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Linear evolution equations in a reflexive Banach space

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§1. INTRODUCTION

In this paper we discuss the construction of an evolution system associated with the well posed problem in the sense of Hadamard for the time-dependent differential equation in a Banach space $X$

$$(DE)_s \left\{ \begin{array}{l} (d/dt)u(t) = A(t)u(t) \text{ for } t \in [s,T] \\ u(s) = x, \end{array} \right.$$ 

where $s \in [0,T)$, $u(\cdot)$ stands for an $X$-valued unknown function on the interval $[s,T]$ and $\{A(t) : t \in [0,T]\}$ is a given family of linear operators in $X$.

Assume for the moment that there exist a dense subspace $Y$ of $X$ and an injective bounded linear operator $C_1$ on $X$ such that $Y \subset D(A(t))$ for $t \in [0,T]$ and the following conditions hold:

1) For $s \in [0,T]$ and $x \in C_1(Y)$, there exists a unique solution $u(t;s,x)$ such that $u(t;s,x) \in Y$ for $t \in [s,T]$.

2) For $x \in C_1(Y)$, $u(t;s,x)$ is continuous for $0 \leq s \leq t \leq T$.

3) If $\{u(t;s,x_n)\}$ is a sequence of solutions with $x_n \to 0$ in the $C_1^{-1}$-graph norm as $n \to \infty$ then $u(t;s,x_n)$ converges to zero uniformly with respect to $t$ and $s$.

Here we note that in the special case where $A(t) = A$, $s = 0$, $Y = D(A)$ and $C_1 = R(c : A)^n$ ($n \in \mathbb{N} \cup \{0\}$ and $c \in \rho(A)$), the concept of the above well posed problem is equal to that of the well posed problem in the sense of Hadamard in the autonomous case (see [5,8]), which several authors [1,4,9,10,11,12] recently have studied via the theory of integrated semigroups or $C$-semigroups.
Now we turn to the above well posed problem. We define a linear subspace $D(s)$ of $X$ and a linear operator $U(t, s)$ on $D(s)$ by

$$\begin{aligned}
D(s) &= \{ x \in X : \text{the} \ (DE) \text{ has a unique solution } u(t; s, x) \} \\
U(t, s)x &= u(t; s, x) \text{ for } x \in D(s).
\end{aligned}$$

Then, from the uniqueness of the solutions it follows that $U(t, s) : D(s) \to D(t)$ and $U(t, r)U(r, s) = U(t, s)$ on $D(s)$ for $0 \leq s \leq r \leq t \leq T$. Formally, the two parameter family $\{U(t, s) : 0 \leq s \leq t \leq T\}$ may have the properties

\begin{align*}
(1.1) \quad & (\partial/\partial t)U(t, s) = A(t)U(t, s) \\
(1.2) \quad & (\partial/\partial s)U(t, s) = -U(t, s)A(s)
\end{align*}

(this property is useful to show the existence of the solutions),

We define $\{V_1(t, s) : 0 \leq s \leq t \leq T\}$ by

$$V_1(t, s)y = U(t, s)C_1y \ (= u(t; s, C_1y)) \text{ for } y \in Y.$$ 

Since $Y$ is dense in $X$ one can see by the condition 3) that $V_1(t, s)$ is extended to a bounded linear operator on $X$, which we denote by the same symbol. Then, the two parameter family $\{V_1(t, s) : 0 \leq s \leq t \leq T\}$ has the properties

(i) for $x \in X$, $(t, s) \to V_1(t, s)x$ is continuous for $0 \leq s \leq t \leq T$,

(ii) $V_1(t, s)(Y) \subset Y$ for $0 \leq s \leq t \leq T$,

(iii) $(\partial/\partial t)V_1(t, s)y = A(t)V_1(t, s)y$ for $y \in Y$, and $V_1(s, s) = C_1$.

We also consider the following important property to show the uniqueness of the solutions:

(iv) $(\partial/\partial s)V_2(t, s)y = -V_2(t, s)A(s)y$ for $y \in Y$, and $V_2(s, s) = C_2$.

Multiplying (1.2) by the injective bounded linear operator $C_2$ from the left-hand side, and then defining $V_2(t, s)$ by $C_2U(t, s)$ we obtain the property (iv).
Moreover, the following relation between $V_1(t, s)$ and $V_2(t, s)$ holds:

\[(v) \quad C_2 V_1(t, s) = V_2(t, s) C_1 \quad \text{for} \quad 0 \leq s \leq t \leq T.\]

In §2 we will construct a pair of evolution systems \(\{V_1(t, s)\}, \{V_2(t, s)\}\) having the properties (i) - (v) in order to investigate the well posed problem in the sense of Hadamard for the time-dependent differential equation \(DE\).

As an application we also consider the second order differential equation in a reflexive Banach space $X$

\[
(DE)_2^s \quad \left\{ \begin{align*}
    u''(t) &= A u(t) + B(t) u(t) & \text{for} & \quad t \in [s, T] \\
    u(s) &= x, \quad u'(s) = y,
\end{align*} \right.
\]

where $A$ is the infinitesimal generator of a cosine family and \(\{B(t) : t \in [0, T]\}\) is a given family of linear operators in $X$.

§2. CONSTRUCTION OF EVOLUTION SYSTEMS

Let $X$ and $Y$ be Banach spaces with norm $\| \cdot \|$ and $\| \cdot \|_Y$ respectively. We write $B(Y, X)$ for the set of all bounded linear operators on $Y$ to $X$ and denote $B(X, X)$ by $B(X)$. For each $i = 1, 2$, let $C_i$ be an injective operator in $B(X)$.

Throughout this paper we will assume that

\[(H_1) \quad Y \text{ is reflexive},\]

\[(H_2) \quad Y \text{ is densely and continuously imbedded in } X, \text{ that is, } Y \text{ is a dense subspace of } X \text{ and there is a constant } L \text{ such that } \|y\| \leq L\|y\|_Y \text{ for } y \in Y,\]

\[(H_3) \quad C_1(Y) \subset Y \text{ and } C_1(Y) \text{ is } \| \cdot \|_Y \text{-dense in } Y.\]

We will make the following assumptions on a family $\{A(t) : t \in [0, T]\}$ of closed linear operators in $X$:

\[(A_1) \text{ There are constants } M_1 \geq 0 \text{ and } \omega_1 \geq 0 \text{ such that }\]

\[
(\omega_1, \infty) \subset \rho(A(t)) \text{ for } t \in [0, T] \text{ and } \\
\| \lambda^m \left( \prod_{i=1}^m R(\lambda : A(t_i)) \right) C_1 \| \leq M_1 \text{ for } \lambda > \omega_1
\]
and every finite sequence \( \{t_i\}_{i=1}^{m} \) such that \( 0 \leq t_1 \leq \cdots \leq t_m \leq T \) and \( m \) with \( 0 \leq m/\lambda \leq T \).

\((A_2)\) There are constants \( M_2 \geq 0 \) and \( \omega_2 \geq \omega_1 \) such that
\[
\left( \prod_{i=1}^{m} R(\lambda : A(t_i)) \right) C_1(Y) \subset Y \quad \text{and} \quad \left\| \lambda^m \left( \prod_{i=1}^{m} R(\lambda : A(t_i)) \right) C_1 \right\|_Y \leq M_2 \quad \text{for} \quad \lambda > \omega_2
\]
and every finite sequence \( \{t_i\}_{i=1}^{m} \) such that \( 0 \leq t_1 \leq \cdots \leq t_m \leq T \) and \( m \) with \( 0 \leq m/\lambda \leq T \).

\((A_3)\) There are constants \( M_3 \geq 0 \) and \( \omega_3 \geq \omega_1 \) such that
\[
\left\| C_2 \left( \lambda^m \left( \prod_{i=1}^{m} R(\lambda : A(t_i)) \right) \right) \right\| \leq M_3 \quad \text{for} \quad \lambda > \omega_3
\]
and every finite sequence \( \{t_i\}_{i=1}^{m} \) such that \( 0 \leq t_1 \leq \cdots \leq t_m \leq T \) and \( m \) with \( 0 \leq m/\lambda \leq T \).

\((A_4)\) For \( t \in [0,T] \), \( D(A(t)) \supset Y \) and \( D(C_1^{-1}A(t)C_1) \supset Y \), and the function \( t \to A(t) \) is continuous in the \( B(Y,X) \) norm \( \| \cdot \|_{Y \to X} \) and \( M_4 = \sup \{ \| C_1^{-1}A(t)C_1 \|_{Y \to X} : t \in [0,T] \} < \infty \).

The main result of this paper is given by

**Theorem 2.1.** If the family \( \{A(t) : t \in [0,T]\} \) of closed linear operators in \( X \) satisfies \((A_1)-(A_4)\) then there exists a unique pair \( \{(V_1(t,s)), (V_2(t,s))\} \) of strongly continuous families of bounded linear operators defined on the triangle \( \Delta = \{(t,s) : 0 \leq s \leq t \leq T\} \) with the following properties:

(a) For \( i = 1,2 \), \( V_i(s,s) = C_i \) on \([0,T]\) and \( C_2 V_1(t,s) = V_2(t,s)C_1 \) on \( \Delta \).
(b) \( V_1(t,s)(Y) \subset Y \) for \( 0 \leq s \leq t \leq T \).
(c) For \( y \in Y \) and \( y^* \in Y^* \), \( (t,s) \to (y^*, V_1(t,s)y) \) is continuous on \( \Delta \).
(d) \[ \langle x^*, V_1(t,s)y - V_1(r,s)y \rangle = \int_r^t \langle x^*, A(\tau)V_1(\tau,s)y \rangle \, d\tau \]
for $y \in Y, x^* \in X^*$ and $0 \leq s \leq r \leq t \leq T$. In particular, $(\partial/\partial t)V_1(t, s)y$ exists for almost every $t \in [s, T]$ and equals $A(t)V_1(t, s)y$.

(e) $V_2(t, r)y - V_2(t, s)y = -\int_s^r V_2(t, \tau)A(\tau)y d\tau$

for $y \in Y$ and $0 \leq s \leq r \leq t \leq T$.

Remarks. 1) In the case where $A(t) \subset C_1^{-1}A(t)C_1$ for $t \in [0, T]$, the condition $(A_3)$ is automatically satisfied with $C_2 = C_1$ if the condition $(A_1)$ is satisfied.

2) In the case where $C_1 = C_2 = I$ (the identity operator on $X$), Theorem 2.1 is [6, Theorem 5.1].

Before proving Theorem 2.1 we prepare three lemmas. Let $s \in [0, T)$ and let $\lambda > 0$ be such that $\lambda \omega_3 < 1$. Set

$$P_{\lambda, k}(s) = \prod_{i=1}^k J_\lambda(s + i\lambda) \text{ for } 0 \leq k \leq [(T - s)/\lambda],$$

where $[\ ]$ denotes the Gaussian bracket and $J_\lambda(t) = (1 - \lambda A(t))^{-1} = \lambda^{-1}R(\lambda^{-1}; A(t))$ for $t \in [0, T]$.

Now we define $A_{k, l}$ and $B_{k, l}$ by

$$A_{k, l}x = P_{\lambda, k}(s)C_1x - P_{\mu, l}(s)C_1x \text{ for } x \in X,$$

$$B_{k, l}y = \mu(A(s + k\lambda) - A(s + l\mu))P_{\mu, l}(s)C_1y \text{ for } y \in Y.$$ 

Here we note by the conditions $(A_2)$ and $(A_4)$ that $B_{k, l}$ is well defined because $P_{\mu, l}(s)C_1(Y) \subset Y \subset D(A(t))$ for $t \in [0, T]$.

Using the resolvent identity we obtain by a standard argument

**Lemma 2.2.** Let $s \in [0, T)$ and $\lambda, \mu > 0$ be such that $\lambda \omega_3, \mu \omega_3 < 1$. Then, for $y \in Y$ we have

$$A_{k, l}y = J_\mu(s + k\lambda)(\alpha A_{k-1, l-1}y + \beta A_{k, l-1}y + B_{k, l}y)$$

(2.1)
for $0 \leq k \leq [(T - s)/\lambda]$ and $0 \leq l \leq [(T - s)/\mu]$, where $\alpha = \frac{k}{\lambda}$ and $\beta = \frac{\lambda - \mu}{\lambda}$.

Let $s \in [0, T)$ and $\lambda, \mu > 0$ be such that $\lambda \omega_3, \mu \omega_3 < 1$. Let $k$ and $j$ be nonnegative integers. We denote by $H(m, k)$ the set of all operators $Q$ obtained by multiplying $k$ operators $J_{\mu}(t_i)$ ($i = 1, \cdots, k$) in the family $\{J_{\mu}(s + i\lambda) : i = 1, \cdots, m\}$ such that $Q = \prod_{i=1}^{k} J_{\mu}(t_i)$ for $0 \leq s + \lambda \leq t_1 \leq \cdots \leq t_k \leq s + m\lambda \leq T$; $H(m, 0) = H(0, k) = \{ \text{the identity operator} \}$. By $H(m, k, j)$ we denote the set of all sums of $j$ operators $Q_i$ ($i = 1, \cdots, j$) in $H(m, k)$, where in $j$ operators $Q_1, \cdots, Q_j$, same operators are allowed to appear repeatedly.

Using the relation (2.1) and then taking account of the definition $H(\cdot, \cdot, \cdot)$ we obtain by a routine calculation the following crucial estimate:

**Lemma 2.3.** Let $s \in [0, T)$ and let $\lambda, \mu > 0$ such that $\lambda \omega_3, \mu \omega_3 < 1$. Then, for $y \in Y$ we have

$$
A_{m,n}y \in \sum_{i=0}^{(m-1)\wedge n} \alpha^i \beta^{n-i} H\left(m, n, \binom{n}{i}\right) A_{m-i,0}y + \sum_{i=m}^{n} \alpha^m \beta^{i-m} H\left(m, i, \binom{i-1}{m-1}\right) A_{0,n-i}y + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \alpha^i \beta^{j-i} H\left(m, j+1, \binom{j}{i}\right) B_{m-i,n-j}y
$$

for $0 \leq m \leq [(T - s)/\lambda]$ and $0 \leq n \leq [(T - s)/\mu]$, where $\alpha = \frac{k}{\lambda}$, $\beta = \frac{\lambda - \mu}{\lambda}$, $l \wedge k = \min(l, k)$ and $\binom{j}{i}$ is the binomial coefficient.

**Lemma 2.4.** (I) Let $s \in [0, T)$ and let $\lambda > \mu > 0$ be such that $\lambda \omega_3 < 1$. Then, there exists a positive constant $K$, depending only on $M_i (i = 1, 2, 3, 4)$, such that

$$
\|C_p^2 P_{\lambda,m}(s) C_{11}y - C_p^2 P_{\mu,n}(s) C_{11}y\| \leq K\|y\| \left\{ 2((n\mu - m\lambda)^2 + T(\lambda - \mu))^{1/2} + T(\rho(\delta) + \rho(\delta)) + \frac{T^2}{\delta^2} \rho(T)(\lambda - \mu) \right\}
$$

for $1 \leq m \leq [(T - s)/\lambda]$, $1 \leq n \leq [(T - s)/\mu]$, $y \in Y$ and $\delta > 0$, where $\rho(r) = \sup\{\|A(t) - A(s)\|_{Y \rightarrow X} : t, s \in [0, T], |t - s| \leq r\}$ for $r \geq 0$. 

6
Let $0 \leq f \leq s \leq T_{\partial J1}d$ and let $\lambda > 0$ be such that $\lambda \omega_3 < 1$. Then there exists a positive constant $K$, depending only on $M_i(i=2,3)$, such that

$$(2.3) \quad \|C_2P_{\lambda,m}(s)C_1y - C_2P_{\lambda,m}(r)C_1y\| \leq KT\|y\|_Y\rho(s - r)$$

for $1 \leq m \leq \lfloor (T - s) / \lambda \rfloor$ and $y \in Y$.

**Proof:** By virtue of Lemma 2.3 we can show (2.2) in the same way as in the proof of [2, Theorem 2.1]. To prove (2.3), let $0 \leq r \leq s \leq T$ and let $\lambda > 0$ be such that $\lambda \omega_3 < 1$. For $1 \leq k \leq \lfloor (T - s) / \lambda \rfloor$ we define $A_k$ and $B_k$ by

$$\begin{align*}
A_k x &= P_{\lambda,k}(s)C_1x - P_{\lambda,k}(r)C_1x \quad \text{for } x \in X, \\
B_k y &= \lambda(A(s + k\lambda) - A(r + k\lambda))P_{\lambda,k}(s)C_1y \quad \text{for } y \in Y.
\end{align*}$$

Then, by a simple computation we have

$$A_k y = (J_\lambda(s + k\lambda) - J_\lambda(r + k\lambda))P_{\lambda,k-1}(s)C_1y + J_\lambda(r + k\lambda)(P_{\lambda,k-1}(s)C_1y - P_{\lambda,k-1}(r)C_1y)$$

$$= J_\lambda(r + k\lambda)(A_{k-1}y + B_k y)$$

for $y \in Y$. By solving this we find

$$A_m y = \sum_{i=1}^{m} \left( \prod_{k=1}^{m} J_\lambda(r + k\lambda) \right) B_i y$$

for $y \in Y$ and $1 \leq m \leq \lfloor (T - s) / \lambda \rfloor$. Therefore, we obtain the desired estimate (2.3) by the conditions $(A_2)$ and $(A_3)$. Q.E.D.

**Proof of Theorem 2.1:** Let $s, r \in [0, T)$ and let $\lambda > \mu > 0$ be such that $\lambda \omega_3 < 1$. Let $m$ and $n$ be integers such that $0 \leq s + m\lambda, r + n\mu \leq T$ and let $y \in Y$. If $s \leq r$ then $0 \leq s + n\mu \leq T$, so that $P_{\mu,n}(s)$ is well defined. Similarly, $P_{\lambda,m}(r)$ is well defined if $s \geq r$. Therefore, $C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(r)C_1y$ can be written as

$$\begin{align*}
C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(s)C_1y + (C_2P_{\mu,n}(s)C_1y - C_2P_{\mu,n}(r)C_1y) & \text{ if } s \leq r \\
C_2P_{\lambda,m}(s)C_1y - C_2P_{\lambda,m}(r)C_1y + (C_2P_{\lambda,m}(r)C_1y - C_2P_{\mu,n}(r)C_1y) & \text{ if } s \geq r.
\end{align*}$$
Applying Lemma 2.4 to this we see that there exists a positive constant $K$, depending only on $M_{i}(i = 1, 2, 3, 4)$, such that

$$\begin{align*}
\|C_{2}P_{\lambda_{n},m}(s)C_{1}y - C_{2}P_{\mu_{n},n}(r)C_{1}y\| \\
\leq K\|y\|Y \left\{ 2((n\mu - m\lambda)^{2} + T(\lambda - \mu))^{1/2} + T(\rho(|n\mu - m\lambda|) \\
+ \rho(\delta) + \rho(|r - s|) + \frac{T^{2}}{\delta^{2}}\rho(T)(\lambda - \mu) \right\}
\end{align*}$$

for $\delta > 0$ and $y \in Y$. Since $C_{1}(Y)$ is dense in $X$ and $\|C_{2}P_{\lambda_{n},m}(s_{n})\| \leq M_{3}$ for $n \geq 1$ it follows that

(2.4) $V_{2}(t,s)x = \lim_{n\to\infty}C_{2}\left( \prod_{i=1}^{n}J_{\lambda_{n}}(s_{n} + i\lambda_{n}) \right)x$

exists for $x \in X$ if $\{s_{n}\}$ is a sequence of nonnegative numbers with $\lim_{n\to\infty}s_{n} = s$ and $\{\lambda_{n}\}$ is a sequence such that $0 \leq s_{n} + n\lambda_{n} \leq T$ and $s_{n} + n\lambda_{n} \to t - s > 0$ as $n \to \infty$. Here we have used the fact that $\rho(\delta) \to 0$ as $\delta \to 0+$. We note that the limit is independent of $\{s_{n}\}$ and $\{\lambda_{n}\}$.

Let $\{s_{n}\}$ be a sequence of nonnegative numbers such that $\lim_{n\to\infty}s_{n} = s$ and let $\{\lambda_{n}\}$ be a sequence such that $0 \leq s_{n} + n\lambda_{n} \leq T$ and $s_{n} + n\lambda_{n} \to t - s > 0$ as $n \to \infty$. We then define $V_{1}^{(n)}(t,s)$ on $X$ by

$$V_{1}^{(n)}(t,s) = \begin{cases} C_{1} & \text{for } t = s, \\
(\prod_{i=1}^{n}J_{\lambda_{n}}(s_{n} + i\lambda_{n}))C_{1} & \text{for } s < t.
\end{cases}$$

Then, by the condition $(A_{2})$ we have

$$V_{1}^{(n)}(t,s)(Y) \subset Y \text{ and } \|V_{1}^{(n)}(t,s)\|Y \leq M_{2} \text{ for } 0 \leq s \leq t \leq T \text{ and } n \geq 1.$$ 

We now show that for $y \in Y$ and $y^{*} \in Y^{*}$, $\langle y^{*}, V_{1}^{(n)}(t,s)y \rangle$ is convergent. Let $\{n_{k}\}$ be any subsequence of $\{n\}$. Since $Y$ is reflexive there exists a subsequence $\{n_{k}'\}$ of $\{n_{k}\}$ and $y(t,s) \in Y$, depending upon $\{n_{k}'\}$, such that

$$\langle y^{*}, V_{1}^{(n_{k}')}t,s)y \rangle \to \langle y^{*}, y(t,s) \rangle$$

for $y^{*} \in Y^{*}$ as $n \to \infty$. In particular, for $x^{*} \in X^{*}$ we have

$$\langle C_{2}^{*}x^{*}, V_{1}^{(n_{k}')}t,s)y \rangle \to \langle C_{2}^{*}x^{*}, y(t,s) \rangle = \langle x^{*}, C_{2}y(t,s) \rangle$$
as $n \to \infty$, since $C_{2}^{*}x|_{Y} \in Y^{*}$. On the other hand, by (2.4) we obtain for $x^{*} \in X^{*}$,

$$
(C_{2}^{*}x^{*}, V_{1}^{(n_{k}')}^{(t,s)}y) = (x^{*}, C_{2}V_{1}^{(n_{k}')}^{(t,s)}y) \to (x^{*}, V_{2}(t,s)C_{1}y)
$$

as $n \to \infty$. Hence $C_{2}y(t,s) = V_{2}(t,s)C_{1}y$, so that $y(t,s)$ is independent of $\{n_{k}'\}$. Therefore it is proved that

$$
\lim_{n \to \infty} (y^{*}, V_{1}^{(n)}(t,s)y) = (y^{*}, C_{2}^{-1}V_{2}(t,s)C_{1}y)
$$

for $y \in Y$. By this together with the fact that $x^{*}|_{Y} \in Y^{*}$ we have for $x^{*} \in X^{*}$,

$$
(x^{*}, C_{2}^{-1}V_{2}(t,s)C_{1}y) = \lim_{n \to \infty} (x^{*}, V_{1}^{(n)}(t,s)y) \quad \text{for} \quad y \in Y.
$$

Hence

$$
\|C_{2}^{-1}V_{2}(t,s)C_{1}y\| \leq M_{1}\|y\|
$$

for $y \in Y$ and $0 \leq s \leq t \leq T$. Since $Y$ is dense in $X$ we see by the closed graph theorem that $C_{2}^{-1}V_{2}(t,s)C_{1} \in B(X)$ and $\|C_{2}^{-1}V_{2}(t,s)C_{1}\| \leq M_{1}$ for $0 \leq s \leq t \leq T$.

We now define $V_{1}(t,s)$ on $X$ by

$$
V_{1}(t,s) = C_{2}^{-1}V_{2}(t,s)C_{1} \quad \text{for} \quad 0 \leq s \leq t \leq T.
$$

Then, it follows from the fact which has been proved above that $\|V_{1}(t,s)\| \leq M_{1}, V_{1}(t,s)(Y) \subset Y, \|V_{1}(t,s)\|_{Y} \leq M_{2}$ and $C_{2}V_{1}(t,s) = V_{2}(t,s)C_{1}$ for $0 \leq s \leq t \leq T$. Moreover, we have

$$
\lim_{n \to \infty} \left< y^{*}, (\prod_{i=1}^{n} J_{\lambda_{n}}(s_{n}+i\lambda_{n}))C_{1}y \right> = \left< y^{*}, V_{1}(t,s)y \right>
$$

for $y \in Y$ and $y^{*} \in Y^{*}$ if $\{s_{n}\}$ is a sequence of nonnegative numbers such that $\lim_{n \to \infty} s_{n} = s$ and $\{\lambda_{n}\}$ is a sequence such that $0 \leq s_{n} + n\lambda_{n} \leq T$ and $s_{n} + n\lambda_{n} \to t - s > 0$ as $n \to \infty$.  


To prove that for $x \in X$, $(t,s) \rightarrow V_1(t,s)x$ is continuous on $\Delta$, since $Y$ is dense in $X$ and $\|V_1(t,s)\| \leq M_1$ on $\Delta$ it suffices to show that

\[(2.5) \quad \|V_1(t,s)y - V_1(\tau,s)y\| \leq K(t-\tau)\|y\|_Y \]

for $y \in Y$ and $0 \leq s \leq \tau \leq t \leq T$,

\[(2.6) \quad \|V_1(t,s+h)y - V_1(t,s)y\| \leq Kh\|y\|_Y \]

for $y \in Y$ and $0 \leq s \leq s+h \leq t \leq T$.

To prove (2.5), let $y \in Y$ and $0 \leq s \leq \tau \leq t \leq T$ and let $\lambda > 0$ be such that $\lambda \omega_3 < 1$. If $n$ and $m$ be integers such that $m < n \leq \lfloor (T-s)/\lambda \rfloor$ then

\[(2.7) \quad \langle x^*, P_{\lambda,n}(s)C_1y-P_{\lambda,m}(s)C_1y \rangle \]

\[= \left\langle x^*, \sum_{k=m}^{n-1} (P_{\lambda,k+1}(s)C_1y - P_{\lambda,k}(s)C_1y) \right\rangle \]

\[= \left\langle x^*, \lambda \sum_{k=m}^{n-1} A(s+(k+1)\lambda)P_{\lambda,k+1}(s)C_1y \right\rangle \quad \text{for} \quad x^* \in X^*, \]

from which it follows that

\[|\langle x^*, P_{\lambda,n}(s)C_1y - P_{\lambda,m}(s)C_1y \rangle| \]

\[\leq \|x^*\| \lambda(n-m) \cdot \sup \{\|A(t)\|_{Y \rightarrow X} : t \in [0,T]\} \cdot M_2 \|y\|_Y \]

for $x^* \in X^*$. Setting $n = \lfloor (t-s)/\lambda \rfloor$ and $m = \lfloor (\tau-s)/\lambda \rfloor$, and then letting $\lambda \rightarrow \infty$ we obtain the desired estimate (2.5).

To prove (2.6) let $y \in Y$ and $0 \leq s < s+h < t \leq T$, and choose a sequence $\{k(n)\}$ of integers such that $k(n)h/n \leq t - (s+h)$ and $k(n)h/n \rightarrow t - (s+h)$ as $n \rightarrow \infty$. Then, since

\[(2.8) \quad \left( \prod_{i=1}^{k(n)} J_{h/n}(s+h+ih/n) \right)y - \left( \prod_{i=1}^{n+k(n)} J_{h/n}(s+ih/n) \right)y \]

\[= \sum_{j=1}^{n} \left\{ \left( \prod_{i=j+1}^{n+k(n)} J_{h/n}(s+ih/n) \right)y - \left( \prod_{i=j}^{n+k(n)} J_{h/n}(s+ih/n) \right)y \right\} \]

\[= -(h/n) \sum_{j=1}^{n} \left( \prod_{i=j}^{n+k(n)} J_{h/n}(s+ih/n) \right)A(s+jh/n)y, \]
it follows from the conditions \((A_1)\) and \((A_4)\) that

\[
|x^*, P_{h/n,k(n)}(s+h)C_1y - P_{h/n,n+k(n)}(s)C_1y| \leq hM_1M_4\|y\|Y\|x^*\|
\]

for \(x^* \in X^*\). Passing to the limit as \(n \to \infty\) we obtain (2.6).

The strongly continuity of \(V_2(t,s)\) immediately follows from the strongly continuity of \(V_1(t,s)\) and the relation that \(C_2V_1(t,s) = V_2(t,s)C_1\), since \(C_1(X)\) is dense in \(X\) and \(\|V_2(t,s)\| \leq M_3\) on \(\Delta\).

Since \(Y\) is reflexive, using the strongly continuity of \(V_1(t,s)\) together with the facts that \(V_1(t,s)(Y) \subset Y\) and \(\|V_1(t,s)\|_Y \leq M_2\) on \(\Delta\) we see by a standard argument that for \(y \in Y\) and \(y^* \in Y^*\), \((t,s) \to \langle y^*, V_1(t,s)y \rangle\) is continuous for \(0 \leq s \leq t \leq T\).

To prove that \(\{V_1(t,s) : 0 \leq s \leq t \leq T\}\) has the property (d), let \(y \in Y, x^* \in X^*\) and \(0 \leq s \leq r < t \leq T\). Setting \(n = [(t-s)/\lambda]\) and \(m = [(r-s)/\lambda]\) in (2.7) we have

\[
\langle x^*, P_{\lambda,[(r-s)/\lambda]}(s)C_1y - P_{\lambda,[(t-s)/\lambda]}(s)C_1y \rangle = \int^{s+[(t-s)/\lambda]}_{s+[(r-s)/\lambda]} (\tilde{A}(\tau)^*x^*, P_{\lambda,[(\tau-s)/\lambda]+1}(\tau)s)C_1y d\tau,
\]

where \(\tilde{A}(t)^*: X^* \to Y^*\) denotes the adjoint of the restriction \(\tilde{A}(t)\) of \(A(t)\) to \(Y\). The condition \((A_4)\) implies that \(t \to \tilde{A}(t)^*\) is continuous in the \(B(X^*, Y^*)\) norm; thus passing to the limit as \(\lambda \to \infty\) we see by Lebesgue's convergence theorem that

\[
\langle x^*, V_1(t,s)y - V_1(r,s)y \rangle = \int^{t}_{r} (\tilde{A}(\tau)^*x^*, V_1(\tau,s)y) d\tau.
\]

This shows that the property (d) is satisfied.

We next show that \(\{V_2(t,s) : 0 \leq s \leq t \leq T\}\) has the property (e). Let \(0 \leq s < s + h < t \leq T\) and choose a sequence \(\{k(n)\}\) of integers such that
$k(n)h/n \leq t - (s + h)$ and $k(n)h/n \rightarrow t - (s + h)$ as $n \rightarrow \infty$. By (2.8) we have

$$C_{2}P_{h/n,k(n)}(s + h)y - C_{2}P_{h/n,n+k(n)}(s)y$$

$$= - \sum_{j=1}^{n} \int_{s+(j-1)h/n}^{s+jh/n} C_{2}P_{h/n,n+k(n)-j+1}(s+(j-1)h/n)A(s+jh/n)y \, dr$$

$$= - \int_{s}^{s+h} C_{2}P_{h/n,n+k(n)-r(n)}(s+r(n)h/n)A(s+(r(n)+1)h/n)y \, dr$$

for $y \in Y$, where $r(n) = \lfloor (r - s)/(h/n) \rfloor$. Letting $n \rightarrow \infty$ in this equality we see that the property (e) is satisfied.

Suppose that $\{W_{1}(t, s)\}, \{W_{2}(t, s)\}$ is a pair of strongly continuous families of bounded linear operators defined on the triangle $\Delta$ with the properties (a) - (e). Then, by the properties (d) and (e) we see that for $y \in Y$, the function $r \rightarrow V_{2}(t, r)W_{1}(r, s)y$ is Lipschitz continuous and $(\partial/\partial r)V_{2}(t, r)W_{1}(r, s)y = 0$ for almost every $r \in [s, T]$. Integrating this from $s$ to $t$ we obtain

$$C_{2}W_{1}(t, s)y = V_{2}(t, s)C_{1}y$$

for $y \in Y$. By the property (a), $W_{2}(t, s)$ is equal to $V_{2}(t, s)$ on the dense subspace $C_{1}(Y)$ of $X$, so that $\{W_{1}(t, s)\}, \{W_{2}(t, s)\}$ is the only pair of strongly continuous families of bounded linear operators defined on the triangle $\Delta$ with the properties (a) - (e). Q.E.D.

**Definition 2.1.** A function $u(\cdot; s, x)$ on $[s, T]$ is a strong solution of $(DE)$, if

(i) $u(\cdot; s, x) \in A^{1,1}(s, T; X)$,

(ii) $u(\cdot; s, x)$ satisfies $(DE)$, almost everywhere.

Here we denote by $A^{k,p}(a, b; X)$ the space of all absolutely continuous functions $u : [a, b] \rightarrow X$ for which $d^{j}u/dt^{j}$ exist (and are defined almost everywhere) for $j = 1, 2, \ldots, k$ such that $d^{j}u/dt^{j}, j = 1, 2, \ldots, k-1$, are all absolutely continuous and $d^{k}u/dt^{k} \in L^{p}(a, b; X)$.

Existence and uniqueness of the strong solutions of the time-dependent differential equation $(DE)$, is provided by
Theorem 2.5. If the family \( \{A(t) : t \in [0, T]\} \) of closed linear operators in \( X \) satisfies the conditions \((A_1) - (A_4)\) then, for every initial data \( x \in C_1(Y) \) the \((DE)_s\) has a unique strong solution satisfying \( u(t;s, x) \in Y \) for \( t \in [s, T] \) and 
\[
sup\{\|u(t;s, x)\|_Y : t \in [s, T]\} < \infty.
\]

Proof: By Theorem 2.1 there exists a unique pair \( (\{V_1(t, s)\}, \{V_2(t, s)\}) \) of strongly continuous families of bounded linear operators defined on the triangle 
\[ \Delta = \{(t, s) : 0 \leq s \leq t \leq T\} \] with the properties (a) - (e). Let \( x \in C_1(Y) \) and set \( u(t; s, x) = V_1(t, s)C_1^{-1}x \) for \( 0 \leq s \leq t \leq T \). Then, it is easy to see that \( u(t; s, x) \) is a strong solution of \((DE)_s\) satisfying \( u(t; s, x) \in Y \) for \( t \in [s, T] \) and 
\[
sup\{\|u(t;s, x)\|_Y : t \in [s, T]\} < \infty.
\]
To prove the uniqueness of the solutions, let \( v(t; s, x) \) be a strong solution of \((DE)_s\) satisfying \( v(t; s, x) \in Y \) for \( t \in [s, T] \) and 
\[
sup\{\|v(t;s, x)\|_Y : t \in [s, T]\} < \infty.
\]
Then, we deduce from the property (e) that 
\[
(r \rightarrow V_2(t, r)(u(r;s,x) - v(r;s,x))) \text{ is absolutely continuous on } [s, T] \text{ and }
\]
\[
(\partial/\partial r)V_2(t, r)(u(r;s,x) - v(r;s,x)) = 0
\]
for almost every \( r \in [s,T] \). Integrating this equality from \( s \) to \( t \) we have 
\[
C_2(u(t;s, x) - v(t; s, x)) = 0,
\]
which shows that \( u(t; s, x) = v(t; s, x) \) for \( t \in [s, T] \), since \( C_2 \) is injective. Q.E.D.

We next consider the second order differential equation in a reflexive Banach space \( X \)
\[(DE)_s^2 \begin{cases} u''(t) = Au(t) + B(t)u(t) & \text{for } t \in [s, T] \\ u(s) = x, \ u'(s) = y, \end{cases}
\]
where \( A \) is the infinitesimal generator of a cosine family and \( \{B(t) : t \in [0, T]\} \) is a family of linear operators in \( X \) satisfying the following conditions:
(B1) $D(A) \subset D(B(t))$ for $t \in [0, T]$.

(B2) There are constants $M \geq 0$ and $\omega \geq 0$ such that $\{\lambda^2 : \lambda > \omega\} \subset \rho(A)$, for $t \in [0, T]$ $B(t)R(\lambda^2 : A)$ is strongly infinitely differentiable in $\lambda > \omega$ and satisfies

$$
\|(1/n!)(\lambda - \omega)^{n+1}(d/d\lambda)^{n}B(t)R(\lambda^2 : A)x\| \leq M\|x\|
$$

for $x \in X, \lambda > \omega$ and $n = 0, 1, \cdots$.

(B3) $\lim_{\ell \to \epsilon} \sup\{\|B(t)x - B(s)x\| : x \in D(A), \|x\| + \|Ax\| \leq 1\} = 0$.

(B4) There exists $\lambda_0 > \omega$ such that $(\lambda_0^2 - A)B(t)R(\lambda_0^2 : A) = B(t) + P(t)$, where $\{P(t) : t \in [0, T]\}$ is a strongly continuous family of bounded linear operators on $X$.

Definition 2.2. A function $u(\cdot; s, x, y)$ on $[s, T]$ is a strong solution of $(DE)_{\epsilon}^{2}$ if
(i) $u(\cdot; s, x, y) \in A^{2,1}(s, T; X)$,
(ii) $u(\cdot; s, x, y)$ satisfies $(DE)_{\epsilon}^{2}$ almost everywhere.

Without proof we state the existence and uniqueness theorem of the strong solutions of the second order differential equation $(DE)_{\epsilon}^{2}$ which is obtained by applying Theorem 2.5 with $A(t) = \begin{pmatrix} 0 & 1 \\ A + B(t) & 0 \end{pmatrix}$ and $C_1 = C_2 = \begin{pmatrix} 0 \\ A - \lambda_0^2 \end{pmatrix}$.

Theorem 2.6. Assume that $A$ is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0, T]\}$ is a family of linear operators in $X$ satisfying the conditions $(B_1)$ - $(B_4)$. Then, for every initial data $x \in D(A)$ and $y \in D(A)$ the $(DE)_{\epsilon}^{2}$ has a unique strong solution $u(t; s, x, y)$ such that $u(t; s, x, y) \in D(A)$ for $t \in [s, T]$ and $\sup\{\|Au(t; s, x, y)\| : t \in [s, T]\} < \infty$.

References


