CONVERGENCE THEOREMS FOR THE PSEUDO–CONFORMALLY INVARIANT NONLINEAR SCHRÖDINGER EQUATION

HAYATO NAWA

ABSTRACT. This paper is concerned with the Cauchy problem for the nonlinear Schrödinger equation;

\[
(C(p)) \quad \begin{cases} 
2i\frac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u = 0, \\
u(0,x) = u_0(x), 
\end{cases} \\
(t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
\begin{array}{l} 
x \in \mathbb{R}. 
\end{array}
\]

If \(1 < p < 1 + \frac{4}{N}\), there exists a global solution \(u_p \in C_b(\mathbb{R};H^1(\mathbb{R}^N))\), for any \(u_0 \in H^1(\mathbb{R}^N)\). If \(p \geqq 1 + \frac{4}{N}\), there is a singular solution exploding its \(L^2\) norm of the gradient in a finite time for some \(u_0 \in H^1(\mathbb{R}^N)\). Suppose that \(u_0\) leads to such a singular solution for \(p = 1 + \frac{4}{N}\). Let \(\{u_p\} \subseteq C(\mathbb{R};H^1(\mathbb{R}^N))\) be solutions to \((C(p))\) for \(1 < p < 1 + \frac{4}{N}\). We study the behavior of \(u_p\) as \(p \uparrow 1 + \frac{4}{N}\), and we apply the result to the blow-up problem for solutions of \(C(1 + \frac{4}{N})\).

0. INTRODUCTION

This paper is concerned with the Cauchy problem for the nonlinear Schrödinger equation;

\[
(C(p)) \quad \begin{cases} 
2i\frac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u = 0, \\
u(0,x) = u_0(x), 
\end{cases} \\
(t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
\begin{array}{l} 
x \in \mathbb{R}. 
\end{array}
\]

Here \(i = \sqrt{-1}\), \(u_0 \in H^1(\mathbb{R}^N)\) and \(\Delta\) is the Laplace operator on \(\mathbb{R}^N\).

The local existence theory for \((C(p))\) is well known for \(1 < p < 2^* - 1\) \(\left(2^* = \frac{2N}{N-2}\right)\) if \(N \geqq 3\), = arbitrary number larger than 1 if \(N = 1, 2\); for any \(u_0 \in H^1(\mathbb{R}^N)\), there are \(T_m \in (0, \infty)\) (maximal existence time) and a unique solution \(u(\cdot) \in C([0,T_m);H^1(\mathbb{R}^N))\). Furthermore \(u(\cdot)\) satisfies

\[
(0.1) \quad \|u(t)\| = \|u_0\|, \\
(0.2) \quad E_{p+1}(u(t)) = \|\nabla u(t)\|^2 - \frac{2}{p+1}\|u(t)\|_{p+1}^{p+1} = E_{p+1}(u_0).
\]

for \(t \in [0,T_m]\). For this theory, see e.g. [6] and [9]. Here \(\| \cdot \|\) and \(\| \cdot \|_{p+1}\) denotes the \(L^2\) norm and \(L^{p+1}\) norm respectively.

We know (see [6] [8] [9]);

\[
\]
(i) If $1 < p < 1 + \frac{4}{N}$, there exists a global solution $u_p \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$, for any $u_0 \in H^1(\mathbb{R}^N)$.

(ii) If $p \geq 1 + \frac{4}{N}$, there is a singular solution exploding its $L^2$ norm of the gradient in a finite time for some $u_0 \in H^1(\mathbb{R}^N)$.

Suppose that $u_0$ leads to such a singular solution for $p = 1 + \frac{4}{N}$. Let $\{u_p\} \subset C(\mathbb{R}; H^1(\mathbb{R}^N))$ be solutions to $(C(p))$ for $1 < p < 1 + \frac{4}{N}$. As we have seen above, the number $p = 1 + \frac{4}{N}$ is the critical number for the existence of blow-up solutions to $C(p)$. It is a natural question to investigate the behavior of $u_p$ as $p \uparrow 1 + \frac{4}{N}$.

We note that it can occur that

\[
\limsup_{p \uparrow 1 + \frac{4}{N}} \|u_p(t)\|_\sigma = \infty.
\]

Let

\[
\lambda_p = \frac{1}{\sup_{t \in \mathbb{R}}\|u_p(t)\|_\sigma^{\sigma/2}}
\]

where $\sigma = 2 + \frac{4}{N}$.

We will consider the rescaling function;

\[
u_p^\lambda(t, x) = \lambda_p^{N/2}u(\lambda_p^2t, \lambda_p x)
\]

and analyze the behavior of $u_p^\lambda(t, x)$ as $p \uparrow 1 + \frac{4}{N}$ in $L^\infty(\mathbb{R}; L^\sigma(\mathbb{R}^N))$. We are lead in a natural way to the consideration of a function satisfying the following pseudo-conformally invariant nonlinear Schrödinger equation (see e.g. [19]);

\[(NS-\lambda)
\]

\[
2i\frac{\partial u}{\partial t} + \Delta u + \lambda|u|^{p-1}u = 0,
\]

where

\[
(0 \neq) \lambda \equiv \lim_{p \uparrow 1 + \frac{4}{N}} \lambda_p^{-N(p+1-\sigma)/2}(\leq 1).
\]

Now we explain other motivations of our analysis. The nonlinear Schrödinger equation of the form $(NS-\lambda)$ (with $N = 2$) arises in a theory of the stationary self-focussing of a laser beam propagating along the $t$-axis in a nonlinear medium (see e.g. [1] [2] [26]).

(i) In [1] and [2], Akhmanov et al analyzed a laser beam producing two foci on the $t$-axis. In their papers, "producing two foci of a laser beam" is explained as follows; (roughly speaking) a solution to $(NS-\lambda)$ blows up at a time $T_m$, and it continues beyond $T_m$ and blows up again. Their argument, however, seems to be "physics" not "mathematics". We try to give a mathematical meaning to the phenomenon of "producing two foci of a laser beam" by our subcritical approximated approach. (See § 4 Proposition 4.3 and Conclusion.)
In previous papers [15], [16] and [17], we have been studying the formation of singularities in solutions to the nonlinear Schrödinger equation of the form (NS-$\lambda$) and the like. Now we know that one can understand the focus of a laser beam as "mass concentration" phenomena in blow-up solutions to (NS-$\lambda$). However the shape of blow-up solutions has not been investigated well. Our subcritical approximated approach may obtain more information about the shape of blow-up solution near the blow-up time. (See § 3 Theorem C.)

Our subcritical approximated approach is inspired by the work of Yamabe [25].

For the simplification of arguments below, in this paper we assume

**Assumption.**

If $u$ is a semi global solution of (NS-$\lambda$) such that $u \in C_b((T, \infty); H^1(\mathbb{R}^N))$ or $u \in C_b((\infty, T)); H^1(\mathbb{R}^N))$ for some $T \in \mathbb{R}$, then $E_{\sigma}^\lambda(u) \geq 0$.

**Remark.** If $N=1$ or $u_0 \in H^1(\mathbb{R}^N) \cap L^2(|x|^2\,dx)$, this assumption is true (see Ogawa and Y. Tsutsumi [20] [21]).

Our main theorem is

**Theorem A.** Let $\{p_n\}$ be a sequence such that $p_n \uparrow 1 + \frac{4}{N}$ and $u_{p_n} \in C(\mathbb{R}; H^1(\mathbb{R}^N))$ be a solution to $C(p_n)$. Suppose that

\begin{equation}
\lim_{n \to \infty} \sup_{t \in \mathbb{R}^N} \| \nabla u_{p_n}(t) \| = \lim_{n \to \infty} \sup_{t \in \mathbb{R}^N} \| u_{p_n}(t) \|_{\sigma} = \infty.
\end{equation}

We put

\begin{align}
(\text{A.2}) & \quad \lambda_n = \lambda_{p_n}, \quad u_n(t, x) = \lambda_n^{N/2} u_{p_n}(\lambda_n^2 t, \lambda_n x), \\
(\text{A.3}) & \quad E_{\sigma}^\lambda(v) = \| \nabla v \|^2 - \frac{2}{\sigma} \lambda \| v \|_\sigma^\sigma.
\end{align}

Then there exists a subsequence of $\{u_n\}$ (we still denote it by $\{u_n\}$) which satisfies the following properties: one can find $L \in \mathbb{N}$, nontrivial solutions $\{u^j\}$ of (NS-$\lambda$) in $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$ with $E_{\sigma}^\lambda(u^j) = 0$ and sequences $\{(s_n, y_{n}^j)\} \subset \mathbb{R} \times \mathbb{R}^N$ for $1 \leq j \leq L$ such that

\begin{align}
(\text{A.4}) & \quad \lim_{n \to \infty} \| (s_n, y_n^j) - (s_n, y_n^k) \| = \infty \quad (j \neq k), \\
(\text{A.5}) & \quad u_{n}^1 \equiv u_n(\cdot + s_n, \cdot + y_n^1) \rightharpoonup u^1 \text{ in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \\
(\text{A.6}) & \quad u_{n}^j \equiv (u_{n}^{j-1} - u^{j-1})(\cdot, \cdot + y_n^j) \rightharpoonup u^j \text{ (} j \geq 2 \text{) in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \\
(\text{A.7}) & \quad \lim_{n \to \infty} \int_I \{ E_{\sigma}^\lambda(u_{n}^j) - E_{\sigma}^\lambda(u_{n}^j - u^j) - E_{\sigma}^\lambda(u^j) \} \, dt = 0, \text{ for any } I \subset \mathbb{R}, \\
(\text{A.8}) & \quad \lim_{n \to \infty} \| u_{n}^L(0) - u^L(0) \|_{\sigma} = 0.
\end{align}

**Remarks.** (1) It is worth while to note that if

\begin{align}
(\text{0.7}) & \quad \lim_{n \to \infty} \sup_{t \in \mathbb{R}} \| u_n(t + s_n, \cdot) - \sum_{j=1}^L u^j(t, \cdot - \sum_{k=1}^j y_n^k) \|_{\sigma} > 0,
\end{align}
there exists \( \{(s_{n}^{2}, y_{n}^{2,1})\} \in \mathbb{R} \times \mathbb{R}^{N} \) such that

\[
(0.8) \quad (u_{n}^{L} - u^{L})(\cdot + s_{n}^{2}, \cdot + y_{n}^{2,1}) \nrightarrow u^{2,1} \quad \text{in} \quad L^{\infty}(\mathbb{R}; H^{1}(\mathbb{R}^{N})).
\]

One can see that \( u^{2,1} \) is almost a solution to (NS-\( \lambda \)) near \( t = 0 \).

(2) If Assumption were not true for \( N \geq 2 \), it could occur \( L = \infty \) in Theorem A.

(3) If \( u_{0} \) is radially symmetric or \( ||u_{0}|| = ||Q|| \), we have \( L = 1 \) in Theorem A without Assumption. Here \( Q(x) \) is a nontrivial minimal \( L^{2} \) norm solution to (NSF)

\[
\begin{cases}
\Delta Q - Q + |Q|^{4}\pi Q = 0 \\
Q \in H^{1}(\mathbb{R}^{N})
\end{cases}
\]

We note that if \( ||u_{0}|| = ||Q||, \lambda = 1 \) in (0.6). For (NSF), see e.g. \([5][23]\).

(4) Theorem A seems to be closely related to a phenomenon which has been observed in various nonlinear problem by the name of bubble theorem or concentration-compactness theorem (for example, see \([3][11][12][22]\)).

The rest of paper is arranged as follows;

1. Lemmata
   The proof of Theorem A is inspired by the work of Brézis and Coron \([3]\). One may see the underlying idea being the method of concentration-compactness due to Lions \([11][12]\). We, however, do not use the general method of it. In this section we prepare several lemmata to prove Theorem A.

2. Proof of Theorem A
   We conclude the proof of Theorem A.

3. Application to the blow-up problem for \( C(1 + \frac{4}{N}) \)
   Using the idea of section 1, we study the shape of blow-up solution to \( C(1 + \frac{4}{N}) \) near the blow-up time.

   We finish with a suggestion that how understand the "two foci" of a laser beam as a mathematical theory.

Acknowledgement. The author would like to express his deep gratitude to professors D. Fujiwara and A. Inoue for having interest in this study and helpful discussions. The author is grateful to professor Y. Kametaka who brought papers \([1][2]\) to his attention. The author also grateful to professor A. Matsumura who kindly showed his unpublished numerical results.

1. Lemmata and related results
In this section we prepare several lemmata which is crucial for the proof of Theorem A. One may find that the argument in their proofs are closely related to the weak compactness result due to Lieb \([10]\) and Brézis and Lieb's lemma \([4]\).

We will use the following notations;

\( \mu = \text{Lebesgue measure on } \mathbb{R}^{N} \),
\[ [f > \varepsilon] = \{x \in \mathbb{R}^{N}; f(x) > \varepsilon \} \text{ (or = the characteristic function of this set)}, \]
\[ B(y; R) = \{x \in \mathbb{R}^{N}; |x - y| \leq R \}. \]
Lemma 1.1. Let \( 1 < \alpha < \beta < \gamma \) and let \( g(t, x) \) be a measurable function on \( \mathbb{R} \times \mathbb{R}^N \) such that, for some positive constants \( C_\alpha, C_\beta, C_\gamma \),

\[
\begin{align*}
&\sup_{t\in\mathbb{R}} \|g(t)\|_\alpha^\alpha \leq C_\alpha, \\
&\sup_{t\in\mathbb{R}} \|g(t)\|_\beta^\beta \geq C_\beta > 0, \\
&\sup_{t\in\mathbb{R}} \|g(t)\|_\gamma^\gamma \leq C_\gamma.
\end{align*}
\]

Then one has

\[
\sup_{t\in\mathbb{R}} \mu(\{|g(t, \cdot)| > \eta\}) > C
\]

for some \( \eta, C > 0 \) depending on \( \alpha, \beta, \gamma, C_\alpha, C_\beta, C_\gamma \), but not on \( g \).

Proof. Simple calculation with (1.1) - (1.3) implies that, for sufficiently small \( \eta > 0 \),

\[
\begin{align*}
\int_{\mathbb{R}^N} |g(t, x)|^\beta \, dx &= \int_{|g(t, \cdot)| < \eta} |g(t, x)|^\beta \, dx + \int_{\eta < |g(t, \cdot)| < \frac{1}{\eta}} |g(t, x)|^\beta \, dx + \int_{|g(t, \cdot)| > \frac{1}{\eta}} |g(t, x)|^\beta \, dx \\
&\leq \frac{C_\beta}{4C_\alpha} \int_{|g(t, \cdot)| < \eta} |g(t, x)|^\alpha \, dx + \int_{\eta < |g(t, \cdot)| < \frac{1}{\eta}} |g(t, x)|^\beta \, dx \\
&\quad + \frac{C_\beta}{4C_\gamma} \sup_{t\in\mathbb{R}} \|g(t)\|_\gamma^\gamma \\
&\leq \frac{C_\beta}{2} \eta^\beta + \mu(\{|g(t, \cdot)| > \eta\})(\frac{1}{\eta})^\beta.
\end{align*}
\]

Thus we have (1.4) with \( C = \frac{C_\beta}{2} \eta^\beta \).

Lemma 1.2. Let \( 1 \leq \alpha < \infty \) and let \( v \) be a function such that \( v(\cdot) \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)) \), \( \sup_{t\in\mathbb{R}} \|\nabla v(t, \cdot)\|_\alpha \leq C_1 \) and \( \sup_{t\in\mathbb{R}} \mu(\{|v(t, \cdot)| > \eta\}) > C_2 \) for some positive constants \( C_1, \eta, C_2 \). Then there exists a shift \( T_{s,y}v(t, x) = v(t + s, x + y) \) such that, for some constant \( \delta = \delta(C_1, C_2, \eta) \),

\[
\mu(\{B(0;1) \cap |T_{s,y}v(0, \cdot)| > \frac{\eta}{2}\}) > \delta.
\]

Proof. We borrow the idea of Brézis in Lieb [10]. Let \( f \) be a function such that \( f(\cdot) \in L^\infty(\mathbb{R}; L^\alpha_{loc}(\mathbb{R}^N)) \), \( \sup_{t\in\mathbb{R}} \|\nabla f(t, \cdot)\|_\alpha \leq 1 \). First we claim that there exists a point \( (s, y) \in \mathbb{R} \times \mathbb{R}^N \) such that

\[
\int_{C_s} |\nabla f(s, x)|^\alpha \, dx < K \int_{C_s} |f(s, x)|^\alpha \, dx,
\]

where \( K > 0 \).
where
\[ K = 1 + \frac{1}{\sup_{t \in \mathbb{R}} \| f(t) \|_{\alpha}^\alpha}, \]
\[ C_y = \text{cube in } \mathbb{R}^N \text{ with center } y \text{ and the side length } \frac{1}{\sqrt{2}}. \]

One can easily show (1.6) by simple contradiction argument. By (1.6) one has
\[ \int_{C_y} |\nabla f(s, x)|^\alpha + |f(s, x)|^\alpha \, dx < (K + 1) \int_{C_y} |f(s, x)|^\alpha \, dx. \]

On the other hand, by Sobolev's inequality we have
\[ \int_{C_y} |\nabla f(s, x)|^\alpha + |f(s, x)|^\alpha \, dx \geq S \left( \int_{C_y} |f(s, x)|^\alpha \, dx \right)^{\frac{\alpha^*}{\alpha}}, \]
where \( \frac{1}{\alpha} + \frac{1}{N} = \frac{1}{\alpha} \) if \( \alpha < N \) and, if \( \alpha \geq N \), \( \alpha^* \) is arbitrary with \( \alpha < \alpha^* < \infty \). S depends only on \( \alpha, \alpha^* \).

Combining (1.7), (1.8) and Hölder's inequality we obtain
\[ S < (K + 1) \mu(C_y \cap \text{suppf}(s, \cdot))^{1-g_{\alpha}-}. \]

Now we put \( f(t, x) = \max(v(t, x) - \frac{\eta}{2}, 0) \). For simplicity we assume that \( \| \nabla v(t) \|_{\alpha} \leq 1 \) so that \( \sup_{t \in \mathbb{R}} \| \nabla f(t, \cdot) \|_{\alpha} \leq 1 \). From the assumption of this lemma we have
\[ \sup_{t \in \mathbb{R}} \| v(t) \|_{\alpha}^\alpha \geq \left( \frac{\eta}{2} \right)^{\alpha} \sup_{t \in \mathbb{R}} \mu(\{ |v(t, \cdot)| > \frac{\eta}{2} \}) \geq \left( \frac{\eta}{2} \right)^{\alpha} C_2, \]

and thus \( K \leq 1 + \frac{2^2}{\eta^2 C_2} \). From (1.9) we deduce (1.5) for some point \((s, y) \in \mathbb{R} \times \mathbb{R}^N \) and some constant \( \delta \) depending only on \( N, \alpha, \eta, C_2, \) and \( C_1 \).

Combining above two lemmata, we have by Ascoli-Arzela lemma

**Lemma 1.3.** Let \( 1 < \alpha < \beta < \gamma \) and let \( \{v_n(t, x)\} \) be a uniformly equibounded family in \( C_b(\mathbb{R}; W^{1,\alpha}(\mathbb{R}^N)) \) such that, for some positive constants \( C_\alpha, C_\beta, C_\gamma \),
\begin{align*}
(1.11) & \sup_{t \in \mathbb{R}} \| v_n(t) \|_{\alpha}^\alpha \leq C_\alpha, \\
(1.12) & \sup_{t \in \mathbb{R}} \| v_n(t) \|_{\beta}^\beta \geq C_\beta > 0, \\
(1.13) & \sup_{t \in \mathbb{R}} \| v_n(t) \|_{\gamma}^\gamma \leq C_\gamma. 
\end{align*}

Suppose that \( \{v_n(t, x)\} \) is a uniformly equicontinuous family in \( C_b(\mathbb{R}; L^\alpha(\mathbb{R}^N)) \). Then there exist a family of shifts \( \{(s_n, y_n)\} \subset \mathbb{R} \times \mathbb{R}^N \) such that,
\begin{align*}
(1.14) & v_n(\cdot + s_n, \cdot + y_n) \rightarrow v \neq 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \\
(1.15) & v_n(\cdot + s_n, \cdot + y_n) \rightarrow v \neq 0 \quad \text{strongly in } C(I; L^\alpha(\Omega)), 
\end{align*}
for some \( v \in C_b(\mathbb{R}; W^{1,\alpha}(\mathbb{R}^N)) \) (modulo subsequence). Here \( I \times \Omega \subset \mathbb{R} \times \mathbb{R}^N \).

Following proposition will play a very important roll in our analysis.
Proposition 1.4. Let \( \{f_n(x)\} \) be a bounded sequence of functions in \( H^1(\mathbb{R}^N) \) such that, for some positive constants \( C_\sigma \),

\[
\|f_n(t)\|_\sigma \geq C_\sigma > 0,
\]

\[
\lim_{n \to \infty} \sup_{n} E_\sigma^\lambda(f_n) = \lim_{n \to \infty} \sup_{n} \left( \|\nabla f_n\|^2 - \frac{2}{\sigma} \lambda \|f_n\|_\sigma^\sigma \right) \leq 0
\]

Then there exists a subsequence of \( \{f_n\} \) (we still denote it by \( \{f_n\} \)) which satisfies the following properties: one can find \( L \in \mathbb{N} \cup \{\infty\} \) and sequences \( \{y_n^j\} \subset \mathbb{R}^N \) for \( 1 \leq j < L \) such that

\[
\lim_{n \to \infty} |y_n^j - y_n^k| = \infty \quad (j \neq k),
\]

\[
f_n^j \equiv f_n(\cdot + y_n^j) \to f^j \neq 0 \text{ weakly in } H^1(\mathbb{R}^N) \quad (j \geq 2),
\]

\[
\lim_{n \to \infty} \{E_\sigma^\lambda(f_n^j) - E_\sigma^\lambda(f_n^j - f^j) - E_\sigma^\lambda(f^j)\} = 0,
\]

\[
\lim_{n \to \infty} \|f_n^L - f^L\|_\sigma = 0 \text{ if } L < \infty,
\]

\[
\lim_{j \to L} \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y;R)} |f_n^j(x) - f^j(x)|^2 \, dx = 0 \text{ if } L = \infty.
\]

Proposition 1.4 is a time independent version of Lemma 1.3 with \( \alpha = 2^*, \beta = \sigma, \gamma = 2 \) and the extra condition (1.16). For its proof, we also need Brézis-Lieb's lemma [4] (see Lemma 1.5 below). In fact (1.16) together with Brézis-Lieb's lemma implies (1.20) and (1.21). One can find a complete proof in Nawa [16].

Remarks. (1) Proposition 1.4 asserts that \( f_n \) behaves like a superposition of several parts \( f_n^1, f_n^2, \cdots, f_n^L \) (\( L \) may be infinite) as \( n \to \infty \).

(2) Above arguments are somewhat related to those used in Lions [11] [12], Brézis and Coron [3] and Struwe [22].

Proposition 1.4 is very useful to study "mass concentration" phenomena in solutions to \( (C(1 + \frac{4}{N})) \). In [16], we proved following theorem by using Proposition 1.4 (with \( \lambda = 1 \)) and the characterization of minimal \( L^2 \) norm solution to (NSF) (see Remark below).

Theorem B. Let \( u(t) \) be a blow-up solution to \( (C(1 + \frac{4}{N})) \) which blows up at time
Let \( \{ t_n \} \) be any sequence such that \( t_n \to T_m \) as \( n \to \infty \). Set

\[
\tilde{\lambda}_n \equiv \frac{1}{\| u(t_n) \|^{\sigma/2}},
\]

\[
u_n(x) \equiv \frac{1}{\| u(t_n) \|^{\sigma/2}} u(t_n, \tilde{\lambda}_n x).
\]

Then there exists a subsequence of \( \{ t_n \} \) (we still denote it by \( \{ t_n \} \)) which satisfies the following properties: one can find a sequence \( \{ y_n \} \) in \( \mathbb{R}^N \) such that, for any \( \epsilon \), there is a positive constant \( K > 0 \);

\[
\liminf_{n \to \infty} \int_{B_n} |u(t_n, x)|^2 dx \geq (1 - \epsilon) \| Q \|^2,
\]

where \( B_n = \{ x \in \mathbb{R}^N; |x - \tilde{\lambda}_n y_n| \leq K \tilde{\lambda}_n \} \) and \( Q \) is a minimal \( L^2 \) norm solution to (NSF).

For the proof of this theorem, we employ Proposition 1.4 with putting \( f_n = u_n \). One can find a complete proof in Nawa [16]. More precise study for "path" \( y(t) \) (not sequence \( \{ y_n \} \)) is found in Nawa [15] [18].

Remark. The minimal \( L^2 \) norm solution to (NSF) is a solution to the following variational problem; Find \( Q \in H^1(\mathbb{R}^N) \) such that

\[
\| Q \| = \inf_{v \in H^1(\mathbb{R}^N)} \left\{ \| v \| ; E_\sigma(v) = \| \nabla v \|^2 - \frac{2}{\sigma} \| v \|^\sigma \leq 0 \right\}.
\]

Using Proposition 1.4, we can solve this variational problem (see Theorem D in Appendix of this paper).

We conclude this section with Brézis-Lieb's lemma [4] and its variant adopted to our problem for convenience.

Lemma 1.5. Let \( \{ v_n(t, x) \} \) be an bounded family in \( L^\sigma(I \times \Omega) \) where \( I \times \Omega \subset \mathbb{R} \times \mathbb{R}^N \). Suppose that \( v_n \to v \) a.e. in \( I \times \Omega \). Then

\[
|v_n|^\sigma v_n - |v_n - v|^\sigma (v_n - v) - |v|^\sigma v \to 0 \quad \text{in} \quad L^\sigma(I \times \Omega),
\]

where \( \frac{1}{\sigma} + \frac{1}{\sigma'} = 1 \), and we have

\[
\lim_{n \to \infty} \int_{I \times \Omega} |v_n|^\sigma - |v_n - v|^\sigma - |v|^\sigma |dt \, dx = 0.
\]

2. PROOF OF THEOREM A

The purpose of this section is to prove Theorem A. For simplicity we suppose \( N \geq 3 \). First we note that the rescaled function \( u_n(t, x) = \lambda_n N/2 u_p(\lambda_n^2 t, \lambda_n x) \) belongs to \( C_b(\mathbb{R}; H^1(\mathbb{R}^N)) \) and satisfies

\[
2i \frac{\partial u_n}{\partial t} + \Delta u_n + \lambda_n^{-N(p_n+1-\sigma)/2} |u_n|^{p_n-1} u_n = 0.
\]
For one can easily check that

\[(2.2) \quad \|u_n(t)\| = \|u_0\|,\]
\[(2.3) \quad \sup_{t \in \mathbb{R}} \|u_n(t)\|_{\sigma} = 1,\]
\[(2.4) \quad E_{\sigma}^{\lambda}(u_n) = \lambda_{n}^{2}E_{p_{n}+1}(u_0) + \lambda_{n}^{-N(p_{n}+1-\sigma)/2} \frac{2}{p_{n}+1} \|u_n(t)\|_{p_{n}^{\sigma}}^{p_{n}+1} \ddagger \frac{2}{\sigma} \|u_n(t)\|_{\sigma}^{\sigma}.\]

$H^1$ boundedness follows from (2.2) - (2.3) with the help of Hölder inequality. We have from $H^1$ boundedness,

\[(2.5) \quad \sup_{t \in \mathbb{R}} \|\tilde{u}_n(t)\|_{2^*} \leq C_{2^*},\]

for some constant $C_{2^*} > 0$. We note that $\{u_n(t, x)\}$ is a uniformly equicontinuous family in $C_b(\mathbb{R}; L^2(\mathbb{R}^N))$, and form a uniformly equibounded family in $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$.

We are now in a position to apply Lemma 1.3 to $\{u_n(t, x)\}$.

**Lemma 2.1.** There exist a family of shifts $\{(s_n, y_n^1)\} \subset \mathbb{R} \times \mathbb{R}^N$ such that,

\[(2.6) \quad u_n^1 \equiv u_n(\cdot + s_n, \cdot + y_n^1) \rightarrow u^1 \neq 0 \text{ in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)),\]
\[(2.7) \quad u_n^1 \equiv u_n(\cdot + s_n, \cdot + y_n^1) \rightharpoonup u^1 \neq 0 \text{ strongly in } C(I; L^2(\Omega)),\]

for some $u^1 \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$ (modulo subsequence). Here $I \times \Omega \Subset \mathbb{R} \times \mathbb{R}^N$.

Lemma 2.1 is, of course, valid for a subsequence. We shall however often extract subsequence without explicitly mentioning this fact.

**Lemma 2.2.** The limit function $u^1$ in Proposition 2.1 solves (NS-$\lambda$) in the sense of distribution. Thus $u^1 \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$.

**Proof.** By (2.7), we have

\[(2.8) \quad u_n^1 \equiv u_n(t + s_n, x + y_n^1) \rightarrow u^1 \neq 0 \text{ a.e. } \mathbb{R} \times \mathbb{R}^N.\]

Thus, by classical argument (see e.g. [7]), one can see from (2.8)

\[(2.9) \quad \lambda_{n}^{-N(p_{n}+1-\sigma)/2} \|u_n\|_{p_{n}-1} \|u_n(\cdot + s_n, \cdot + y_n) - \lambda |u^1|^{\frac{1}{p_{n}}} u^1(\cdot, \cdot)\|_{L^\infty(\mathbb{R} \times \mathbb{R}^N)},\]

so that, by the weak form of (NS-$\lambda$), $u^1$ solves (NS-$\lambda$). The last assertion follows from the uniqueness theorem of solution to (NS-$\lambda$) (see Kato [9]).

Furthermore we have by Lemma 1.5 (putting $v_n(t, x) = u_n^1(t, x)$ and $\Omega = \mathbb{R}^N$) and the weakly* convergence of $\nabla u_n^1$ to $\nabla u^1$ in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$.  

9
Lemma 2.3. We have

\[(2.10) \quad |u_{n}^{1}|^{\frac{1}{\sigma}}u_{n}^{1} - |u_{n}^{1} - u^{1}|^{\frac{1}{\sigma}}(u_{n}^{1} - u^{1}) - |u^{1}|^{\frac{1}{\sigma}}u^{1} \rightarrow 0 \quad \text{in } L^{\sigma'}(I \times \mathbb{R}^{N}),\]

where \(\frac{1}{\sigma} + \frac{1}{\sigma'} = 1\), and we have

\[(2.11) \quad \lim_{n \rightarrow \infty} \int_{I \times \mathbb{R}^{N}} |u_{n}^{1}|^{\sigma} - |u_{n}^{1} - u^{1}|^{\sigma} - |u^{1}|^{\sigma} \, dx = 0.\]

\[(2.12) \quad \lim_{n \rightarrow \infty} \int_{I} \{E_{\sigma}^{\lambda}(u_{n}^{1}) - E_{\sigma}^{\lambda}(u_{n}^{1} - u^{1}) - E_{\sigma}^{\lambda}(u^{1})\} \, dt = 0,\]

for any \(I \in \mathbb{R}\).

The proof of Theorem A consists of iterating the constructions of Lemma 2.1, Lemma 2.2 and Lemma 2.3. Now we explain how to carry out this iteration.

It is worth while to note that we have by Lemma 1.2,

\[(2.13) \quad \mu(B(0;1) \cap [|u_{n}(0+s_{n}, \cdot+y_{n})| > \frac{\eta}{2}] > \delta\]

for some positive constants \(\eta\) and \(\delta\). From (2.4) and (2.13), one can easily obtain

\[(2.14) \quad \lim_{n \rightarrow \infty} \sup_{\infty} E_{\sigma}^{\lambda}(u_{n}(0+s_{n}, \cdot)) \leq 0.\]

One can also see, from (2.6) and (2.7)

\[(2.15) \quad u_{n}^{1}(0, \cdot) \equiv u_{n}(0+s_{n}, \cdot+y_{n}) \rightarrow u^{1}(0, \cdot) \neq 0 \quad \text{in } H^{1}(\mathbb{R}^{N}).\]

Therefore \(\{u_{n}(0+s_{n}, \cdot+y_{n})\} \subset H^{1}(\mathbb{R}^{N})\) enjoys the properties of \(\{f_{n}\}\) in Proposition 1.4.

Suppose that

\[(2.16) \quad \lim_{n \rightarrow \infty} \|u_{n}^{1}(0) - u^{1}(0)\|_{\sigma} \neq 0.\]

So at this stage, we consider \(\varphi_{n}^{1}(t, x) = (u_{n}^{1} - u^{1})(t, x)\). Here we note that

\[(2.17) \quad \lim_{n \rightarrow \infty} \sup_{t \in \mathbb{R}} \|\varphi_{n}^{1}(t)\|_{\sigma} > 0.\]

Then, by Lemma 1.3 and Proposition 1.4 again, there exists a family of shifts \(\{y_{n}^{2}\} \subset \mathbb{R}^{N}\) such that,

\[(2.18) \quad u_{n}^{2} \equiv \varphi_{n}^{1}(\cdot, \cdot+y_{n}^{2}) \quad \text{\(\in L^{\infty}(\mathbb{R}; H^{1}(\mathbb{R}^{N})),\)}\]

\[(2.19) \quad u_{n}^{2} \equiv \varphi_{n}^{1}(\cdot, \cdot+y_{n}^{2}) \rightarrow u^{2} \neq 0 \quad \text{strongly in } C(I; L^{2}(\Omega))\]

\[(2.20) \quad u_{n}^{2}(0, \cdot) \equiv \varphi_{n}^{1}(0, \cdot+y_{n}^{2}) \rightarrow u^{2}(0, \cdot) \neq 0 \quad \text{in } H^{1}(\mathbb{R}^{N})\]

for some \(u^{2} \in L^{\infty}(\mathbb{R}; H^{1}(\mathbb{R}^{N})).\)
Lemma 2.4. The limit function $u^2$ in (2.18) is in $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$, and is a solution to $(NS-\lambda)$.

Proof. Since $u_n^1$ satisfies the equation of the form (2.1) and $u^1$ solves $(NS-\lambda)$, we have by Lemma 2.1, (2.10) and (2.11),

\begin{equation}
2i \frac{\partial u_n^2}{\partial t} + \Delta u_n^2 + \lambda |u_n^2|^\delta u_n^2 \\
= \lambda |v^1|^\delta v^1 + \lambda |u_n^2|^\delta u_n^2 - \lambda |v_n^1|^\delta v_n^1 \\
+ \lambda(|v_n^1|^\delta v_n^1 - |v_n^1|^{p_n-1}v_n^1) \\
+ (\lambda - \lambda_n^{-N(p_n+1-\sigma)/2})|v_n^1|^{p_n-1}v_n^1 \\
\rightarrow 0 \text{ strongly in } L^\sigma(I \times \mathbb{R}^N)
\end{equation}

for any $I \subseteq \mathbb{R}$ as $n \to \infty$, where $u_n^1(t, x) = u_n^1(t, x + y_n^2)$ and $u^1(t, x) = u^1(t, x + y_2^2)$. Here we have used the fact that (2.10) and (2.11) hold true, even if we replace $u_n^1(t, x)$ and $u^1(t, x)$ by $u_n^1(t, x + y_n^2)$ and $u^1(t, x + y_2^2)$ respectively. (2.18), (2.19) and (2.21) lead us to show that $u^2$ solves $(NS-\lambda)$. Thus $u^2 \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$.

Proof of Theorem A concluded. Repeating this procedure (according to the proof of Proposition 1.4), we obtain sequences $\{y_n^j\}_{n}^{s}$ ($j = 1, 2, \cdots$) in $\mathbb{R}^N$ such that

\begin{equation}
\lim_{n \to \infty} |y_n^j - y_n^k| = \infty (j \neq k)
\end{equation}

and corresponding functions

\begin{align}
(2.22) \\
&u_n^j \equiv (u_n^{j-1} - u^{j-1})(\cdot, \cdot + y_n^j) \not\equiv 0 \text{ in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)) \\
(2.23) \\
&u_n^j(0, \cdot) \equiv (u_n^{j-1} - u^{j-1})(0, \cdot + y_n^j) \not\equiv 0 \text{ in } H^1(\mathbb{R}^N)
\end{align}

where $j \geq 2$ and $u_n^j$ satisfies

\begin{equation}
\lim_{n \to \infty} \{E_\sigma^\lambda(u_n^j(0, \cdot)) - E_\sigma^\lambda((u_n^j - u^j)(0, \cdot)) - E_\sigma^\lambda(u^j(0, \cdot))\} = 0,
\end{equation}

so that we have

\begin{equation}
\lim_{n \to \infty} E_\sigma^\lambda((u_n^j - u^j)(0, \cdot)) \leq - \sum_{k=1}^{j} E_\sigma^\lambda(u^k(0, \cdot)).
\end{equation}

Hence we obtain the main assertions of Theorem A without the assertions $L < \infty$ and $E_\sigma^\lambda(u^j) = 0$ for $1 \leq j \leq L$. Therefore it remains only to prove the following lemma.

Lemma 2.5. The above procedure requires only a finite number of steps (under Assumption), i.e. $L < \infty$, so that we have $E_\sigma^\lambda(u^j) = 0$ for $1 \leq j \leq L$.

Proof. Suppose $L = \infty$. We have by (2.25),

\begin{equation}
\lim_{n \to \infty} 2 \frac{\lambda}{\sigma} \|(u_n^j - u^j)(0, \cdot)\|_{L^\sigma} \geq \sum_{k=1}^{j} E_\sigma^\lambda(u^k(0, \cdot)).
\end{equation}
Letting $j \to L = \infty$ in (2.26), we have (see (1.24))

\begin{equation}
(2.27) \sum_{k=1}^{L} E^\lambda_{\sigma}(u^k(0, \cdot)) \leq 0.
\end{equation}

We remark that $E^\lambda_{\sigma}(u^j) \geq 0$ by Assumption. Thus (2.27) implies that

\begin{equation}
(2.28) E^\lambda_{\sigma}(u^j) = 0 \quad \text{for} \quad 1 \leq j \leq L,
\end{equation}

so that we have,

\begin{equation}
(2.29) \|u^j(0)\| \geq \|Q_{\lambda}\| \quad \text{for} \quad 1 \leq j \leq L,
\end{equation}

where $Q_{\lambda}$ is the nontrivial minimal $L^2$ norm solution of

\begin{equation}
\begin{cases}
\Delta Q - Q + \lambda |Q|^p Q = 0, & x \in \mathbb{R}^N, \\
Q \in H^1(\mathbb{R}^N).
\end{cases}
\end{equation}

which is characterized as

\begin{equation}
\|Q_{\lambda}\| = \inf_{v \in H^1(\mathbb{R}^N), v \not= 0} \left\{ \|v\| : E^\lambda_{\sigma}(v) = \|\nabla v\|^2 - \frac{2}{\sigma} \lambda \|v\|_\sigma^\sigma \leq 0 \right\}.
\end{equation}

(For this, see Remark below Theorem B in § 1.) Since $\sum_{k=1}^{L} \|u^k(0)\|^2 \leq \|u_0\|^2$, we reach a contradiction. The second assertion also follows from the formula (2.27) and Assumption.

3. APPLICATION TO THE BLOW-UP PROBLEM FOR $C(1 + \frac{4}{N})$

In this section we investigate the shape of blow-up solution to the following Cauchy problem for the pseudo-conformally invariant nonlinear Schrödinger equation:

\begin{equation}
(1 + \frac{4}{N}) \begin{cases}
2i \frac{\partial u}{\partial t} + \Delta u + |u|^p u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\end{equation}

Suppose that the initial datum $u_0(x)$ leads to the solution $u(t, x)$ of $C(1 + \frac{4}{N})$ which blows up at time $T_m \in (0, \infty)$, i.e.

\begin{equation}
(3.1) \lim_{t \to T_m} \|\nabla u(t)\| = \infty.
\end{equation}

We fix such an initial datum $u_0 \in H^1(\mathbb{R}^N)$.

Let $\{u_p(t, x)\}$ be the family of solution to $C(p)$ (see § 0) for $1 < p < 1 + \frac{4}{N}$. We note that $u_p(0, x) = u_0(x)$. As we mentioned in § 0, $u_p \in C_0(\mathbb{R}; H^1(\mathbb{R}^N))$ for $1 < p < 1 + \frac{4}{N}$.

By using the space-time estimate in Kato [9] and the classical compactness argument as in Ginibre-Velo [7], one can show
Proposition 3.1. Let \( \{u_p(t,x)\} \) be the family of solution to \( C(p) \) for \( 1 < p < 1 + \frac{4}{N} \), and let \( u(t,x) \) be the blow-up solution of \( C(1 + \frac{4}{N}) \) (satisfying (3.1) for some \( T_m \in (0,\infty) \)). We note again \( u_p(0,x) = u(0,x) = u_0(x) \). Then, for any \( T \in (0,T_m) \), we have

\[
(3.2) \quad u_p \rightharpoonup u \quad \text{strongly in } C([0,T];H^1(\mathbb{R}^N))
\]

as \( p \uparrow 1 + \frac{4}{N} \).

Therefore we may expect that \( \{u_p(t,x)\} \) brings us some information about the shape of blow-up solution near the blow-up time \( T_m \).

Let \( \{p_n\} \) be a sequence such that \( p_n \uparrow 1 + \frac{4}{N} \) and \( u_{p_n} \in C(\mathbb{R};H^1(\mathbb{R}^N)) \) be a solution to \( C(p_n) \). We may assume by Proposition 3.1,

\[
(3.3) \quad \limsup_{n \to \infty} \sup_{t \in [0,T_m)} \|u_{p_n}(t)\|_{\sigma} = \infty.
\]

We consider the rescaling function

\[
(3.4) \quad u_n(t,x) = \lambda_n^{N/2} u_{p_n}(\lambda_n^2 t + T_m, \lambda_n x),
\]

where

\[
(3.5) \quad \lambda_n = \frac{1}{\sup_{t \in [0,T_m)} \|u_{p_n}(t)\|_{\sigma}^{\sigma/2}}.
\]

We note that \( u_n \in C_0([-\frac{T_m}{\lambda_n}, 0]) \) and solves

\[
(3.6) \quad 2i \frac{\partial u_n}{\partial t} + \Delta u_n + \lambda_n^{-N(p_n+1-\sigma)/2} |u_n|^{p_n-1} u_n = 0.
\]

on \([-\frac{T_m}{\lambda_n}, 0]\). We extend \( u_n \)'s domain to the whole line as follows;

\[
(3.7) \quad \tilde{u} = \begin{cases} 
    u_n(-\frac{T_m}{\lambda_n}, x) = \lambda_n^{N/2} u_{p_n}(0, \lambda_n x) & \text{if } t \in (-\infty, -\frac{T_m}{\lambda_n}), \\
    u_n(t,x) & \text{if } t \in [-\frac{T_m}{\lambda_n},0), \\
    u_n(0,x) = \lambda_n^{N/2} u_{p_n}(T_m, \lambda_n x) & \text{if } t \in [0,\infty).
\end{cases}
\]

We note that \( \{\tilde{u}_n(t,x)\} \) is a uniformly equicontinuous family in \( C_0(\mathbb{R};L^2(\mathbb{R}^N)) \), and form a uniformly equibounded family in \( C_0(\mathbb{R};H^1(\mathbb{R}^N)) \).

In the same way as proving Theorem A, we have

**Theorem C.** Then there exists a subsequence of \( \{u_n\} \) (we still denote it by \( \{u_n\} \)) which satisfies the following properties: one can find \( L \in \mathbb{N} \), nontrivial solutions \( \{u^j\} \)
of (NS-$\lambda$) in $C_b(R; H^1(R^N))$ with $E_\sigma^\lambda(u^j) = 0$ and sequences $\{(s_n^j, y_n^j)\} \subset R \times R^N$ for $1 \leq j \leq L$ such that

(C.1) \[ s_n^j \geq 0 \quad \text{and} \quad \lim_{n \to \infty} |s_n^j \lambda_n^2| = 0 \]
(C.2) \[ \lim_{n \to \infty} |(s_n^j, y_n^j) - (s_n^k, y_n^k)| = \infty \quad (j \neq k), \]
(C.3) \[ u_n^j \equiv (u_n^{j-1} - u_n^{j-1})(\cdot, \cdot + y_n^j) \hookrightarrow u^j \quad (j \geq 2) \quad \text{in} \quad L^\infty(I_s; H^1(R^N)), \]
(C.4) \[ u_n^j \equiv u_n^j(\cdot, \cdot - s_n^1, \cdot + y_n^j) \hookrightarrow u^j \quad \text{in} \quad L^\infty(I_s; H^1(R^N)), \]
(C.5) \[ \lim_{n \to \infty} \int_I \{ E_\sigma^\lambda(u_n^j) - E_\sigma^\lambda(u_n^j - u^j) - E_\sigma^\lambda(u^j) \} dt = 0, \quad \text{for any} \quad I \Subset L, \]
(C.6) \[ \lim_{n \to \infty} \|u_n^L(0) - u^L(0)\| = 0, \]

where

(C.7) \[ I_s = \begin{cases} R & \text{if} \quad \lim_{n \to \infty} s_n^1 = \infty, \\
( -\infty, T_s] & \text{if} \quad \lim_{n \to \infty} s_n^1 = T_s < \infty. \]

Remarks. (1) It is worth while to note that if

(3.8) \[ \lim_{n \to \infty} \sup_{t \in \mathbb{R}} \|u_n(t - s_n^1, \cdot) - \sum_{j=1}^L u_n^j(t, \cdot - \sum_{k=1}^j y_n^k)\| > 0, \]

there exists $\{(s_n^2, y_n^{2,1})\} \in R^+ \times R^N$ such that

(3.9) \[ s_n^2 \geq 0 \quad \text{and} \quad \lim_{n \to \infty} |s_n^2 \lambda_n^2| = 0 \]
(3.10) \[ (u_n^L - u^L)(\cdot, \cdot - s_n^2, \cdot + y_n^{2,1}) \not\hookrightarrow u^{2,1} \neq 0 \quad \text{in} \quad L^\infty(R; H^1(R^N)). \]

One can see that $u^{2,1}$ is almost a solution to (NS-$\lambda$) near $t = 0$. (See next section.) Therefore Theorem C suggests that the blow-up solution of $C(1 + \frac{4}{N})$ has a self-similar structure around singularities.

(2) If Assumption were not true for $N \geq 2$, it would be consistent with $L = \infty$ in Theorem A. (3) If $u_0$ is radially symmetric or $\|u_0\| = \|Q\|$, we have $L = 1$ in Theorem C without Assumption. Here $Q(x)$ is a nontrivial minimal $L^2$ norm solution of (NSF).

(4) If $T_s < \infty$ in (C.7), we can take $s_n^1 = 0$ and $I_s = (-\infty, 0]$.

4. "TWO FOCI" OF A LASER BEAM.

For simplicity we assume $N \geq 2$ and $u_0(x)$ (the initial datum in $C(p)$) is radially symmetric, so that the corresponding solution of $C(p)$ ($1 < p < 2^*$) is also radially symmetric. In this case, we do not need Assumption. Suppose that $u_0(x)$ leads to the blow-up solution to $u(t, x)$ of $C(1 + \frac{4}{N})$ such that $\lim_{t \to T_m} \|\nabla u(t)\| = \infty$ for some $T_m \in (0, \infty)$.
Let \( \{p_n\} \) be a sequence such that \( p_n \uparrow 1 + \frac{4}{N} \) and \( u_{p_n} \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \) be a solution to \( C(p_n) \). We may assume

\[
\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \|u_{p_n}(t)\|_{\sigma} = \infty.
\]

We consider the rescaling function

\[
u_{n}(t, x) = \lambda_n^{N/2}u_{P\mathfrak{n}}(\lambda_n^{2}t, \lambda_n x),\]

where

\[
\lambda_n = \frac{1}{\sup_{t \in \mathbb{R}}\|u_{P\mathfrak{n}}(t)\|_{\sigma}^{\sigma/2}}.
\]

(We recall \( \sigma = 2 + \frac{4}{N} \).)

By Theorem A and the radial symmetricity of \( u_n \)'s (using well known radial compactness lemma in Proposition 1.4), we have

**Lemma 4.1.** There exist a family of shifts \( \{s_n^1\} \subset \mathbb{R} \) such that,

\[
\begin{align*}
(4.4) & \quad u_n^1(t, x) = u_{n}(\cdot + s_n^1, \cdot) \rightharpoonup u^1 \neq 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)) , \\
(4.5) & \quad u_n^1(t, x) \rightharpoonup u^1 \neq 0 \quad \text{strongly in } C(I; L^2(\Omega)) , \\
(4.6) & \quad u_n^1(0, \cdot) \rightharpoonup u^1(0, \cdot) \neq 0 \quad \text{strongly in } L^\sigma(\mathbb{R}^N),
\end{align*}
\]

for some \( u^1 \in C_b(\mathbb{R}; H^1(\mathbb{R}^N)) \). Here \( I \times \Omega \subset \mathbb{R} \times \mathbb{R}^N \). Furthermore \( u^1 \) solves \( (NS-\lambda) \).

Now suppose that

\[
\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \|u_n^1(t) - u^1(t)\|_{\sigma} > 0.
\]

We put \( \varphi_n^1(t, x) = (u_n^1 - u^1)(t, x) \). One has from Lemma 1.5,

\[
2i \frac{\partial \varphi_n^1}{\partial t} + \Delta \varphi_n^1 + |\varphi_n^1|^\frac{4}{\sigma} \varphi_n^1 = \lambda |u_n^1|^\frac{4}{\sigma} u_n^1 + \lambda |\varphi_n^1|^\frac{4}{\sigma} \varphi_n^1 - \lambda |u_n^1|^\frac{4}{\sigma} u_n^1 + \lambda (|u_n^1|^\frac{4}{\sigma} u_n^1 - |u_n^1|^{p_n-1}u_n^1) + (\lambda - \lambda_n^{-N(p_n+1-\sigma)/2})|u_n^1|^{p_n-1}u_n^1
\]

\[
\rightarrow 0 \quad \text{strongly in } L^\sigma(I \times \mathbb{R}^N)
\]

for any \( I \subset \mathbb{R} \), where \( \frac{1}{\sigma} + \frac{1}{\sigma'} = 1 \).

From (4.7), (4.8) and Lemma 1.3, we have
Lemma 4.2. There exists a family of shifts $\{s_n^2\} \subset \mathbb{R}$ such that,

(4.9) $u_n^2 \equiv \varphi_n^1(\cdot + s_n^2, \cdot) \rightharpoonup u^2 \neq 0$ in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$,

(4.10) $u_n^2 \equiv \varphi_n^1(\cdot + s_n^2, \cdot) \rightarrow u^2 \neq 0$ strongly in $C(I; L^2(\Omega))$

(4.11) $u_n^2(0, \cdot) \equiv \varphi_n^1(0 + s_n^2, \cdot) \rightarrow u^2(0, \cdot) \neq 0$ strongly in $L^\sigma(\mathbb{R}^N)$.

It is worth while to note that, in general, we have

(4.12) $2i \frac{\partial u_n^2}{\partial t} + \triangle u_n^2 + \lambda |u_n^2|^{4} u_n^2 \rightarrow 0$,

regardless of (4.8), since it is not obvious whether

$([u^1] \overset{\#}{\star} u^1 + [\varphi_n^1] \overset{\#}{\star} \varphi_n^1 - [u_n^1] \overset{\#}{\star} u_n^1)(\cdot + s_n^2, \cdot) \rightarrow 0$

or not. So we consider the function $h_n$ which satisfies

(4.13) $2i \frac{\partial h_n}{\partial t} + \triangle h_n + \lambda |u_n^2 + h_n|^{4} (u_n^2 + h_n)

= \lambda^{-N(p_n+1-\sigma)/2} \|v_n^1\|^{4} v_n^1

- \lambda |v_n^1| \overset{\#}{\star} v_n^1$

with initial condition $h_n(0, x) = 0$, where $v_n^1(t, x) = u_n^1(t + s_n^2, x)$. We can solve this Cauchy problem, at least, locally in time (uniformly in $n$) in $H^1(\mathbb{R}^N)$. Putting $\psi_n = u_n^2 + h_n$, we see $\psi_n$ solves

(4.14) $2i \frac{\partial \psi_n}{\partial t} + \triangle \psi_n + \lambda |\psi_n|^{4} \psi_n = 0$

in a neighborhood $I_0$ of $t = 0$ (uniformly in $n$) by (4.8) and (4.13). One can show

(4.15) $\psi_n \rightharpoonup \psi \neq 0$ in $L^\infty(I_0; H^1(\mathbb{R}^N))$,

(4.16) $\psi_n \rightarrow \psi \neq 0$ strongly in $C(I_0; L^2(\Omega))$

for some $\psi \in C_b(I_0; H^1(\mathbb{R}^N))$ such that $\psi$ solves

$$
\left\{
\begin{array}{ll}
2i \frac{\partial \psi}{\partial t} + \triangle \psi + |\psi|^{4} \psi = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
\psi(0, x) = u^2(0, x), & x \in \mathbb{R},
\end{array}
\right.
$$

on $I_0$.

Summing up, we have

**Proposition 4.3.** Suppose we have (4.7), then there exist a family of shifts $\{s_n^2\} \subset \mathbb{R}$ and a local solution $\psi$ of (NS-$\lambda$) defined on a neighborhood of $t = 0$ such that

(4.17) $u_n^2 \equiv \varphi_n^1(\cdot + s_n^2, \cdot) \rightharpoonup u^2 \neq 0$ in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$,

(4.18) $\lim_{t \downarrow 0} \|u^2(t) - \psi(t)\|_{H^1(\mathbb{R}^N)} = 0$.

We close this section with the following
Conclusion. If we have

\begin{align*}
\lim_{n \to \infty} \lambda_n^2 |s_n^1 - s_n^2| &> 0 \\
\lim_{n \to \infty} \lambda_n^2 |s_n^1| &< \infty, \quad \lim_{n \to \infty} \lambda_n^2 |s_n^2| < \infty,
\end{align*}

we may conclude that the laser beam described by the blow-up solution \( u \) to \( C(1 + 1/N) \) have two focus points on \( t \)-axis, around which the beam has an approximately self-similar structure.

**APPENDIX**

As an application of Proposition 1.4 \((\lambda = 1)\), we can show the following theorem.

**Theorem D.** Let

\begin{align*}
(D.1) \quad m &= \inf_{v \in H^1(\mathbb{R}^N), v \neq 0} \left\{ \|v\| ; E_\sigma(v) = \|\nabla v\|^2 - \frac{2}{\sigma} \|v\|_\sigma^\sigma \leq 0 \right\}, \\
(D.2) \quad \frac{1}{C_N} &= \inf_{v \in H^1(\mathbb{R}^N), v \neq 0} \frac{\|v\|^{\frac{\sigma}{2}} \|\nabla v\|^2}{\|v\|_\sigma^{\sigma}} = \inf_{v \in H^1(\mathbb{R}^N), v \neq 0} J(v).
\end{align*}

There is a function \( Q \in H^1(\mathbb{R}^N) - \{0\} \) such that

\begin{align*}
(D.3) \quad \|Q\| &= m, \\
(D.4) \quad \Delta Q - Q + |Q|^\frac{\sigma}{2} Q &= 0, \\
(D.5) \quad \frac{2}{\sigma} \|Q\|^{\frac{\sigma}{2}} &= \frac{1}{C_N}.
\end{align*}

Remark. The constant \( C_N \) in (D.2) is the best constant for the Gagliardo-Nirenberg inequality, so that

\begin{align*}
(G-N) \quad \|v\|_\sigma^\sigma &\leq C_N \|v\|^{\frac{\sigma}{2}} \|\nabla v\|^2
\end{align*}

holds true for any \( v \in H^1(\mathbb{R}^N) \).

**Proof of Theorem C.** First we note that \( m > 0 \), more precisely

\begin{align*}
(1) \quad \frac{2}{\sigma} m^{\frac{\sigma}{2}} &\geq \frac{1}{C_N}
\end{align*}

by the Gagliardo-Nirenberg inequality (G-N).

Let \( \{v_n\} \subset H^1(\mathbb{R}^N) \) be a minimizing sequence for (D.1), i.e.

\begin{align*}
(2) \quad \lim_{n \to \infty} \|v_n\| &= m, \\
(3) \quad E_\sigma(v_n) &\leq 0 \quad \text{for any } n \in \mathbb{N}.
\end{align*}
It is worth while to note that the boundedness of \( \{v_n\} \) in \( H^1(\mathbb{R}^N) \) is not known. So we rescale \( v_n \) as follows:

\[
Q_n(x) = \nu_n^{N/2} v(\nu_n x), \quad \nu_n = \frac{1}{\|v_n\|_\sigma^{\sigma/2}},
\]

so that we have

\[
\|Q_n\| = \|v_n\| \to m \quad \text{as} \quad n \to \infty,
\]

\[
\|Q_n\|_\sigma = \|v_n\|_\sigma, \quad E_\sigma(Q_n) = \nu_n^2 E_\sigma(v_n).
\]

Thus we get a \( H^1 \)-bounded minimizing sequence \( \{Q_n\} \) for (D.1).

We shall apply Proposition 1.4 (with \( \lambda = 1 \)) to this \( \{Q_n\} \); There exists a subsequence of \( \{Q_n\} \) (we still denote it by \( \{Q_n\} \)) which satisfies

\[
Q_n^1 \equiv Q_n(\cdot + y_n^1) \rightharpoonup Q^1 \neq 0 \quad \text{weakly in} \quad H^1(\mathbb{R}^N),
\]

\[
\lim_{n \to \infty} \{E_\sigma(Q_n^1) - E_\sigma(Q_n^1 - Q^1) - E_\sigma(Q^1)\} = 0,
\]

\[
\lim_{n \to \infty} (\|Q_n^1\|^2 - \|Q_n^1 - Q^1\|^2 - \|Q^1\|^2) = 0,
\]

for some \( \{y_n^1\} \subset \mathbb{R}^N \). Noting that \( Q_n^1 \) is also a \( H^1 \)-bounded minimizing sequence of (D.1), we have from (7) and (8) (by simple contradiction argument),

\[
E(Q^1) \leq 0.
\]

It follows from (9) and the definition of \( m \) that \( \|Q^1\| \geq m \), so that we have

\[
\|Q^1\| = m,
\]

since \( Q_n^1 \rightharpoonup Q^1 \) weakly in \( L^2(\mathbb{R}^N) \). Thus we get \( \lim_{n \to \infty} \|Q_n^1 - Q^1\| = 0 \). (So we have \( L = 1 \) in the terminology of Proposition 1.4.)

Let \( \{w_n\} \subset H^1(\mathbb{R}^N) \) be a minimizing sequence for (D.2). We rescale \( w_n \) as follows:

\[
W_n(x) = w_n(\frac{x}{\tilde{\nu}_n}), \quad \tilde{\nu}_n = \sqrt{\frac{\sigma\|\nabla w_n\|^2}{2\|w_n\|_\sigma^2}}.
\]

Then one has

\[
J(W_n) = J(w_n),
\]

\[
E_\sigma(W_n) = \tilde{\nu}_n^{N-2}(\|\nabla w_n\|^2 - \tilde{\nu}_n^2\frac{2}{\sigma}\|w_n\|_\sigma^2) = 0,
\]

so that

\[
\frac{1}{C_N} = \lim_{n \to \infty} \frac{2}{\sigma}\|W_n\|_\sigma^2, \quad E_\sigma(W_n) = 0.
\]
Thus by the definition of $m$, we have 
\[
\frac{2}{\sigma} m \leq \frac{1}{C_N}.
\]
Hence we obtain, by (1),
\[
\frac{2}{\sigma} m \leq \frac{1}{C_N}.
\]
Thus $Q^1$ is a critical point of $J(\cdot)$. Since $|\nabla|Q^1|| \leq |\nabla Q^1|$, we may assume $Q^1 \equiv 0$. So we have
\[
\frac{d}{dt} J(Q^1 + t\varphi) \bigg|_{t=0} = 0
\]
for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Hence $Q^1$ satisfies
\[
\Delta Q^1 - \left( \frac{2||\nabla Q^1||^2}{N||Q^1||^2} \right) Q^1 + |Q^1|^p Q^1 = 0.
\]
in the sense of distribution.

Taking
\[
Q(x) = \tilde{\nu}^{N/2} Q^1(\tilde{\nu} x), \quad \tilde{\nu} = \sqrt{\frac{N||Q^1||^2}{2||\nabla Q^1||^2}},
\]
one can easily verifies that this $Q$ satisfies (D.4) and $\|Q\| = \|Q^1\| = m$.

Remark  Considering the continuous curve $Q_s : (0, \infty) \ni s \mapsto Q^1(\frac{x}{s}) \in H^1(\mathbb{R}^N)$, we have
\[
0 \leq \lim_{s \uparrow 1} E_\sigma(Q_s) = E_\sigma(Q^1) \leq 0,
\]
since $E_\sigma(Q_s) > 0$ if $s \in (0, 1)$. Thus we have $\lim_{n \to \infty} \|Q_n^1 - Q^1\|_{H^1(\mathbb{R}^N)} = 0$. Therefore we obtain an extra property of $Q$ such that
\[
E_\sigma(Q) = 0.
\]

REFERENCES
13. Merle, F., Construction of solutions with exactly k blow-up points for the Schrödinger equation with the critical power nonlinearity, preprint.
15. ______, “Mass concentration” phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity, preprint.
21. ______, Blow-up of $H^1$-solution for the one dimensional nonlinear Schrödinger equation with critical power nonlinearity, preprint.