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Kyoto University
Shape Optimization in Multi-Phase Stefan Problem

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1. Formulation of the optimization problem

Let us consider the enthalpy formulation of Stefan problem described as follows:

$$
SP(\Omega) \left\{
\begin{array}{ll}
u_t - \Delta \beta(u) = f & \text{in } Q(\Omega) := (0,T) \times \Omega, \\
u(0, \cdot) = u_0 & \text{in } \Omega, \\
\beta(u) = g & \text{on } \Sigma(\Omega) := (0,T) \times \partial \Omega,
\end{array}
\right.
$$

where $\hat{\Omega}$ is a fixed smooth bounded domain in $R^N (N \geq 2)$, and $\Omega$ is a smooth subdomain of $\hat{\Omega}$, $0 < T < \infty$, $\hat{Q} := (0,T) \times \hat{\Omega}$ and $\hat{\Sigma} := (0,T) \times \partial \hat{\Omega}$; $\beta : R \rightarrow R$ is a nondecreasing function on $R$ such that

$$
\beta(0) = 0, |\beta(r)| \geq C_0 |r| - C_0', \quad |\beta(r) - \beta(r')| \leq L_0 |r - r'| \quad \text{for all } r, r' \in R,
$$

(1.1)

where $C_0 > 0$, $C_0' \geq 0$ and $L_0 > 0$ are constants. Here we suppose that $f \in L^2(\hat{Q})$, $g \in W^{2,2}(0,T; L^2(\hat{\Omega})) \cap L^2(0, T; H^2(\hat{\Omega}))$ and $u_0 \in L^2(\hat{\Omega})$. In this paper, $u$ represents the enthalpy and $\beta(u)$ the temperature.

Now we give the weak formulation of $SP(\Omega)$.

**DEFINITION 1.1.** A function $u : [0,T] \rightarrow L^2(\Omega)$ is a weak solution of $SP(\Omega)$, if the following three conditions $(w1) - (w3)$ are satisfied:

$(w1)$ $u \in C^0_w([0,T]; L^2(\Omega))$, $u(0) = u_0$;

$(w2)$ $\beta(u) \in L^2(0,T; H^1(\Omega))$ and $\beta(u) - g \in L^2(0,T; H^1_0(\Omega))$;

$(w3)$ $- \int_{Q(\Omega)} u \eta_t dx dt + \int_{Q(\Omega)} \nabla \beta(u) \nabla \eta dx dt = \int_{Q(\Omega)} f \eta dx dt$

for all $\eta \in L^2(0,T; H^1_0(\Omega))$ with $\eta_t \in L^2(Q(\Omega))$ and $\eta(0, \cdot) = \eta(T, \cdot) = 0$.

**REMARK 1.1.** (1) In $(w3)$ of Definition 1.1, it is enough to take as test function $\eta$
any smooth function of the form $\rho z$, with $\rho \in D(0,T)(= \{\rho \in C^\infty(R); \text{supp } \rho \subset (0,T)\})$ and $z \in H^1_0(\Omega)$.

(2) We denote by $C_w([0,T]; L^2(\Omega))$ the space of all weakly continuous functions from $[0,T]$ to $L^2(\Omega)$ and by $\langle \cdot, \cdot \rangle_{\Omega}$ the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Now we introduce the notion of convergence of closed convex sets in a Banach space $X$, which is due to Mosco [13]. Let $\{K_n\}$ be a sequence of closed convex sets in $X$ and $K$ be a closed convex set in $X$. Then we say "$K_n \to K$ in $X$ as $n \to \infty$ (in the sense of Mosco)" if the following two conditions (M1) and (M2) are satisfied:

(M1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_{n_k} \in K_{n_k}$, and $z_{n_k} \to z$ weakly in $X$ as $k \to \infty$, then $z \in K$.

(M2) For any $z \in K$ there is a sequence $\{z_n\} \subset X$ such that $z_n \in K_n$, $n = 1, 2, \ldots$, and $z_n \to z$ in $X$ as $n \to \infty$.

We denote by $\chi_\Omega$ the characteristic function of $\Omega$ in $\hat{\Omega}$ for any subset $\Omega$ of $\hat{\Omega}$. We put

$$O := \{\Omega \subset \hat{\Omega}; \Omega \text{ is a smooth subdomain of } \hat{\Omega}\}$$

and for each $\Omega \in O$ denote by $V(\Omega)$ the set

$$\{z \in H^1_0(\hat{\Omega}); z = 0 \text{ a.e. on } \hat{\Omega} - \Omega\}.$$ 

Clearly $V(\Omega)$ is a closed linear subspace of $H^1_0(\hat{\Omega})$.

We consider the shape optimization problem for any non-empty subset $O_c$ of $O$ which is compact in the following sense:

$$\{\text{For any sequence } \{\Omega_n\} \subset O_c \text{ there is a subsequence } \{\Omega_{n_k}\} \text{ of } \{\Omega_n\} \text{ with } \Omega \in O_c \text{ such that } \chi_{\Omega_{n_k}} \to \chi_{\Omega} \text{ in } L^1(\hat{\Omega}) \text{ as } k \to \infty \text{ and } V(\Omega_{n_k}) \to V(\Omega) \text{ in } H^1_0(\hat{\Omega}) \text{ as } k \to \infty \text{ (in the sense of Mosco)}. \}$$

We give below typical examples of $O_c$, which are very important in the application of our main results.

**EXAMPLE 1.1.** (1) Let $\hat{\Omega}$ and $O$ be the same as stated before. Let $\Theta$ be the class of...
all $C^1$-diffeomorphisms from $\overline{\Omega}$ onto itself. Here we give $\Theta$ the topology induced from $C^1(\overline{\Omega})$.

Let $\Omega'$ be a smooth subdomain of $\widehat{\Omega}$ with $\overline{\Omega'} \subset \widehat{\Omega}$. For a given a non-empty compact subset $\Theta_c$ of $\Theta$, we put

$$O_c = \{\theta(\Omega'); \theta \in \Theta_c\}. \quad (1.2)$$

Then this subset $O_c$ of $O$ satisfies condition $(C)$.

Let $\{\Omega_n = \theta_n(\Omega')\}$ be any sequence in $O_c$. Then, by the compactness of $\Theta_c$, there is a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \rightarrow \theta$ in $C^1(\overline{\Omega})$ as $k \rightarrow \infty$ for some $\theta \in \Theta_c$. We see easily that $\chi_{\Omega_{n_k}} \rightarrow \chi_{\Omega}$, with $\Omega = \theta(\Omega')$, in $L^1(\widehat{\Omega})$ as $k \rightarrow \infty$. Moreover, $V(\Omega_{n_k}) \rightarrow V(\Omega)$ in $H^1_0(\Omega)$ as $k \rightarrow \infty$ (in the sense of Mosco). In fact, if $z_k \rightarrow z$ weakly in $H^1_0(\widehat{\Omega})$ as $k' \rightarrow \infty$ for a subsequence $\{n_{k'}\}$ and $z_{k'} \in V(\Omega_{n_{k'}})$, then $\overline{z_{k'}}(x) = z_{k'}(\theta_{n_{k'}}, 0\theta^{-1}(x)) \in V(\Omega)$ and $\overline{z_{k'}} \rightarrow z(\theta \circ \theta^{-1}) = z$ weakly in $H^1_0(\widehat{\Omega})$. So we see that $z \in V(\Omega)$. Also, let $z \in V(\Omega)$ and put $z_k(x) := z(\theta \circ \theta^{-1}(x)) \in V(\Omega_{n_k})$. Then, clearly, we have $z_k \rightarrow z$ in $H^1_0(\widehat{\Omega})$.

**EXAMPLE 1.2.** Let $\widehat{\Omega} := \{x; |x| < 2\} \subset R^3$, $\Omega_a := \{x; a < |x| < 1\}$ for any $0 < a \leq \frac{1}{2}$ and $\Omega := \{x; |x| < 1\}$. Here we put $O_c := \{\Omega_a; 0 < a \leq \frac{1}{2}\} \cup \{\Omega\}$. Then, we see that this subset $O_c$ of $O$ satisfies condition $(C)$.

In fact, by [13; Lemma 1.8], the 2-capacity of any singleton is zero. Then, by [13], we see that $V(\Omega_a) \rightarrow V(\Omega)$ in $H^1_0(\widehat{\Omega})$ in the sense of Mosco as $a \rightarrow 0$. In the other hand, by the same argument as in Example 1.1, we obtain that $V(\Omega_{a'}) \rightarrow V(\Omega_a)$ in $H^1_0(\widehat{\Omega})$ in the sense of Mosco as $a' \rightarrow a$. Hence $O_c$ satisfies condition $(C)$.  

In the case of Example 1.1, problems $SP(\Omega)$ can be reformulated as degenerate parabolic equations on the fixed domain $\Omega'$ by using the variable transformation $y = \theta^{-1}(x)$. However, in the case of Example 1.2, the situation is quite different, because there is no $C^1$-diffeomorphism between domains $\Omega_a$ and $\Omega$.

Based on an abstract result of [1] about the solvability of $SP(\Omega)$, we consider a shape optimization problem. For a given non-empty subset $O_c$ of $O$, our optimization problem,
denoted by $P(O_c)$, is formulated as follows:

$$P(O_c) \quad \text{Find } \Omega_* \in O_c \text{ such that } J(\Omega_*) = \inf_{\Omega \in O_c} J(\Omega),$$

where

$$J(\Omega) = \frac{1}{2} \int_{Q(\Omega)} |\beta(u_{\Omega}) - \beta_d|^2 \, dx \, dt + \frac{1}{2} \int_{\hat{Q} - Q(\Omega)} |g|^2 \, dx \, dt \quad \text{for } \Omega \in O,$$

$u_{\Omega}$ is the weak solution of $SP(\Omega)$, and $\beta_d$ is a given function in $L^2(\hat{Q})$.

In real problem, the driving variables are $f, g$ and $\Omega$. But, in this paper, we are interested in the effect of the domain $\Omega$ for the shape optimization. So, we fix the functions $f$ and $g$, and take $\Omega$ as the driving variable.

The main results are stated in the following theorems. To prove the existence of solutions to $P(O_c)$, an important part is to show the continuous dependence of weak solution $u = u_{\Omega}$ to $SP(\Omega)$ upon $\Omega \in O$.

**THEOREM 1.1.** Let $\{\Omega_n\} \subset O$ and $\Omega \in O$ such that $V(\Omega_n) \to V(\Omega)$ in $H^1_0(\hat{\Omega})$ as $n \to \infty$ (in the sense of Mosco) and $\chi_{\Omega_n} \to \chi_{\Omega}$ in $L^1(\hat{\Omega})$ as $n \to \infty$. Also, denote by $u_n$ and $u$ the weak solutions of $SP(\Omega_n)$ and $SP(\Omega)$, respectively. Then, as $n \to \infty$,

$$u_n(t), z)_{\Omega_n} \to (u(t), z)_{\Omega} \quad \text{for any } z \in L^2(\hat{\Omega})$$

and

$$\tilde{\beta}(u_n) \to \tilde{\beta}(u) \quad \text{in } L^2(\hat{Q}).$$

Here we denote by $(\cdot, \cdot)_{\Omega'}$ the inner product in $L^2(\Omega')$ and put

$$\tilde{\beta}(u_{\Omega'}) = \begin{cases} \beta(u_{\Omega'}) & \text{in } Q(\Omega'), \\ g & \text{in } \hat{Q} - Q(\Omega'), \end{cases}$$

for any $\Omega' \in O$.

The next theorem is concerned with the existence of a solution to $P(O_c)$. 

THEOREM 1.2. Problem $P(O_c)$ has at least one optimal solution $\Omega_*$.

We shall prove Theorems 1.1 and 1.2 in section 3.

2. Uniform estimates for the weak solutions to $SP(\Omega)$

In this section, we obtain some results from [1] on the existence, uniqueness and uniform estimates for weak solutions to $SP(\Omega)$. We use the following notations.

For simplicity, we denote by $H$ the space $L^2(\Omega)$ and by $X$ the Sobolev space $H_0^1(\Omega)$. Moreover, $|\cdot|_H$ stands for the norm in $H$ and $(\cdot, \cdot)$ the inner product in $H$. For each $\Omega \in O$, we define a bilinear form $a_\Omega(\cdot, \cdot)$ on $H^1(\Omega)$ by

$$a_\Omega(u, v) := \int_\Omega \nabla u \nabla v \, dx$$

for all $u, v \in H^1(\Omega)$, and denote by $F_\Omega$ the duality mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ which is given by the formula

$$\langle F_\Omega v, z \rangle := a_\Omega(v, z)$$

for all $v, z \in H_0^1(\Omega)$, where $\langle \cdot, \cdot \rangle_\Omega$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H^1(\Omega)$. In particular, we put $a(\cdot, \cdot) := a_\Omega(\cdot, \cdot)$.

According to the abstract result of [1; Theorem 2.1], problem $SP(\Omega)$ has a unique weak solution $u$ such that $u \in W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ and $\beta(u) - g \in L^2(0, T; H_0^1(\Omega))$ for any $\Omega \in O$. In fact, the weak solution $u$ is obtained as a unique solution of the following evolution problem in $H^{-1}(\Omega)$:

\begin{equation}
\left\{
\begin{array}{ll}
u'(t) + F_\Omega(\beta(u(t))) - g(t) = f(t) + \Delta g(t) & \text{for a.e. } t \in [0, T], \\
u(0) = u_0.
\end{array}
\right.
\end{equation}

We give some uniform estimates for weak solutions of $SP(\Omega)$ with respect to $\Omega \in O$.

LEMMA 2.1 There exists a positive constant $M_1$ independent of $\Omega$ such that

\begin{equation}
|u_\Omega|_{L^\infty(0, T; L^2(\Omega))} \leq M_1, \quad |\beta(u_\Omega)|_{L^2(0, T; L^2(\Omega))} \leq M_1
\end{equation}

and

\begin{equation}
|t^{1/2} \frac{d}{dt} \beta(u_\Omega)|_{L^2(0, T; L^2(\Omega))} \leq M_1, \quad |t^{1/2} \beta(u_\Omega)|_{L^\infty(0, T; H^1(\Omega))} \leq M_1
\end{equation}
for all $\Omega \in O$, where $u_\Omega$ is the weak solution of $SP(\Omega)$.

**Proof.** As was seen in [1], problem $SP(\Omega)$ is able to be approximated by non-degenerated problem $SP(\Omega)^\epsilon$, $\epsilon \in (0,1]$:

$$
SP(\Omega)^\epsilon \begin{cases}
    u_t - \Delta \beta^\epsilon(u) = f & \text{in } Q(\Omega), \\
    u(0,\cdot) = u_0 & \text{in } \Omega, \\
    \beta^\epsilon(u) = g & \text{on } \Sigma(\Omega),
\end{cases}
$$

where $\beta^\epsilon(r) = \beta(r) + \epsilon r, r \in \mathbb{R}$.

In fact, this problem has one and only one weak solution $u^\epsilon \in C([0,T];L^2(\Omega))$ such that $t^{1/2} \frac{d}{dt} \beta^\epsilon(u^\epsilon) \in L^2(Q(\Omega))$ and $\beta^\epsilon(u^\epsilon) \in L^2(0,T;H^1(\Omega))$. Moreover, we see that $u^\epsilon \to u_\Omega$ in $C_w([0,T];L^2(\Omega))$ and $\beta^\epsilon(u^\epsilon) \to \beta(u_\Omega)$ weakly in $L^2(0,T;H^1(\Omega))$ as $\epsilon \to 0$. After some calculations, we obtain that there is a positive constant $C'$ independent of $\epsilon$ and $\Omega$ such that

$$
\sup_{0 \leq t \leq T} |u^\epsilon(t)|_{L^2(\Omega)}^2 + \int_0^T |\nabla(\beta^\epsilon(u^\epsilon(t)))|_{L^2(\Omega)}^2 dt \leq C'.
$$

Moreover, multiply both sides of $u_t - \Delta \beta^\epsilon(u^\epsilon) = f$ by $t \frac{d}{dt}(\beta^\epsilon(u^\epsilon) - g)$ and integrate over $Q(\Omega)$. Then, by (2.4), we have

$$
|t^{1/2} \beta^\epsilon(u^\epsilon)|_{L^\infty(0,T;H^1(\Omega))} \leq C'',
$$

for any $\epsilon \in (0,1]$ and $\Omega \in O$,

where $C''$ is a constant independent of $\epsilon \in (0,1]$ and $\Omega \in O$. Therefore, letting $\epsilon \to 0$, we see that (2.2) and (2.3) hold.

3. **Proofs of Theorems 1.1 and 1.2**

First we prove Theorem 1.1.

**Proof of THEOREM 1.1.** Let consider the function $u_g \in L^\infty(0,T;H)$ such that $g(t,x) = \beta(u_g(t,x))$ in $\hat{Q}$. Here, we put

$$
\bar{u}_n = \begin{cases}
    u_n & \text{in } Q_n := Q(\Omega), \\
    u_g & \text{in } \hat{Q} - Q_n.
\end{cases}
$$
Then, we see that $\tilde{u}_n \in L^\infty(0, T; H)$. Moreover, we put $v_n := \beta(\tilde{u}_n)$ in $\tilde{Q}$. By using Lemma 2.1, there exist a subsequence $\{n_k\}$ of $\{n\}$, $v \in L^2(0, T; H^1(\tilde{\Omega}))$ and $\tilde{u} \in L^\infty(0, T; H)$ such that

\[(3.1)\quad \tilde{u}_{n_k} \rightharpoonup \tilde{u} \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0, T; H)\]

and

\[(3.2)\quad \begin{cases} v_{n_k} \rightharpoonup v & \text{weakly in} \quad L^2(0, T; H^1(\tilde{\Omega})), \\ v_{n_k}(t) \rightarrow v(t) & \text{weakly in} \quad H^1(\tilde{\Omega}) \quad \text{for all} \quad t \in (0, T]. \end{cases}\]

By using Ascoli-Arzela's theorem and Lemma 2.1, we easily verify that

$v_{n_k} \rightarrow v$ in $L^2(0, T; H)$ as $k \rightarrow \infty$.

Since $v_{n_k} = \beta(\tilde{u}_{n_k})$ in $\tilde{Q}$, from (3.1) and (3.2) we show that $v = \beta(\tilde{u})$ and that $\beta(\tilde{u}(t)) - g(t) \in V(\Omega)$ for any $t \in (0, T]$.

Next, let $z$ be any function in $V(\Omega)$ and $\rho$ be any function in $\mathcal{D}(0, T)$. By the assumptions of Theorem 1.1, there exists a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \rightarrow z$ in $X$. Then by the definition of solution to $SP(\Omega)$ we have

$$- \int_0^T (u_{n_k}(t), z_n(t))_{\Omega_{n_k}} \rho'(t) dt + \int_0^T a_{\Omega_{n_k}}(v_{n_k}(t), z_n(t)) \rho(t) dt = \int_0^T (f(t), z_n(t))_{\Omega_{n_k}} \rho(t) dt.$$

Letting $k \rightarrow \infty$, by $z_{n_k} = 0$ a.e. on $\tilde{\Omega} - \Omega_{n_k}$ we obtain

$$- \int_0^T (\tilde{u}(t), z) \rho'(t) dt + \int_0^T a(v(t), z) \rho(t) dt = \int_0^T (f(t), z) \rho(t) dt.$$

This shows that $u = \tilde{u} |_{Q(\Omega)}$ is the solution of $SP(\Omega)$. 

Proof of THEOREM 1.2. Since $J(\Omega) \geq 0$, there exists a minimizing sequence $\{\Omega_n\}$ in $O_c$ such that

$J(\Omega_n) \rightarrow J_* := \inf\{J(\Omega); \Omega \in O_c\}$
Then, by the compactness of $O_c$, there are a subsequence $\{\Omega_n\}$ of $\{\Omega_n\}$ and $\Omega_* \in O_c$ such that $V(\Omega_n) \to V(\Omega_*)$ in $X$ (in the sense of Mosco) for some $\Omega_* \in O_c$ and $\chi_{\Omega_n} \to \chi_{\Omega_*}$ in $L^1(\hat{\Omega})$ as $k \to \infty$. Now, denote by $u_k$ the weak solution of $SP(\Omega_n)$ and by $u_*$ the weak solution of $SP(\Omega_*)$. Then put

$$v_k := \begin{cases} \beta(u_k) & \text{in } Q_k = \Omega_n, \\ g & \text{in } \hat{Q} - Q_k, \end{cases}$$

and

$$v := \begin{cases} \beta(u_*) & \text{in } Q = \Omega_*, \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

From Theorem 1.1, it follows that $v_k \to v$ in $L^2(0,T;H)$ as $k \to \infty$. Then we see that

$$J(\Omega_n) \to J(\Omega_*).$$

Therefore $J(\Omega_*) = J_*$. Hence $\Omega_*$ is a solution of $P(O_c)$.

4. Approximations for $SP(\Omega)$ and $P(O_c)$

In this section, from some numerical points of view, we discuss approximations of $SP(\Omega)$ and $P(O_c)$ by smooth problems. At first, we introduce the approximation $\beta^\epsilon$ and $\chi_{\Omega}^\nu$, respectively.

Let $\{\beta^\epsilon\} = \{\beta^\epsilon; 0 < \epsilon \leq 1\}$ be a family of (smooth) functions $\beta^\epsilon : R \to R$ such that

$$\begin{cases} |\beta^\epsilon(r) - \beta(r)| \leq \epsilon |r| + 1, \\ \beta^\epsilon(0) = 0, \\ \frac{d}{dr}\beta^\epsilon(r) \geq \epsilon \end{cases}$$

for all $r \in R$ and $r, r' \in R$ for a.e. $r \in R$, where $\tilde{L}_0 > 0$ is a constant independent of $\epsilon$.

Next, let $\{\chi_{\hat{\Omega}}^\nu\} = \{\chi_{\hat{\Omega}}^\nu; 0 < \nu \leq 1, \Omega \in O_c\}$ be a family of smooth functions on $\hat{\Omega}$ and suppose that the following two conditions $(\chi 1)$ and $(\chi 2)$ hold:

$(\chi 1)$ $0 \leq \chi_{\hat{\Omega}} \leq \chi_{\hat{\Omega}}^\nu \leq 1$ in $\hat{\Omega}$ and $\supp(\chi_{\hat{\Omega}}^\nu) \subset \{x \in \hat{\Omega}; dist(x, \Omega) \leq \nu\}$ for any $\nu \in (0,1]$ and $\Omega \in O_c$.

$(\chi 2)$ For each $\nu \in (0,1]$, $\{\chi_{\hat{\Omega}}^\nu; \Omega \in O_c\}$ is compact in $L^1(\hat{\Omega})$. 

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We give below typical examples of approximations $\beta^* \epsilon$ and $\chi_{\Omega}^\nu$ for $\beta$ and $\chi_{\Omega}$, respectively, which satisfy the conditions mentioned above.

**EXAMPLE 4.1.** (1) We define $\beta^*: \mathbb{R} \rightarrow \mathbb{R}$ by $\beta^*(r) = \beta(r) + \epsilon r$ for any $r \in \mathbb{R}$. Then, the family of $\{\beta^*\}$ satisfies the condition $(\beta)$ for $\tilde{L}_0 = L_0 + 1$ where $L_0$ is the constant of (1.1).

(2) Let $\hat{\Omega}$, $\Omega'$ and $O_c$ be the same as in Example 1.1. Now, for each $\nu \in (0, 1]$ and $\Omega \in O_c$, we denote by $\Omega(\frac{\nu}{2})$ the set $\{x \in \hat{\Omega}; \text{dist}(x, \Omega) \leq \frac{\nu}{2}\}$. Let $\chi_{\Omega}^\nu$ be the regularization of $\chi_{\Omega(\frac{\nu}{2})}$ by means of usual mollifiers on $\hat{\Omega}$. Clearly, we see that $(\chi 1)$ holds. Also, we obtain that $(\chi 2)$ holds. Because we can prove that

\[(\nu)\quad \text{if } \Omega_n = \theta_n(\Omega'), \theta_n \rightarrow \theta \text{ in } C^1(\overline{\hat{\Omega}}) \text{ and } \Omega = \theta(\Omega'), \text{ then } \chi_{\Omega_n} \rightarrow \chi_{\Omega} \text{ in } L^1(\hat{\Omega}).\]

Now, we define the approximate problem $SP(\Omega)^{e\nu\mu}$, $\epsilon, \nu, \mu \in (0, 1]$, by using the penalty method for $SP(\Omega)$:

$$\begin{align*}
SP(\Omega)^{e\nu\mu} \{ u_t - \Delta \beta^*(u) = f - \frac{1 - \chi_{\hat{\Omega}}^\nu}{\mu} (\beta^*(u) - g) & \quad \text{in } \hat{Q}, \\
u(0, \cdot) = u_0 & \quad \text{in } \hat{\Omega}, \\
\beta^*(u) = g & \quad \text{on } \hat{\Sigma}.
\end{align*}$$

Here we give the weak formulation of $SP(\Omega)^{e\nu\mu}$.

**DEFINITION 4.1.** A function $u : [0, T] \rightarrow H$ is a solution of $SP(\Omega)^{e\nu\mu}$, if the following three conditions $(aw1) - (aw3)$ are satisfied:

$(aw1)$ $u \in C([0, T]; H) \cap W^{1,2}_{\text{loc}}((0, T]; H) \cap L^2(0, T; H^1(\hat{\Omega}))$, $u(0) = u_0$ in $\hat{\Omega}$;

$(aw2)$ $\beta^*(u(t)) - g(t) \in X$ for a.e. $t \in [0, T]$;

$(aw3)$ $\langle u'(t), z \rangle_{\hat{\Omega}} + a(\beta^*(u(t)), z) = (f(t) - \frac{1 - \chi_{\hat{\Omega}}^\nu}{\mu} (\beta^*(u(t)) - g(t)), z)$

for any $z \in X$, a.e. $t \in [0, T]$.

According to the abstract result in [9; Chapter 2] (or [10]), problem $SP(\Omega)^{e\nu\mu}$ has a unique solution $u$.

Our approximate optimization problem $P(O_c)^{e\nu\mu}$, associated with $SP(\Omega)^{e\nu\mu}$, is formu-
lated as follows:

\[ P(O_c)^{e\nu\mu} \quad \text{Find } \Omega_{*}^{e\nu\mu} \in O_c \text{ such that } J^{e\nu\mu}(\Omega_{*}^{e\nu\mu}) = \inf_{\Omega \in O_c} J^{e\nu\mu}(\Omega), \]

where

\[ J^{e\nu\mu}(\Omega) = \frac{1}{2} \int_{\hat{Q}} \chi_{\Omega}^{\nu} | \beta^{e}(u_{\Omega}^{e\nu\mu}) - \beta_d |^2 \, dx \, dt + \frac{1}{2} \int_{\hat{Q}} (1 - \chi_{\Omega}^{\nu}) | g |^2 \, dx \, dt, \]

\[ u_{\Omega}^{e\nu\mu} \] is the solution of \( SP(\Omega)^{e\nu\mu} \).

Next, we give the convergence results in the following theorem.

**THEOREM 4.1.** We have the following statements (1) and (2):

(1) For each \( \epsilon, \nu, \mu \in (0, 1] \), \( P(O_c)^{e\nu\mu} \) has at least one solution.

(2) Let \( \{\epsilon_n\}, \{\nu_n\}, \{\mu_n\} \) be null sequences and let \( \{\Omega_n\} \subset O_c \) and \( \Omega \in O_c \) such that \( V(\Omega_n) \rightarrow V(\Omega) \) in \( X \) as \( n \rightarrow \infty \) (in the sense of Mosco), \( \chi_{\Omega_n}^{\nu} \rightarrow \chi_{\epsilon}^{\nu} \) in \( L^1(\hat{\Omega}) \) as \( n \rightarrow \infty \).

Denote by \( u_n \) the solution of \( SP(\Omega_n)^{\epsilon_n\nu_n\mu_n} \). Then as \( n \rightarrow \infty \),

\[
\begin{align*}
\chi_{\Omega_n}^{\nu} u_n &\rightarrow \chi_{\Omega}^{\nu} u \text{ weakly* in } L^\infty(0,T;H), \\
\beta^{e}(u_n) &\rightarrow v \text{ in } L^2(0,T;H) \text{ and weakly in } L^2(0,T;H^1(\hat{\Omega})),
\end{align*}
\]

Moreover \( u \) is the weak solution of \( SP(\Omega) \) and

\[
v = \begin{cases} 
\beta(u) & \text{in } Q = (0,T) \times \Omega, \\
g & \text{in } \hat{Q} - Q.
\end{cases}
\]

In particular, if \( \Omega_n \) is a solution of \( P(O_c)^{e\nu\mu} \) with \( \epsilon = \epsilon_n, \nu = \nu_n \) and \( \mu = \mu_n \) for \( n = 1, 2, \ldots, \) then \( \Omega \) is a solution of \( P(O_c) \).

In this theorem, \( \{\epsilon_n\}, \{\nu_n\}, \) and \( \{\mu_n\} \) are chosen independently. This is very convenient for numerical computation. Moreover, we show that \( P(O_c)^{e\nu\mu} \) converges to \( P(\epsilon) \) in some sense.

### 5. Energy estimates for \( SP(\Omega)^{e\nu\mu} \)

For the proof of Theorem 4.1, we prepare some lemmas on energy estimates for solutions of \( SP(\Omega)^{e\nu\mu} \) with respect to \( \epsilon, \nu, \mu \in (0, 1] \) and \( \Omega \in O_c \).
LEMMA 5.1. There is a positive constant $M_2$ such that

\begin{equation}
|u^e_{\Omega}^{\nu\mu}|_{L^{\infty}(0,T;H)} \leq M_2, |\beta^e(u^e_{\Omega}^{\nu\mu})|_{L^2(0,T;H^1(\Omega))} \leq M_2
\end{equation}

and

\begin{equation}
\frac{1}{\mu} \int_{\hat{Q}} (1 - \chi_{\Omega}^{\nu}) |\beta^e(u^e_{\Omega}^{\nu\mu}) - g|^2 \, dx \, dt \leq M_2
\end{equation}

for all $\epsilon, \nu, \mu \in (0,1]$ and $\Omega \in O_c$, where $u^e_{\Omega}^{\nu\mu}$ is the solution of $SP(\Omega)^{\nu\mu}$.

**Proof.** For $0 < \nu, \mu \leq 1, \Omega \in O$, $0 \leq t \leq T$, we introduce a proper lower semi-continuous convex function $\varphi_{\Omega}^{\nu\mu}$ on $H$ as follows:

\begin{equation}
\varphi_{\Omega}^{\nu\mu}(t, z) = \left\{ \begin{array}{ll}
\frac{1}{2} \| \nabla z \|^2_H + \frac{1}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^{\nu}) |z - g(t)|^2 \, dx & \text{for } z - g(t) \in X, \\
+ \infty & \text{otherwise}.
\end{array} \right.
\end{equation}

We easily see that the subdifferential $\partial \varphi_{\Omega}^{\nu\mu}(t, \cdot)$ in $H$ is singlevalued in $H$ and

\begin{equation}
z^* = \partial \varphi_{\Omega}^{\nu\mu}(t, z) \iff \begin{cases}
z - g(t) \in X, z^* \in H, \\
z^* = -\Delta z + \frac{1 - \chi_{\Omega}^{\nu}}{\mu} (z - g(t)) \in H.
\end{cases}
\end{equation}

By using (5.4), we can show that $SP(\Omega)^{\nu\mu}$ can be reformulated by the following evolution problem in $H$:

\begin{equation}
\begin{cases}
u'(t) + \partial \varphi_{\Omega}^{\nu\mu}(t, \beta^e(u(t))) = f(t) & \text{in } H \text{ for a.e. } t \in [0, T], \\
u(0) = u_0.
\end{cases}
\end{equation}

For simplicity, we write $u$ for $u^e_{\Omega}^{\nu\mu}, \chi$ for $\chi_{\Omega}^{\nu}$ and $\varphi(t, \cdot)$ for $\varphi_{\Omega}^{\nu\mu}(t, \cdot)$. Multiplying $\nu'(t) + \partial \varphi(t, \beta^e(u(t))) = f(t) \beta^e(u(t)) - g(t)$, by using (5.4), we obtain

\begin{align*}
(u'(t), \beta^e(u(t)) - g(t)) + a(\beta^e(u(t)), \beta^e(u(t)) - g(t)) + \\
\frac{1}{\mu} \int_{\Omega} (1 - \chi) |\beta^e(u(t)) - g(t)|^2 \, dx = (f(t), \beta^e(u(t)) - g(t)).
\end{align*}

After some calculations, we obtain the following inequality:

\begin{equation}
\begin{align*}
\frac{d}{dt} \left\{ \int_{\Omega} \beta^e(u(t)) \, dx - (g(t), u(t)) \right\} \\
+ R_1 \{ |\nabla(\beta^e(u(t)) - g(t))|^2_H + \frac{1}{\mu} \int_{\Omega} (1 - \chi) |\beta^e(u(t)) - g(t)|^2 \, dx \} \\
\leq R_2 \left\{ \int_{\Omega} \beta^e(u(t)) \, dt - (g(t), u(t)) \right\} \\
+ R_3 (1 + |g(t)|^2_H + |g'(t)|^2_H + |f(t)|^2_H)
\end{align*}
\end{equation}
where $R_i, i = 1, 2, 3$, are positive constants independent of $\epsilon, \nu, \mu$ and $\Omega$. By using Gronwall's inequality and (5.6), we show (5.1) and (5.2) for a positive constant $M_2$ independent of $\epsilon, \nu, \mu \in (0, 1)$ and $\Omega \in O_c$.

**LEMMA 5.2.** There is a positive constant $M_3$ such that

\begin{equation}
|t^{1/2}\beta^e(u^e_{\Omega}^\nu\mu)|_{L^\infty (0,T;H(\Omega))} \leq M_3, |t^{1/2}\frac{d}{dt}\beta^e(u^e_{\Omega}^\nu\mu)|_{L^1(0,T;H)} \leq M_3,
\end{equation}

and

\begin{equation}
\sup_{t \in (0,T]} \frac{t}{\mu} \int_{\Omega} (1 - \chi_{\Omega}^\nu) |\beta^e(u^e_{\Omega}^\nu\mu)(t) - g(t)|^2 dx \leq M_3,
\end{equation}

for all $\epsilon, \nu, \mu \in (0, 1)$ and $\Omega \in O_c$, where $u^e_{\Omega}^\nu\mu$ is the solution of $SP(\Omega)^{e\nu\mu}$.

**Proof.** Simply write $u$ for $u^e_{\Omega}^\nu\mu$ and $\tilde{\beta}$ for $\beta^e(u^e_{\Omega}^\nu\mu)$. Let us consider the convex function $\psi := \psi_{\Omega}^{\nu\mu}$ on $H$ given by

\[ \psi_{\Omega}^{\nu\mu}(z) = \begin{cases} 
\frac{1}{2} \|\nabla z\|_H^2 + \frac{1}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^\nu) |z|^2 dx, & z \in X, \\
+\infty, & \text{otherwise}.
\end{cases} \]

In fact, it is easy to see that $\psi$ is proper lower semicontinuous and convex on $H$, and the subdifferential $\partial \psi$ is singlevalued in $H$. Besides,

\[ z^* = \partial \psi(z) \Leftrightarrow \begin{cases} 
z \in X, z^* \in H, \\
z^* = -\Delta z + \frac{1 - \chi_{\Omega}^\nu}{\mu} z \in H.
\end{cases} \]

Moreover, by the standard argument of convex analysis, we have

\begin{equation}
\frac{d}{dt} \psi(z(t)) = (\partial \psi(z(t)), z'(t)) \text{ for } z \in W^{1,2}(0,T;H).
\end{equation}

Then, by using (5.4) and (5.5), we see that

\[ (u'(t), \tilde{\beta}'(t) - g'(t)) + (-\Delta(\tilde{\beta}(t) - g(t)) + \frac{1 - \chi_{\Omega}^\nu}{\mu} (\tilde{\beta}(t) - g(t)), \tilde{\beta}'(t) - g'(t)) = (f(t) + \Delta g(t), \tilde{\beta}'(t) - g'(t)). \]
Then, by (5.9), we show that
\[
\frac{t}{2L_0} \left| \tilde{\beta}'(t) \right|^2_H + \frac{d}{dt} \left\{ \frac{t}{2} \left| \nabla^2 (\tilde{\beta}(t) - g(t)) \right|^2_H - t(u(t), g'(t)) + \frac{t}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^\nu) \left| \tilde{\beta}(t) - g(t) \right|^2 dx \right\}
\leq T |f(t) + \Delta g(t)|_H \left\{ |g'(t)|_H + \frac{L_0}{2} |f(t) + \Delta g(t)|_H \right\} + T |u(t)|_H \cdot |g^u(t)|_H + \frac{1}{2} |\nabla (\tilde{\beta}(t) - g(t))|^2_H - (u(t), g'(t)) + \frac{1}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^\nu) \left| \tilde{\beta}(t) - g(t) \right|^2 dx.
\]
(5.10)

Here, integrating (5.10) over \([0, t]\) and using Lemma 5.1, we derive the estimates (5.7) and (5.8) for some positive constant \(M_3\) independent of \(\epsilon, \nu, \mu \in (0, 1]\) and \(\Omega \in O_c\).


Now we prove Theorem 4.1.

Proof of (1) of THEOREM 4.1. Fix \(\epsilon, \nu, \mu \in (0, 1]\) and put \(I_*= \inf \{J^{\epsilon\nu\mu}(\Omega); \Omega \in O_c\} \geq 0\). Then, there exists a minimizing sequence \(\{\Omega_n\}\) in \(O_c\) such that
\[
J^{\epsilon\nu\mu}(\Omega_n) \to I_* \quad (\text{as } n \to \infty).
\]

By (\(\chi 2\)), there is a subsequence \(\{\Omega_{n_k}\}\) of \(\{\Omega_n\}\) such that \(V(\Omega_{n_k}) \to V(\Omega)\) in \(X\) (in the sense of Mosco) and \(\chi_{\Omega_{n_k}} \to \chi_{\Omega} =: \chi\) in \(L^1(\hat{\Omega})\) for some \(\Omega \in O_c\). In a similar way to that of the proof of Theorem 1.1, we can prove that the solution \(u_k := u_{\Omega_{n_k}}^{\epsilon\nu\mu}\) converges to the weak solution \(u := u_{\Omega}^{\epsilon\nu\mu}\) of \(SP(\Omega)^{\epsilon\nu\mu}\) in the sense that
\[
\left\{ \begin{array}{l}
u_k \to u \quad \text{in } L^2(0,T; H) \\
\beta^*(u_k) \to \beta^*(u) \quad \text{in } L^2(0,T; H) 
\end{array} \right.
\]
Therefore
\[
I_* = \lim_{k \to \infty} J^{\epsilon\nu\mu}(\Omega_k) = J^{\epsilon\nu\mu}(\Omega),
\]
and we see that \(\Omega\) is a solution of \(P(O_c)^{\epsilon\nu\mu}\).

Proof of (2) of Theorem 4.1. By Lemma 5.1 and Lemma 5.2, we may assume that
\[
u_n \to \bar{\nu} \quad \text{weakly* in } L^\infty([0,T]; H),
\]
(6.1)
and

\[ (6.2) \quad \left\{ \begin{array}{l}
\tilde{\beta}_n := \beta^\epsilon(u_n) \to \beta(\tilde{u}) =: \tilde{\beta} \quad \text{in } C_{\text{loc}}((0, T]; H), \\
\tilde{\beta}_n(t) \to \tilde{\beta}(t) \text{ weakly in } H^1(\hat{\Omega}) \quad \text{for any } t \in (0, T].
\end{array} \right. \]

In fact, (6.1) and (6.2) are obtained in a similar way to the proof of Theorem 1.2. Moreover, by using (5.8) of Lemma 5.2 and (6.2), we have

\[ \left\{ \begin{array}{l}
\chi_{\Omega_n} u_n \to \chi_{\Omega} u \quad \text{weakly* in } L^\infty(0, T; H), \\
\tilde{\beta}_n \to \tilde{\beta} \quad \text{in } L^2(0, T; H), \\
\int_{\hat{\Omega}} (1 - \chi_{\Omega_n}^\nu) \frac{1}{\beta_n(t) - g(t)} \, dx \to 0 = \int_{\hat{\Omega}} (1 - \chi_{\Omega}) |\tilde{\beta}(t) - g(t)|^2 \, dx
\end{array} \right. \]

for any \( t \in (0, T] \), so that

\[ (6.3) \quad \tilde{\beta}(t) - g(t) \in V(\Omega) \quad \text{for any } t \in (0, T]. \]

Next, let \( \rho \) be any function in \( \mathcal{D}(0, T) \). By assumption, for any \( z \in V(\Omega) \), there is a sequence \( \{z_n\} \) such that \( z_n \in V(\Omega_n) \) and \( z_n \to z \) in \( X \). From (5.5) it follows that

\[ -\int_0^T (u_n(t), z_n) \rho(t) \, dt + \int_0^T a(\tilde{\beta}_n(t), z_n) \rho(t) \, dt + \frac{1}{\mu_n} \int_0^T ((1 - \chi_{\Omega_n}^\nu)(\tilde{\beta}_n - g)(t), z_n) \rho(t) \, dt \]

\[ = \int_0^T (f(t), z_n) \rho(t) \, dt. \]

Since \( (1 - \chi_{\Omega_n}^\nu)z_n = 0 \) a.e. on \( \hat{\Omega} \), as \( n \to \infty \), we get that

\[ \int_0^T (\tilde{\beta}(t), z \rho(t)) \, dt + \int_0^T a(\tilde{\beta}(t), z) \rho(t) \, dt = \int_0^T (f(t), z) \rho(t) \, dt. \]

Therefore \( \tilde{u} \) is the weak solution of \( SP(\Omega) \).

In particular, let \( \Omega_n \) be a solution of \( P(O_c)^{e_n \nu_n \mu_n} \) for each \( n \). Just as above

\[ J^{e_n \nu_n \mu_n}(\Omega_n) \to J(\Omega) \]

and

\[ J^{e_n \nu_n \mu_n}(\Omega') \to J(\Omega') \quad \text{for any } \Omega' \in O_c. \]

Therefore, for any \( \Omega' \in O_c, \)

\[ J(\Omega') = \lim_{n \to \infty} J^{e_n \nu_n \mu_n}(\Omega') \geq \lim_{n \to \infty} J^{e_n \nu_n \mu_n}(\Omega_n) = J(\Omega). \]
This shows that $\Omega$ is a solution of $P(O_c)$. ⊙

For the detailed proofs of all results stated in this note, see the forthcoming paper [17].

References


