Bry86] is one of representation forms of logic functions. It is widely used in application programs for logic design verification, test generation and logic synthesis, owing to its properties that there exists a unique canonical form for each logic function and that many practical logic functions are expressible by BDD's of feasible size. However, in the worst case the size of the BDD of a logic function is known to be exponential of the number of input variables. There are few discussions on a problem of what kind of logic functions are expressible by BDD's of feasible size [Bry90]. In this paper we focus on this point so as to clarify the efficiency and limitation of application programs using logic function manipulation based on BDD's. We define a class of logic functions expressible by BDD's whose size (the number of nodes) is bounded by a polynomial of the number of input variables. We derive some properties of this class through the discussion on the relation between polynomial-size BDD's and Turing machines. We also focus on the relation between polynomial-size BDD's and combinational circuits and show an upper bound of the depth of combinational circuits that realize logic functions expressed by polynomial-size BDD's. Main results of this paper are as follows:

(1) Let PolyBDD be a class of logic functions expressible by uniform BDD's whose size is bounded by a polynomial of $n$, where $n$ is the number of the input variables. Let LOGIREG be a class of logic functions which is computable by an LSIA, where an LSIA is a one-way off-line Turing machine which has $O(n)$ bounded working tape and has an ability to know $n$ without reading the input tape. We have shown PolyBDD = LOGIREG.

(2) From the properties of LSIA's we derive that symmetric functions, threshold functions and selector functions etc. are in PolyBDD.

(3) PolyBDD is shown to be included by DLOGSPACE directly from the definition of an LSIA. Since it is know that DLOGSPACE is included by $NC^{2}$ (a class of logic functions realized by uniform combinational circuits of depth $O(n^{2})$), we can conclude that logic functions
expressed by polynomial-size BDD's can be synthesized into $O(\log^2 n)$ depth combinational circuits. We also show a concrete procedure of constructing an $O(\log^2 n)$ depth combinational circuit from a polynomial size BDD.

2 Family of Binary Decision Diagrams

2.1 Binary Decision Diagram (BDD)

We define a binary decision diagram (BDD) over $B = \{0, 1\}$ as follows.

Def 2.1 A binary decision diagram over $B$ is a 6-tuple $B = (X, N, s, l, e^0, e^1)$, where

- $X = \{x_1, x_2, \cdots, x_n\}$ is a totally ordered set of variables, where $x_1 < x_2 < \cdots < x_n$ holds,
- $N$ is a set of nodes,
- $s \in N$ is the initial node,
- $l : N \to (X \cup B)$ represents the label of a node,
- $e^0, e^1 : N \to N$ represents a set of 0-edges and 1-edges, respectively, where

\[
\forall v \in N \text{ s. t. } l(v) \in X : l(e^0(v)) \in B \text{ or } l(v) < l(e^0(v)), \text{ and } l(e^1(v)) \in B \text{ or } l(v) < l(e^1(v)), \text{ and }
\]

\[
\forall v \in N \text{ s. t. } l(v) \in B : e^0(v) = e^1(v) = v. \quad \square
\]

The set $N$ is divided into two subsets; $N_V$ and $N_R$, where $N_V = \{v \mid v \in N, l(v) \in X\}$ and $N_R = \{v \mid v \in N, l(v) \in B\}$. $(N = N_V \cup N_R$ and $N_V \cap N_R = \emptyset$). A node in $N_V$ is called a variable node and a node in $N_R$ is called a value node. $N_R$ consists of just two nodes; $r_0$ and $r_1$, where $l(r_0) = 0$ and $l(r_1) = 1$. We call $r_0$ and $r_1$ the 0-node and the 1-node, respectively. We denote the index of a variable $x_i$ in $X$ as $i(x_i)$. Namely $i(x_i) = i$. A pair of nodes $(v, e^0(v))$ and $(v, e^1(v))$ are called the 0-edge and 1-edge of node $v$. The size of BDD $B$, denoted as size($B$), is defined as the number of nodes of $B$. Namely, size($B$) = |$N$|, where |$N$| is the number of elements in set $N$.

Let $A$ be a set of assignments for $X$, where assignment $a$ for $X$ is a vector in $B^{|X|}$. We denote the $i$-th element of $a$ as $a_i$ and call it an assignment to variable $x_i$ (the $i$-th variable). For a given assignment $a$, we define $T(a)$ which represents the set of nodes reachable from $s$ under the assignment $a$.

\[v \in T(a) \text{ iff } v = s \text{ or } \exists u \in T(a) \text{ s. t. } e^{i_{i(l(v))}}(u) = v.\]

The output value of the logic function represented by a BDD is defined using this set of reachable nodes.

Def 2.2 We define the following $f_B$ as the logic function expressed by BDD $B$:

\[f_B(a) = 1 \text{ iff } r_1 \in T(a). \quad \square\]

Next, we define a description of a BDD. We assume that each element $e$ in $X$ and $N$ has an identifier which is denoted as id($e$). Let us call 4-tuple $D_v = (id(v), id(l(v)), id(e^0(v)), id(e^1(v)))$ a description of node $v \in N$. We can describe all the information of a BDD as the set of descriptions of the nodes of the BDD.

Def. 2.3 We define the $D_B$ as the node set description of BDD $B$, where $D_B = \{D_v \mid v \in N\}$. \square
2.2 BDD Family and Its Uniformity

In order to discuss the size of BDD’s with respect to the number of input variables, we define a family of BDD’s.

Def 2.4 BDD family \( \{B_n\} \) is a sequence of BDD’s \( B_1, B_2, B_3, \cdots \), where \(|X_n| = n\) holds for each \( B_n = (X_n, N_n, s_n, l_n, e_0^n, e_1^n) \).

Let \( \{f_n\} \) be a sequence of logic functions where \( f_n : B^n \rightarrow B \) (an n-variable logic function). We can consider that \( \{f_n\} \) expresses a language \( L \) over \( B \) by the following correspondence:

\[
b_1 b_2 \cdots b_n \in L \iff f_n(b_1, b_2, \cdots, b_n) = 1.
\]

Similarly we define a language for a BDD family.

Def 2.5 The language accepted by BDD family \( \{B_n\} \) is defined as follows and denoted as \( L_{\{B_n\}} \):

\[
b_1 b_2 \cdots b_n \in L_{\{B_n\}} \iff f_{B_n}(b_1, b_2, \cdots, b_n) = 1.
\]

In this paper, we discuss correspondence between BDD families and Turing machines. For this purpose we define uniformity of a BDD family following after the uniformity of a combinational circuit family [Ruz81].

Def 2.6 BDD family \( \{B_n\} \) is uniform if the description of the n-th BDD \( B_n \) can be generated from a binary representation of \( n \) by an \( O(\log \text{size}(B_n)) \) space bounded off-line Turing machine.

As a class of languages accepted by a BDD family of feasible size, we define a class of PolyBDD.

Def 2.7 PolyBDD is a class of languages accepted by a uniform BDD family \( \{B_n\} \), which satisfies \( \text{size}(B_n) \leq \text{poly}(n) \), where \( \text{poly}(n) \) is a polynomial of \( n \).

3 Relation between Polynomial Size BDD’s and Log-Space Automata

3.1 Log-Space Automaton

We will refer a one-way off-line Turing machine with \( O(\log n) \) bounded working tape as a log space automaton (LSA). An input to an LSA is given on its input tape which is read-only. An LSA can read the symbols on the input tape only once in the given order (one-way). Instead, an LSA can read and write the working tape. Namely, an LSA can be regarded as an automaton provided with \( O(\log n) \) working tape. We also define an abstract machine referred to as a log space input-size-look-ahead automata (LSIA): An LSIA is an LSA which has an ability to know the length of a given input sequence without scanning the input sequence. The length of an input sequence is given as the initial value on the working tape in binary representation.

Def 3.1 LOGREG and LOGIREG are classes of languages which can be accepted by an LSA and an LSIA, respectively.
The main result of this paper is that PolyBDD is equivalent to LOGIREG.

**Th 1** PolyBDD = LOGIREG.

(LOGIREG \subseteq PolyBDD): 
Since an LSIA has only \(O(\log n)\) memory, all the states of an LSIA can be represented by \(p\) nodes where \(p \leq \text{poly}(n)\). Then using \(p \times n\) nodes we can construct a state transition diagram without loop, which is the very \(n\)-th BDD. Since transitions are computable by an \(O(\log n)\) space bounded Turing machine, the BDD family is uniform.

(PolyBDD \subseteq LOGIREG): 
Using an \(O(\log n)\) working tape, an LSIA can simulate the \(O(\log n)\) space bounded Turing machine to generate a member of a uniform BDD family. In other words an LSIA can compute the next node of a node for a given assignment. Since the number of nodes in a BDD is bounded by a polynomial of \(n\), \(O(\log n)\) space is enough to reach final node from the initial node \(s\). \(\Box\)

### 3.2 Properties of PolyBDD

We can lead properties of the logic functions represented by polynomial size BDD's directly from properties of LSIA's.

Let REG and DLOGSPACE be classes of languages which can be accepted by a finite automaton and a log-space Turing machine (an off-line Turing machine with \(O(\log n)\) bounded working tape), respectively. Obviously \(\text{REG} \subseteq \text{LOGIREG} \subseteq \text{DLOGSPACE}\). The difference between \(\text{REG}\) and \(\text{LOGIREG}\) is due to the working tape of an LSIA and the difference between \(\text{LOGIREG}\) and \(\text{DLOGSPACE}\) is due to the restriction that an LSIA can read the input tape only once. Logic functions which can be computed by a sequential machine with constant number of registers, such as parity and carry, are expressible by polynomial size BDD's. An LSA has an \(O(\log n)\) working tape besides a finite control. Using this working tape, an LSA can accept more complex languages.

1. \(\{0^n1^n\}\) belongs to PolyBDD. 
   With the \(O(\log n)\) working tape, an LSA can count and compare the number of 0's and 1's in a given sequence.

2. All the symmetric functions belong to PolyBDD. The output of a symmetric function depends only on the number of 1's in the inputs. Then it is computable using the \(O(\log n)\) working tape.

3. Threshold functions belong to PolyBDD if the magnitude of each weight is bounded by a polynomial of \(n\).

4. Let \(\text{int}(w)\) be the integer value of binary representation \(w\), \(\text{weight}(\sigma)\) be the number of 1's in sequence \(\sigma\), and \(\sigma[k]\) be the \(k\)-th alphabet of \(\sigma\). Then the selector function 
   \[\{w \mid |w| = \lfloor \log |\sigma| \rfloor, \sigma[|w| + \text{int}(w) + 1] = 1\}\] belongs to PolyBDD.

The essence of the above properties is that an LSIA can count number up to \(\text{poly}(n)\) using the \(O(\log n)\) working tape. It can count the position or the number of 1's in a given sequences. However it can not memorize a whole input sequence itself because that requires working tape of length \(O(n)\).
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Figure

BDD.

v_1 (=s)

v_2

v_3

v_4 (=r_0)

v_5 (=r_1)

Figure 1: Relations among classes.

Figure 2: A BDD.

(5) \{ww \mid w \in B^*\} and \{ww^R \mid w \in B^*\} (where \ w^R \ is \ a \ reversed \ sequence \ of \ w) \ does \ not
belong \ to \ PolyBDD.

In \ order \ to \ accept \ ww \ by \ scanning \ an \ input \ sequence \ only \ once, \ the \ first \ half \ of \ the
sequence \ must \ be \ memorized, \ which \ requires \ an \ O(n) \ working \ tape.

(6) \ Let \ x_1x_2\cdots x_k \circ \ y_1y_2\cdots y_k = x_1y_1x_2y_2\cdots x_ky_k \ where \ x_i, y_i \in B. \ Then \ the \ shift \ function
defined \ as \ \{sw \mid w = (x_1x_2\cdots x_k) \circ (0^{in1(\ell)}x_1x_2\cdots x_k-in1(\cdot))\} \ does \ not \ belong \ to \ PolyBDD.

Another \ example \ is \ that \ the \ selector \ function \ in \ (4) \ does \ not \ belong \ to \ PolyBDD \ if \ we
define \ the \ selector \ function \ as \ \{sw | \cdots\}. \ This \ is \ also \ an \ example \ to \ show \ that \ the \ size \ of \ the
BDD \ representing \ the \ same \ function \ can \ vary \ depending \ on \ ordering \ of \ input \ variables.

The \ property \ predicts \ that \ it \ may \ be \ difficult \ to \ express \ logic \ functions \ computed \ by
sequential \ circuits \ even \ if \ the \ logic \ functions \ of \ their \ combinational \ part \ are \ expressible \ by
BDD's \ of \ feasible \ size. \ If \ an \ output \ of \ a \ sequential \ circuit \ depends \ on \ input \ sequences \ of
length \ O(n) \ and \ the \ number \ of \ its \ registers \ is \ larger \ than \ O(log n), \ there \ can \ be \ cases \ where \ the
sequential \ function \ does \ not \ belong \ to \ PolyBDD \ even \ if \ the \ combinational \ function \ belongs \ to \ PolyBDD.

4 \ Relation \ between \ BDD's \ and \ Combinational \ Circuits

We \ also \ investigated \ the \ relation \ between \ PolyBDD \ and \ other \ classes \ related \ to \ combinational
circuits. \ Figure \ 1 \ is \ the \ summary \ of \ relation \ among \ the \ classes. \ NC^k \ is \ a \ class \ of \ logic \ functions
which \ can \ be \ expressed \ by \ a \ uniform \ family \ of \ combinational \ circuits \ of \ depth \ O(log^k n) \ and
size \ O(poly(n)) \ under \ fan-in \ restriction \ [Coo85]. \ Since \ PolyBDD \ is \ included \ by \ DLOGSPACE
and \ DLOGSPACE \ is \ included \ by \ NC^2, \ we \ can \ conclude \ that \ logic \ functions \ expressed \ by \ poly-
nomial \ size \ BDD's \ can \ be \ synthesized \ into \ polynomial \ size \ and \ O(log^2 n) \ depth \ combinational
circuits. \ We \ show \ a \ constructive \ proof.

As \ is \ formalized \ in \ section \ 2, \ the \ function \ represented \ by \ a \ BDD \ is \ defined \ as \ the \ reachabil-
ity \ problem \ on \ the \ BDD. \ We \ will \ construct \ a \ combinational \ circuit \ which \ solves \ the \ reachability
problem. \ For \ a \ given \ BDD \ B \ we \ define \ \{N\} \times \{N\} \ matrix \ A_B = [a_{i,j}] \ as \ follows. \ Intuitively \ a_{i,j}
becomes \ 1 \ if \ node \ v_j \ is \ directly \ reachable \ from \ node \ v_i \ under \ a \ given \ assignment. \ We \ call \ A_B \ the
adjacency matrix \ of \ B.

\[a_{i,j} : B^n \rightarrow B, \text{ where}\]
\[a_{i,j} = \overline{x}_k \text{ if } l(v_i) = x_k, \text{ and } e^0(v_i) = v_j,\]
\[a_{i,j} = \overline{x}_k \text{ if } l(v_i) = x_k, \text{ and } e^1(v_i) = v_j,\]
If $l(v_i) \in R$, 
$a_{i,j} = 1$ otherwise.

For example, the adjacency matrix of the BDD in Figure 2 is

$$A_B = \begin{bmatrix}
0 & x_1 & \overline{x}_1 & 0 & 0 \\
0 & 0 & \overline{x}_2 & 0 & x_2 \\
0 & 0 & 0 & \overline{x}_3 & x_3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

Let us denote the $(i, j)$-element of $A_B^n$ (the $n$-th power of $A_B$) as $a^n_{i,j}$ and let $v_s = s$ and $v_r = r_1$. Then $f_B \equiv a^n_{s,r}$ by definition. We will show how to construct a combinational circuit of depth $O(\log^2 n)$ which computes $A_B^n$. A combinational circuit which computes $A_B$ can be realized according to the above definition. Multiplication of $m \times m$ Boolean matrix is computable by a combinational circuit of depth $O(\log m)$ and size $O(m^3)$. By constructing a tree of multiplication circuits we can compute the $n$-th power of $A_B$. Since the depth of the tree is $\lceil \log n \rceil$, the total depth of the circuit is $O(\log n \log m)$. If the size of the BDD is bounded by a polynomial of $n$, it comes to $O(\log^2 n)$.

As for the relation between PolyBDD and $NC^1$ we have not yet obtained significant results. There exists a logic function family which belongs to $NC^1$ but not to PolyBDD. Integer multiplication belongs to $NC^1$ but is known to require exponential nodes in BDD representation [Bry90]. Therefore, there are two possibility; PolyBDD $\subset NC^1$, or PolyBDD and $NC^1$ are incomparable. This problem has a significant importance on the synthesis of multilevel circuits because polynomial size BDD’s can be synthesized into $O(\log n)$ depth combinational circuits if PolyBDD $\subset NC^1$.

5 Conclusion

We have defined a class of logic functions expressible by polynomial-size BDD’s and have investigated its properties through discussions on relation between log-space automata. We also discussed the relationship between BDD’s and combinational circuits and showed a concrete procedure to synthesize $O(\log^2 n)$ depth combinational circuits form polynomial size BDD’s. It remains as a future work to clarify the relation between PolyBDD and $NC^1$.

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References

