数理解析研究所講究録
755巻 1991年 61-72

Approximation of inertial manifolds for semilinear evolution equations

Kazuo Kobayasi (Shonan Institute of Technology)
小林 和夫

1. Introduction.

Several of evolution equations, for example, the Kuramoto-Sivashinsky equation [7], [17], [20], the Cahn-Hilliard equation [17], [20] as well as reaction-diffusion equations [7], [9], [11], [14] have finite dimensional inertial manifolds. The restriction of those partial differential equations to the inertial manifold reduces to finite ordinary differential equations. An inertial manifold for an evolution equation is a smooth and positively invariant manifold under the flow which also attracts exponentially all orbits. Therefore, the long-time behavior of solutions of a partial differential equation possessing an inertial manifold is completely determined by the finite system of ordinary differential equations.

This note discusses the existence and the convergence of inertial manifolds for approximations to the semi-linear evolution equation in a Banach space Y

$$du(t)/dt = Au(t) + F(u(t)), \quad t > 0$$

where A is the infinitesimal generator of a C_o-semigroup $\{S(t);\ t \geq 0\}$ on Y, while F is a nonlinear operator. The approximations to the evolution equation considered here are associated with Chernoff's product formulas (see [2]).

2. Main results.

Let X and Y be Banach spaces with norms $\|\circ\|$ and $|\circ|$, respectively, such that X is continuously embedded in Y. Let $\{S(t);\ t\geq 0\}$ be a C_o-semigroup on Y and $F \in Lip(X,Y) \cap C^1(X,Y)$. Consider the following semilinear evolution equation

$$du(t)/dt = Au(t) + F(u(t)), \qquad t \geq 0, \tag{2.1}$$

$$u(0) = x_0 \tag{2.2}$$

where A is the infinitesimal generator of $\{S(t)\}$. It has been recently shown by several authors (e.g. see [3],[6],[7],[14], [16]) that (2.1) has an inertial manifold M under certain assumptions. More precisely, the following assumptions are considered:

(S1) There are constants $\tilde{M} \geq 1$ and $\tilde{\omega} \geq 0$ such that $|S(t)y| \leq \tilde{M}e^{\tilde{\omega}t}|y|$ for $t \geq 0$ and $y \in Y$.

(S2) $S(t)Y \subset X$ for $t > 0$ and $S(t)x \in C([0,\infty);X)$ for $x \in X$.

(S3) $Y = Y_1 + Y_2$ and $P_iS(t) = S(t)P_i$ for $i = 1,2$ and $t \geq 0$, where Y_i is a closed linear subspace and P_i is a projection from Y onto Y_i.

(S4) $\{S(t)P_i;\ t \geq 0\}$ forms a uniformly continuous semigroup on Y_1.

(S5) There exist constants α, $\beta > 0$, $\gamma \in [0,1)$, $\eta < -\max\{\alpha,\beta\}$ and M_1, M_2, M_3, $M_4 \geq 0$ such that

$$|e^{-\eta t}S(t)P_1y| \leq M_1e^{\alpha t}|y|, \qquad t \leq 0,\ y \in Y, \tag{2.3}$$

$$\|e^{-\eta t}S(t)P_1y\| \leq M_2e^{\alpha t}|y|, \qquad t \leq 0,\ y \in Y, \tag{2.4}$$

$$\|e^{-\eta t}S(t)P_2x\| \leq M_3e^{-\beta t}\|x\|, \qquad t \geq 0,\ x \in X, \tag{2.5}$$

(2.6) $\qquad \|e^{-\eta t}S(t)P_2y\| \le (M_3t^{-\gamma} + M_4)e^{-\beta t}|y|, \qquad t > 0,\ y \in Y.$

We note that the above assumptions ensure the unique mild solution $u(t;x_0) \in C([0,\infty);\ X)$ of (2.1) and (2.2) for each $x_0 \in X$ (e.g. see [18; Chapter 6]). Under the assumptions (S1)-(S5), Chow and Lu [3] proved that if

(2.7) $\qquad K(\alpha,\beta)\mathrm{Lip}F < 1 \quad \text{and} \quad \dfrac{M_1M_3K(\alpha,\beta)\mathrm{Lip}F}{1 - K(\alpha,\beta)\mathrm{Lip}F} < 1$

where

(2.8) $\qquad K(\alpha,\beta) = \hat{M}\{M_2\alpha^{-1} + M_3\Gamma(1-\gamma)\beta^{\gamma-1} + M_4\beta^{-1}\}$

then there exists an inertial manifold M for (2.1) satisfying

(a) $M = \{\ \xi + h(\xi);\ \xi \in Y_1\}$ for some $h \in C^1(Y_1,\ P_2X)$,

(b) if $x_0 \in M$ then $u(t;x_0) \in M$ for all $t > 0$,

(c) for each $x_0 \in X$ there exists a unique element $x_0^* \in M$ such that

$$\sup_{t\ge0} e^{-\eta t}\|u(t;x_0) - u(t;x_0^*)\| < \infty.$$

Let $\{C(\lambda)\}_{\lambda>0}$ be a family of bounded linear operators in Y. We now consider the following type of approximation to (2.1)

(2.9) $\qquad x_{n+1} = C(\lambda)x_n + \lambda J_{\nu(\lambda)}F(x_n), \qquad n \ge 0,$

where $\lambda > 0$, $\nu \in C(\mathfrak{R}^+,\mathfrak{R}^+)$ with $\nu(0) = 0$, and $J_\nu = (I - \nu A)^{-1}$, the resolvent of A. We shall make the following assumptions on the family $\{C(\lambda)\}$:

(C1) There are constants $\tilde{M} \ge 1$ and $\tilde{\omega} \ge 0$ such that $|C(\lambda)^ny| \le$

$\tilde{M}e^{\tilde{\omega}n\lambda}|y|$ for $\lambda > 0$, $n \geq 0$ and $y \in Y$.

(C2) $\lim_{\lambda\downarrow 0} (I - h\lambda^{-1}(C(\lambda) - I))^{-1}y = J_h y$ in Y for each $y \in Y$ and $h > 0$ sufficiently small.

(C3) $C(\lambda)C(\mu) = C(\mu)C(\lambda)$ and $P_iC(\lambda) = C(\lambda)P_i$ for λ, $\mu > 0$ and $i = 1,2$, where P_i is a continuous projection from Y onto Y_i satisfying $P_1 + P_2 = I$.

(C4) $[C(\lambda)P_1]^{-1} \in B(Y_1)$ for $\lambda > 0$.

(C5) There exist constants α, $\beta > 0$, $\gamma \in [0,1)$, $\eta < -\max\{\alpha,\beta\}$ and M_1, M_2, M_3, $M_4 \geq 0$ such that

$$|e^{\eta n\lambda}[C(\lambda)P_1]^{-n}P_1y| \leq M_1e^{-\alpha n\lambda}|y|, \tag{2.10}$$

$$\|e^{\eta n\lambda}[C(\lambda)P_1]^{-n}P_1y\| \leq M_2e^{-\alpha n\lambda}|y|, \tag{2.11}$$

$$\|e^{-\eta n\lambda}C(\lambda)^nP_2x\| \leq M_3e^{-\beta n\lambda}\|x\|, \tag{2.12}$$

$$\|e^{-\eta n\lambda}C(\lambda)^nJ_{\nu(\lambda)}P_2y\| \leq \{M_3((n+1)\lambda)^{-\gamma} + M_4\}e^{-\beta n\lambda}|y| \tag{2.13}$$

for $n \geq 0$, $x \in X$ and $y \in Y$.

Our first result is the following

Theorem 2.1. Assume that (C1)-(C5) are satisfied. If $F \in Lip(X,Y) \cap C^1(X,Y)$ and (2.7) holds, then for all sufficiently small $\lambda > 0$ there exists a manifold M_λ satisfying

$(a)_\lambda$ $M_\lambda = \{ \xi + h_\lambda(\xi);\ \xi \in Y_1\}$ for some $h_\lambda \in C^1(Y_1,P_2X)$,

$(b)_\lambda$ if $x_0 \in M_\lambda$, then $x_n \in M_\lambda$ for all $n \geq 1$,

$(c)_\lambda$ for each $x_0 \in X$ there exists a unique element $x_0^* \in M_\lambda$ such that

$$\sup_{n\geq 0} e^{-\eta n\lambda}\|x_n - x_n^*\| < \infty,$$

where $\{x_n^*\}$ is the sequence defined by (2.9) with initial x_0^* instead of x_0.

The second result concerns the stability of inertial manifolds for (2.1) with respect to the approximation (2.9).

Theorem 2.2. Assume that (S1)-(S5) and (C1)-(C5) are satisfied. If $F \in Lip(X,Y) \cap C^1(X,Y)$ and (2.7) holds, then

$$\lim_{\lambda\downarrow 0} \|h_\lambda(\xi) - h(\xi)\| = 0 \qquad \text{for } \xi \in Y_1,$$

and the convergence is uniform with respect to ξ belonging to each compact subset of Y_1. Here h_λ and h are the functions appearing in $(a)_\lambda$ and (a), respectively.

As an application of the above theorems we shall consider the family

$$C(\lambda) = (I + (1-\theta)\lambda A)(I - \theta\lambda A)^{-1} \tag{2.14}$$

In this case (2.9) with $\nu(\lambda) = \theta\lambda$ is written as

$$\lambda^{-1}(x_{n+1} - x_n) = A(\theta x_{n+1} + (1-\theta)x_n) + F(x_n) \tag{2.15}$$

Theorem 2.3. Let $\theta = 1$. Suppose that $\{S(t)\}$ satisfies (S1)-(S5) with $\alpha \leq -\eta - \|AP_1\|_{L(Y_1)}$. Then the family $\{C(\lambda)\}$ defined by (2.14) satisfies (C1)-(C5).

Theorem 2.4. Let $1/2 < \theta < 1$. Suppose that $\{S(t)\}$ satisfies (S1)-(S5) with $\tilde{M} = M_3 = 1$ and $\alpha \leq -\eta - \|AP_1\|_{L(Y_1)}$. If X and Y are

Hilbert spaces, then the family $\{C(\lambda)\}$ defined by (2.14) satisfies (C1)-(C5).

Remark. Condition (S4) implies that AP_1 can be regarded as a bounded linear operator in Y_1. By combining the above theorems we can conclude that the approximation (2.15) possesses an inertial manifold which converges to the original one for (2.1). When $\theta = 1$, this result has proved by Demengel and Ghidaglia [5] by a different method. When $\theta = 1/2$, (2.15) is called the scheme of Crank-Nicholson in Hilbert spaces. We do not however known if (2.14) with $\theta = 1/2$ satisfies (C1)-(C5).

Conversely, we consider the problem about the existence of inertial manifolds for (2.1) when the approximation (2.9) possesses an inertial manifold.

Theorem 2.5. Assume (C1), (C3), (C4), (C5) and the following conditions:

(X) X is densely and continuously embeded in Y and X is reflexive.

(C2)* There exists a dense subset D of Y such that

$$\lim_{\lambda \downarrow 0} \lambda^{-1}(C(\lambda)y - y) = Ay \quad \text{exists in } Y \text{ for } y \in D,$$

$(I - \lambda_o A)D$ is dense in Y for some $\lambda_o > 0$, and $Y_1 \subset D$.

Then the closure $\bar{A}$ of A generates a C_o-semigroup $\{S(t);t \geq 0\}$ on Y satisfying (S1)-(S5).

The proofs of the above theorems will be given in [12].

3. Examples.

In this section H denotes a Hilbert space with norm $|\cdot|$. Let A be a nonnegative self-adjoint operator in H. $\{S(t)\}$ denotes a C_o-semigroup on H generated by $-A$. Suppose that $(A + \lambda I)^{-1}$ is a compact operator for some $\lambda \geq 0$. Then, the eigenvalues $\tilde{\lambda}_j$ of $A + \lambda I$ satisfy

$$0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_j \leq \cdots \to \infty \qquad \text{as } j \to \infty$$

and the corresponding eigenvectors e_j form an orthonomal basis of H. We note that the eigenvalues λ_j of A is equal to $\tilde{\lambda}_j - \lambda$.

Let $N > 0$ be an integer and P_1 be the projection from H into $\text{span}\{e_1, \cdots, e_N\}$ and $P_2 = I - P_1$. Let $\Lambda = A + \lambda I$. Then, for each $\gamma \in [0.1)$ we have the following properties:

$$|\Lambda^\gamma S(t)P_1x| \leq \tilde{\lambda}_N^\gamma e^{-\lambda_N t} |x|, \qquad t \leq 0,\ x \in H,$$

$$|P_2S(t)x| \leq e^{-\lambda_{N+1} t} |x|, \qquad t \geq 0,\ x \in H,$$

$$|\Lambda^\gamma S(t)P_2x| \leq (t^{-\gamma} + \tilde{\lambda}_{N+1}^\gamma)\, e^{-\lambda_{N+1} t} |x|, \quad t > 0,\ x \in H.$$

Let $Y = H$ and $X = D(\Lambda^\gamma)$ be the Hilbert space with the graph norm. If we set $\eta = -(\lambda_{N+1} + \lambda_N)/2$, $\alpha = \beta = (\lambda_{N+1} - \lambda_N)/2$, $\tilde{M} = M_1 = M_3 = 1$, $M_2 = \tilde{\lambda}_N^\gamma$ and $M_4 = \tilde{\lambda}_{N+1}^\gamma$, then the hypotheses (S1)-(S5) in Section 2 are satisfied. Moreover, we have that $\|AP_1\|_{L(Y_1)} \leq \lambda_N$.

Consider an evolution equation

$$(3.1) \qquad du/dt + Au = R(u), \quad t > 0, \qquad u(0) = x_o$$

in the Hilbert space Y, where $R \in C^1(X,Y)$. Suppose that the following conditions (i) and (ii) are satisfied.

(i) There exists a closed subset D of X such that for each $x_o \in D$ (3.1) has a unique mild solution $u(\cdot;x_o) \in C([0,\infty);D)$.

(ii) For each $x_o \in D$ there exists $T_o = T_o(x_o) > 0$ such that $\|u(t;x_o)\|_X \le \rho$ for all $t \ge T_o$, where ρ is a positive constant independent of x_o.

Then, the following modification

$$(3.2) \qquad du/dt + Au = F(u), \quad t > 0, \quad u(0) = x_o$$

with

$$F(u) = \theta_\varepsilon(\|u\|_X)R(u), \qquad 0 < \varepsilon < 1/2\rho$$

$$\theta_\varepsilon(s) = \theta(s/\varepsilon), \quad \theta \in \dot{C}_o^\infty(\Re), \quad 0 \le \theta \le 1,$$

$$\theta(s) = 1 \text{ for } |s| \le 1, \quad \theta(s) = 0 \text{ for } |s| \ge 2,$$

provides the same asymptotic behavior as $t \to \infty$ on D. Since the norm of X is smooth, $F \in C^1(X,Y) \cap Lip(X,Y)$. Therefore, we can apply our theorems in Section 2 to (3.2). In this case the condition (2.7) holds true whenever

$$(3.3) \qquad \lambda_{N+1}^\gamma(\lambda_{N+1} - \lambda_N)^{-1} \to 0 \qquad \text{as } N \to \infty.$$

Thus we have

Theorem 3.1. Let A and R be defined as above. Assume that (i) and (ii) are satisfied. If (3.3) holds, then (3.1) has an inertial manifold M_D <u>in D</u>, i.e.,

$(A)_D$ $M_D = \{\xi + h(\xi);\ \xi \in P_1D\}$ for some $h \in C^1(P_1Y, P_2X)$.

$(b)_D$ If $x_o \in M_D$ then $u(\cdot;x_o) \in M_D$ for all $t > 0$.

$(c)_D$ For each $x_o \in D$ there exists $x_o^* \in M_D$ such that

$$\sup_{t\geq 0} e^{-\eta t}\|u(t;x_o) - u(t;x_o^*)\|_X < \infty.$$

Moreover, Theorems 2.2-2.5 apply to (3.1) in D.

Example. (The Cahn-Hilliard equation)

Let Ω be a bounded domain in $\Re^d$, $d \leq 2$. We consider the following equation

(3.4) $\quad u_t - \Delta(-\Delta u - \alpha u + \beta u^3) = 0 \quad$ in $\Omega \times \Re^+$

(3.5) $\quad u(\cdot,0) = u_o(\cdot) \quad$ in Ω

and

$(3.6)_n$ $\quad \partial u/\partial n = \partial \Delta u/\partial n \quad$ on $\partial\Omega \times \Re^+$, if $\partial\Omega$ is smooth

or

$(3.6)_p$ $\quad u(x + 2\pi e_i,t) = u(x,t) \quad$ on $\Omega \times \Re^+$, if $\Omega = [-\pi,\pi]^d$

where α and β are positive constants, n is the unit vector normal to $\partial\Omega$ and e_i is the canonical basis of $\Re^d$.

Set $A = \Delta^2$ with $D(A) = \{u \in H^4(\Omega);$ u satisfies $(3.6)_n$ or $(3.6)_p\}$. Then, the operator $\Lambda = A + I$ in $L^2(\Omega)$ satisfies the properties stated above. Let $Y = L^2(\Omega)$, $X = D(\Lambda^{1/2})$ and $R(u) = \Delta(-\alpha u + \beta u^3)$. It is easy to see that $R \in C^1(X,Y)$. For each $\rho > 0$ we set

$$D = \{u \in X;\ |\int_\Omega u\, dx\ | \leq \rho|\Omega|\ \}.$$

It is proved in [17, Proposition 1.1] that the set D satisfies (i) and (ii). The eigenvalues λ_N of A satisfies $\lambda_N \sim N^4$ as $N \to \infty$ if $d = 1$, which implies that (3.3) holds with $\gamma = 1/2$. Hence, Theorem 3.1 applies to (3.4) with $(3.6)_n$ or $(3.6)_p$ if $d = 1$. If $d = 2$, then $\lambda_N \sim N^2$ as $N \to \infty$, which does not impliy (3.3) in general. We know, however, that the eigenvalues μ_j of $-\Delta$ with the periodic boundary condition $(3.6)_p$ have the form

$$\mu_j = k_1^2 + k_2^2$$

where k_1 and k_2 are integers. It follows from [19] that there is a subsequence $\{N_k\}$ such that $\mu_{N_k + 1} - \mu_{N_k} \to \infty$ as $K \to \infty$. Since $\lambda_j \sim \mu_j^2$ as $j \to \infty$, (3.3) (with γ - 1/2) holds true for $N = N_k$. Consequently, Theorem 3.1 applies again to (3.4) with $(3.6)_p$ when $d = 2$.

We can also treat the Kuramoto-Sivashinsky equation, a modified Navier-Stokes equation and a reaction-diffusion equation (see [12]).

References

1. H. Brezis and A. Pazy, Convergence and approximation of semigroups of nonlinear operators in Banach spaces, J. Functional Analysis, 7 (1872), 63-74.
2. P. Chernoff, Note on product formulas for operator semi-groups, J. Functional Analysis, 2 (1968), 63-74.
3. S.N. Chow and K. Lu, Invariant manifolds for flows in Banach spaces, J. Differential Equations, 74(1988), 285-317.
4. P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Integral manifolds and inertial manifolds for dissipative partial differential equations, Springer, New York, 1989.
5. F. Demengel and J.M. Ghidaglia, Discrétisation en temps et variétés dérivées partielles non linéaires, C. R. Acad. Sci. Paris, Sér.I 307 (1988), 453-457.
6. C. Foias, B. Nicolaenko, G.R. Sell and R. Temam, Variétés inertielles pour l'equation de Kuramoto-Sivashinsky, C. R.

Acad. Sci. Paris, Sér.I 301 (1985), 285-288.

7. C. Foias, G.R. Sell and R. Temam, Inertial manifolds for nonlinear evolution equations, J. Differential Equations 73 (1988), 309-353.

8. J.K. Hale, Asymptotic behavior and dynamics in infinite dimensions, Res. Notes in Math. (Hale and Martinez-Amores, eds.) Vol. 132, Pitman, Boston, 1985, 1-41.

9. J.K. Hale, Asymptotic behavior of dissipative systens , Amer. Math. Soc. Providence, RI. 1988.

10. J.K. Hale, L.T. Magalhaes and W.M. Oliva, An introduction to infinite dimensinal dynamical systems= Geometric theory , Springer, New york, 1984.

11. D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Math. No. 840, Springer, New York, 1981.

12. K. Kobayasi, Convergence and approximation of inertial manifolds for evolution equations, to appear.

13. J. Mallet-Paret, Negatively invariant sets of compact maps and an extension of a theorem of Cortwright, J. Differential Equations 22(1976), 331-348.

14. J. Mallet-Paret and G.R. Sell, Inertial manifolds for reaction diffusion equations in higher space dimensions, J. Amer. Math. Soc. 1 (1988), 805-866.

15 R. Mane, On the dimension of the compact invariant sets of certain nonlinear maps, pp.230-242, Lecture Notes in Math., Vol. 898, Springer. New york, 1981.

16. X. Mora, Finite dimensional attracting manifolds in reaction-diffusion equation, Contemporary Math. 17 (1987),

353-360.

17. B. Nicolaenko, B. Scheurer and R. Temam, Some global dynamical properties of a class of pattern formation equations, Comm. in Partial Differential Equations 14 (1989), 245-297.

18. A. Pazy, Semigroups of linear operators and applications to partial differential equations , Springer, New York, 1983.

19. J. Richards, On the gaps between numbers which are the sum of two squares, Adv. Math. 46 (1982), 1-2.

20. R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, Springer, New York, 1988.