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Existence of positive entire solutions for semilinear elliptic systems

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1. Introduction

The existence of entire solutions of semilinear elliptic systems

$$Lu^k = \lambda f^k(x,\mathbf{u},Du^k) \quad \text{in } \mathbf{R}^N,\ N \geq 2,\ k = 1,2,\ldots,M \tag{1}$$

with conditions

$$\mathbf{u}(x) > \mathbf{0} \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{|x|\to\infty}\mathbf{u}(x) = \xi \geq \mathbf{0} \tag{2}$$

will be considered, where L is a uniformly elliptic operator of second order in $\mathbf{R}^N$, λ is a real constant, $\mathbf{u} = (u^1,\ldots,u^M)$, and $\xi = (\xi^1,\ldots,\xi^N)$, $M \geq 1$. A vector valued function $\mathbf{u}$ is said to be an entire solution of (1) if it is of class $C^2(\mathbf{R}^N;\mathbf{R}^M)$ and satisfies (1). Inequalities between vectors are defined to hold componentwise.

For the scalar equations, i.e. M = 1, the existence theory of positive entire solutions of (1) with (2) has been greatly developed by many authors (see eg. [3]-[6], [8], [15], [16] and references therein). On the other hand, for the systems, although there are some interesting works on the existence of positive solutions ([1], [2], [8]-[11], and [16]), most of the literature has been devoted to systems of the form

$$Lu^k = \lambda f^k(x,\mathbf{u}) \quad \text{in } \mathbf{R}^N,\ k = 1,2,\ldots,M, \tag{3}$$

with the structure condition that $\mathbf{f}$ is quasimonotone in $\mathbf{u}$, that is, each f^k is either increasing or decreasing in u^j for $j \neq k$. However, very little is known about systems even for (3) unless $\mathbf{f}$ satisfies monotonicity conditions.

The objective of this paper is to develop the existence theorem on positive entire solutions for general systems of the form (1). More precisely, we give sufficient conditions for system (1) to have entire solutions satisfying (2) regardless the monotonicity condition for $\mathbf{f}$. Further peculity of our consideration is that we can treat system (1) in which f^k contains on the gradient of u^k.

Our results can be applied to the following systems

$$\left\{\begin{aligned} &- \Delta u + c(x)u = \lambda(\Phi_{11}(x)u^{\alpha} + \Phi_{12}(x)v^{\beta} + \Phi_{13}(x)|Du|^{\gamma}) \\ &- \Delta v + c(x)v = \lambda(\Phi_{21}(x)u^{\rho} + \Phi_{22}(x)v^{\sigma} + \Phi_{23}(x)|Dv|^{\tau}) \end{aligned}\right. \tag{A}$$

$$\text{in } \mathbf{R}^N,\ N \geq 3,$$

and

$$\left\{\begin{aligned} &- \Delta u = \lambda\Phi(x)u^{\alpha}v^{\beta}(1 + p_{11}u^{\gamma} + p_{12}v^{\delta}) \\ &- \Delta v = \lambda\Psi(x)u^{\mu}v^{\nu}(1 + p_{21}u^{\sigma} + p_{22}v^{\tau}), \quad \text{in } \mathbf{R}^N,\ N \geq 3, \end{aligned}\right. \tag{B}$$

where Δ is N-dimensional Laplacian, c is a nonnegative function in $\mathbf{R}^N$, λ is a real constant, $\alpha, \beta, \ldots, \tau$ are nonnegative and p_{ij} are constants. Although these systems have very simple forms, the previous existence theorems cannot be applied to (A) provided that either one of Φ_{12} and Φ_{21} changes sign in $\mathbf{R}^N$ or $\Phi_{12}(x)\cdot\Phi_{21}(x) < 0$ at some points in $\mathbf{R}^N$, even if not only $\Phi_{13} \neq 0$ and $\Phi_{23} \neq 0$ but aolso $\Phi_{13} = \Phi_{23} = 0$. For system (B) with positive Φ and Ψ, since the monotonicity condition generally breakes in the case $p_{12} \cdot p_{21} \neq 0$, we cannot directly apply the previous results. Applying our theorems to the systems (A) and (B), we see that there exist infinitely many positive entire solutions of (A) and (B) tending to positive constants as $|x| \to \infty$, provided c, Φ_{ij}, Φ and Ψ satisfy the integral condition such that $\int_0^\infty rh^*(r)dr < \infty$, where $h^*(r) = \max_{|x|=r}|h(x)|$ for continuos function h in $\mathbf{R}^N$. We are also able to see the existence of decaying solutions for (A) and (B). For the detail see Examples 1 and 2 below.

The main tool of the proof of existence theorem is the barrier method (supersolution-subsolution method). The varrier method for systems was first established by Sattinger[18] for the boundary value problems in bounded domains and then extended by Kawano[8] in the entire space case for M = 2. Furthermore, in the recent preprint Kusano and Swanson[11] also generalize the theory to $M \geq 2$, and apply to the study of even order elliptic equations. These barrier methods are based on the monotone iterative technique, and so the quasimonotonicity for $\mathbf{f}$ in $\mathbf{u}$ is essentially assumed. Therefore, we cannot use the previous theory to see the existence of solutions for general systems (1). So, we first prepare the generalized barrier method for systems (1) in Lemma 2 below. This method was first established by Tsai[21] for the boundary value problems in bounded domains (see Lemma 1 below). Especially, if each f^k in (3) is increasing, Lemma 2 covers the results of Kawano[8; Theorems 5.1 and 5.2] and of Kusano and Swanson [11; Theorem 2.1]. To apply the barrier method for (1), we employ the existence and asymptotic behavior at infinity of solutions for second order linear elliptic equations.

Our main results are stated in Section 2 and are proved in Section 3. In the final section, we give some corollaries and typical examples illustrating our theorems.

2. Statement of main results

Let L in (1) be a second order differential operator of the form

$$L = -\left(\sum_{i,j=1}^{N} a_{ij}(x)D_{ij} + \sum_{i=1}^{N} b_i(x)D_i\right) + c\,(x)\cdot,$$

where $D_i = \partial/\partial x_i$ and $D_{ij} = \partial^2/\partial x_i \partial x_j$, $1 \le i,j \le N$. We use the following notation:

$$A(x) = \sum_{i,j=1}^{N} a_{ij}(x)x_i x_j/|x|^2,$$

$$B(x) = \left[\sum_{i=1}^{N} \{b_i(x_i)x_i + a_{ii}(x)\} - A(x)\right]/|x|,$$

$$h^*(r) = \max_{|x|=r} |h(x)|/A(x),\ r > 0,\ \text{for a } h \in C(\mathbf{R}^N;\mathbf{R}).$$

A vector $\mathbf{u} = (u^1,\dots,u^M)$ is said to be positive (or nonnegative) and is denoted by $\mathbf{u} > \mathbf{0}$ (or $\mathbf{u} \ge \mathbf{0}$), if all the components are positve (or nonnegative). Furthermore, we denote the vector $(1,\dots,1)$ by $\mathbf{1}$.

Throughout this paper, we assume that L satisfies the following conditions (H_1)-(H_3):

(H_1) $a_{ij} \in C^{1+\theta}_{loc}(\mathbf{R}^N;\mathbf{R})$, $b_i \in C^{\theta}_{loc}(\mathbf{R}^N;\mathbf{R})$, $c \in C^{\theta}_{loc}(\mathbf{R}^N;\overline{\mathbf{R}}_+)$,

$1 \le i,j \le N$, where $\overline{\mathbf{R}}_+ = [0, \infty)$ and $0 < \theta < 1$.

(H_2) Matrix $(a_{ij}(x))$ is uniformly positive definite in $\mathbf{R}^N$.

(H_3) There exists a function $B_* \in C^\theta_{loc}(\overline{\mathbf{R}}_+;\mathbf{R})$ such that

$$B_*(r) \le \min_{|x|=r} B(x)/A(x) \quad \text{for} \quad r > 0 \quad \text{and}$$

$$\int_1^\infty \exp\left(-\int_1^r B_*(s)ds\right)dr < \infty.$$

For the examples of L satisfying (H_1)-(H_3) we refer to [4]-[6](see Remark 4 below).

To state our main results we need the following functions p_* and π_* defined by, respectively,

$$p_*(r) = \exp\left(\int_1^r B_*(s)ds\right) \quad \text{and} \quad \pi_*(r) = \int_r^\infty ds/p_*(s) \quad \text{for } r \ge 1.$$

For the nonlinear term $\mathbf{f} = (f^1,\ldots,f^M)$ some of the following conditions are assumed to be satisfied:

(F_1) $\mathbf{f} \in C^\theta_{loc}(\mathbf{R}^N\times\mathbf{R}^M\times\mathbf{R}^N;\mathbf{R}^M)$ and for each k and for any bounded domain Ω there exists a nondecreasing function $\psi^k_\Omega \in C(\overline{\mathbf{R}}_+;\overline{\mathbf{R}}_+)$ such that

$$|f^k(x,\mathbf{u},\mathbf{p})| \le \psi^k_\Omega(|\mathbf{u}|)(1 + |\mathbf{p}|^2)$$

for $(x,\mathbf{u},\mathbf{p}) \in \Omega\times\mathbf{R}^M\times\mathbf{R}^N$, where $|\mathbf{u}| = \{(u^1)^2 + \ldots + (u^M)^2\}^{1/2}$ (Nagumo's condition).

(F_2) There exists a bounded function $G \in C^\theta_{loc}(\mathbf{R}^N;\overline{\mathbf{R}}_+)$ together with a positive constant J_0 such that

$$|f^k(x,\mathbf{u},\mathbf{p})| \le G(x), \quad x \in \mathbf{R}^N, \ |\mathbf{u}| \le J_0, \ |\mathbf{p}| \le J_0,$$

and $$\int_1^\infty p_*(r)\pi_*(r)G^*(r)dr < \infty.$$

(F_3) There exist open sets Ω_k, $1 \le k \le M$, constants $J_1 > 0$

and $\gamma \in (0,1)$ such that each f^k satisfies the following conditions (i) and (ii):

(i) $f^k(x,\mathbf{u},\mathbf{p}) \geq 0$ for $x \in \mathbf{R}^N$, $\mathbf{0} \leq \mathbf{u} \leq J_1\mathbf{1}$, $|\mathbf{p}| \leq J_1$,

(ii) $\liminf_{t\to+0} f^k(x,u^1,\ldots,u^{k-1},t,u^{k+1},\ldots,u^M,\mathbf{p})/t^\gamma \geq m$

for some $m > 0$ uniformly in $(x,\mathbf{u},\mathbf{p})$ with $x \in \Omega_k$, $0 < u^j \leq J_1$, $j \neq k$, and $|\mathbf{p}| \leq J_1$.

Our main theorems are as follows.

<u>Theorem</u> 1. <u>Assume that</u> (H_1)-(H_3), (F_1) <u>and</u> (F_2) <u>hold</u>. <u>If</u>

$$\int_1^\infty p_*(r)\pi_*(r)c^*(r)dr < \infty, \tag{4}$$

<u>then there exists</u> $\lambda^* > 0$ <u>such that, for every</u> λ <u>with</u> $|\lambda| < \lambda^*$, (1) <u>has infinitely many positive entire solutions</u> $\mathbf{u}$ <u>satisfying</u> (2) <u>with</u> $\xi = (\xi^1,\ldots,\xi^M) > \mathbf{0}$.

<u>Theorem</u> 2. <u>Assume that</u> (H_1)-(H_3) <u>and</u> (F_1)-(F_3) <u>hold</u>. <u>Then there exists</u> $\lambda^* > 0$ <u>such that for every</u> $\lambda \in (0, \lambda^*)$ (1) <u>has a positive entire solution</u> $\mathbf{u}$ <u>satisfying</u> $\lim_{|x|\to\infty}\mathbf{u}(x) = \mathbf{0}$.

3. Proofs of Theorems

We first give the barrier method for the system in the bounded domain.

<u>Lemma</u> 1. <u>Assume that</u> L <u>satisfy</u> (H_1) <u>and</u> (H_2). <u>Let</u> Ω <u>be a bounded domain in</u> $\mathbf{R}^N$, $\mathbf{f}$ <u>satisfy</u> (F_1) <u>and</u> φ <u>be of class</u> $C^{2+\theta}(\overline{\Omega};\mathbf{R}^M)$. <u>Suppose that there exists a pair of functions</u> $\mathbf{V}$ <u>and</u> $\mathbf{W}$ <u>of class</u> $C^{2+\theta}(\overline{\Omega};\mathbf{R}^M)$ <u>such that</u> $\mathbf{V} \leq \mathbf{W}$ <u>in</u> Ω,

$$LV^k(x) \leq f^k(x,\sigma,DV^k(x))$$

for any $\sigma \in \mathbf{R}^M$ satisfying $V^j(x) \le \sigma^j \le W^j(x)$ for $j \ne k$, $\sigma^k = V^k(x)$ at each fixed $x \in \Omega$,

$$LW^k(x) \ge f^k(x,\tau,DW^k(x))$$

for any $\tau \in \mathbf{R}^M$ satisfying $V^j(x) \le \tau^j \le W^j(x)$ for $j \ne k$, $\tau^k = W^k(x)$ at each fixed $x \in \Omega$, $k = 1,\dots,M$, and $\mathbf{V} \le \varphi \le \mathbf{W}$ on $\partial\Omega$. Then the boundary value problem

$$\begin{cases} Lu^k = f^k(x,\mathbf{u},Du^k) & \text{in } \Omega, \\ u^k = \varphi^k & \text{on } \partial\Omega, \quad k = 1,2,\dots,M \end{cases}$$

has a solution $\mathbf{u} \in C^{2+\theta}(\bar{\Omega};\mathbf{R}^M)$ satisfying $\mathbf{V} \le \mathbf{u} \le \mathbf{W}$ in Ω.

This lemma was proved by Tsai [21; Theorem 2.2] under more general conditions concerning $\mathbf{f}$ by using Leray-Schauder's fixed point theorem. So we omit the proof. See also [12; Theorem 3.4.4] and [14; Theorem 1.4.2].

We now extend Lemma 1 to system (1) in $\mathbf{R}^N$ as follows:

Lemma 2. Let L satisfy (H_1) and (H_2) and assume that $\mathbf{f}$ satisfies (F_1). If there exists a pair of functions $\mathbf{V}$ and $\mathbf{W}$ of class $C^{2+\theta}_{loc}(\mathbf{R}^N;\mathbf{R}^M)$ such that $\mathbf{V} \le \mathbf{W}$ in $\mathbf{R}^N$,

$$LV^k(x) \le \lambda f^k(x,\sigma,DV^k(x)) \tag{5}$$

for any $\sigma \in \mathbf{R}^M$ satisfying $V^j(x) \le \sigma^j \le W^j(x)$, $j \ne k$, $\sigma^k = V^k(x)$ at each fixed $x \in \mathbf{R}^N$, and

$$LW^k(x) \ge \lambda f^k(x,\tau,DW^k(x)) \tag{6}$$

for any $\tau \in \mathbf{R}^M$ satisfying $V^j(x) \le \tau^j \le W^j(x)$, $j \ne k$, $\tau^k = W^k(x)$ at each fixed $x \in \mathbf{R}^N$, $k = 1,2,\dots,M$, then (1) has an entire solution $\mathbf{u}$ with $\mathbf{V} \le \mathbf{u} \le \mathbf{W}$ in $\mathbf{R}^N$.

Proof. It is enough to show the assertion in the case $\lambda = 1$.

For $\ell \in \mathbf{N}$, put $B_\ell = \{x \in \mathbf{R}^N: |x| < \ell\}$ and consider the boundary value problem

$$\begin{cases} Lu^k = f^k(x,\mathbf{u},Du^k) & \text{in } B_\ell, \\ u^k = w^k & \text{on } \partial B_\ell,\ k = 1,2,\ldots,M. \end{cases} \tag{7}$$

The restrictions of $\mathbf{V}$ and $\mathbf{W}$ to $\overline{B}_\ell$ satisfy the conditions in Lemma 1 for problem (7). So, for every $\ell \in \mathbf{N}$ there exists at least one solution $\mathbf{u}_\ell \in C^{2+\theta}(\overline{B}_\ell;\mathbf{R}^M)$ of (7) satisfying $\mathbf{V} \le \mathbf{u}_\ell \le \mathbf{W}$ in $\overline{B}_\ell$. Take a solution $\mathbf{u}_\ell$ of (7), extend it to $\mathbf{R}^N$ by putting equally to $\mathbf{W}$ outside $\overline{B}_\ell$ and denote it by $\mathbf{u}_\ell$ again. Then each $\mathbf{u}_\ell$ is continuous in $\mathbf{R}^N$ and satisfies

$$\mathbf{V} \le \mathbf{u}_\ell \le \mathbf{W} \quad \text{in } \mathbf{R}^N. \tag{8}$$

Therefore the sequence $\{\mathbf{u}_\ell\}$ is locally uniformly bounded in $\mathbf{R}^N$. Furthermore, note that for every $\ell \ge m + 3$, $m \in \mathbf{N}$, $\mathbf{u}_\ell$ satisfies the equations

$$Lu_\ell^k(x) = f^k(x,\mathbf{u}_\ell(x),Du_\ell^k(x)) \quad \text{in } B_{m+3},\ k = 1,\ 2,\ldots,M. \tag{9}$$

Applying the interior estimates of [13; p.266, Theorem 3.1] for the solution of (9) regarded as a single equation for u_ℓ^k, we have

$$\max\{|Du_\ell^k(x)|: x \in \overline{B}_{m+2},\ k = 1,2,\ldots,M\} \le K_1, \tag{10}$$

where K_1 is a constant independent of $\mathbf{u}_\ell$ for $\ell \ge m+3$. Furthermore, by interior L^p-estimates we obtain

$$\|u_\ell^k\|_{2,p,m+1} \le K_2(\|f(x,\mathbf{u}_\ell,Du_\ell^k)\|_{0,p,m+2} + \|u_\ell^k\|_{0,p,m+2}), \tag{11}$$

where $\|\ \|_{j,p,m}$ denotes the norm in the Sobolev space $W^{j,p}(B_m)$. Since by (10) the right hand side of (11) is bounded for $\ell \ge m + 3$, taking p such that $p > N/(1-\theta)$ and using Sobolev's imbedding theorem, we see that the sequence $\{\mathbf{u}_\ell\}_{\ell \ge m+3}$ is bounded in the space $C^{1+\theta}(\overline{B}_{m+1};\mathbf{R}^M)$. Furthermore, Schauder's interior estimates for the solution $\mathbf{u}_\ell$ of (9) regarded as a linear equation for u_ℓ^k imply the boundedness of $\{\mathbf{u}_\ell\}_{\ell\geq m+3}$ in the space $C^{2+\theta}(\overline{B}_m;\mathbf{R}^M)$ for every $m \geq 1$. Therefore by Ascoli-Arzela's theorem we can choose a subsequence $\{\mathbf{u}_{\ell(j)}\}$ of $\{\mathbf{u}_\ell\}$ such that it converges to some function $\mathbf{u} \in C^2(\mathbf{R}^N;\mathbf{R}^M)$ uniformly in any compact set in $\mathbf{R}^N$ together with their derivatives up to the second order. Then by (8) and (9) we see that $\mathbf{u}$ satisfies (1) with $\lambda = 1$ and $\mathbf{V} \leq \mathbf{u} \leq \mathbf{W}$ in $\mathbf{R}^N$.

For the simplicity we consider (1) with $\lambda = 1$ in the following corollaries.

<u>Corollary</u> 1. <u>Let</u> L <u>and</u> $\mathbf{f}$ <u>be</u> <u>as</u> <u>in</u> <u>Lemma</u> 2. <u>Assume</u> <u>that</u> <u>each</u> f^k <u>is</u> <u>nondecreasing</u> <u>in</u> u^j <u>for</u> $j \neq k$. <u>If</u> <u>there</u> <u>exists</u> <u>a</u> <u>pair</u> <u>of</u> <u>functions</u> $\mathbf{V}$ <u>and</u> $\mathbf{W}$ <u>of</u> <u>class</u> $C^{2+\theta}_{loc}(\mathbf{R}^N;\mathbf{R}^M)$ <u>such</u> <u>that</u> $\mathbf{V} \leq \mathbf{W}$ <u>in</u> $\mathbf{R}^N$, <u>and</u>

$$LV^k \leq f^k(x,\mathbf{V},DV^k), \quad LW^k \geq f^k(x,\mathbf{W},DW^k) \quad \text{in } \mathbf{R}^N, \ k = 1,2,\ldots,M,$$

<u>then</u> <u>system</u> (1) <u>with</u> $\lambda = 1$ <u>has</u> <u>a</u> <u>solution</u> $\mathbf{u}$ <u>satisfying</u> $\mathbf{V} \leq \mathbf{u} \leq \mathbf{W}$ <u>in</u> $\mathbf{R}^N$.

<u>Corollary</u> 2. <u>Let</u> L <u>and</u> $\mathbf{f}$ <u>be</u> <u>as</u> <u>in</u> <u>Lemma</u> 2. <u>Assume</u> <u>that</u> <u>eac</u> f^k <u>is</u> <u>nonincreasing</u> <u>in</u> u^j <u>for</u> $j \neq k$. <u>If</u> <u>there</u> <u>exists</u> <u>a</u> <u>pair</u> <u>of</u> <u>functions</u> $\mathbf{V}$ <u>and</u> $\mathbf{W}$ <u>of</u> <u>class</u> $C^{2+\theta}_{loc}(\mathbf{R}^N;\mathbf{R}^M)$ <u>such</u> <u>that</u> $\mathbf{V} \leq \mathbf{W}$ <u>in</u> $\mathbf{R}^N$, <u>and</u>

$$LV^k \leq f^k(x,W^1,\ldots,W^{k-1},V^k,W^{k+1},\ldots,W^M,DV^k) \quad \text{in } \mathbf{R}^N,$$
$$LW^k \geq f^k(x,V^1,\ldots,V^{k-1},W^k,V^{k+1},\ldots,V^M,DW^k) \quad \text{in } \mathbf{R}^N,$$

$k = 1,2,\ldots,M$, <u>then</u> (1) <u>with</u> $\lambda = 1$ <u>has</u> <u>a</u> <u>solution</u> $\mathbf{u}$ <u>satisfying</u> $\mathbf{V} \leq \mathbf{u} \leq \mathbf{W}$ <u>in</u> $\mathbf{R}^N$.

<u>Remark</u> 1. (i) Corollary 1 is the usual supersolution-subsolution method and was proved by Kawano [8] and by Kusano and Swanson [11] in the special case where $L = -\Delta$ and each f^k is independent of Du^k.

(ii) Corollary 2 contains the result of Kawano [8; Theorem 5.2] as a special case.

We are now ready to prove our theorems.

<u>Proof of Theorem</u> 1. By (F_2) and (2), Theorem 2.2 in [5] guarantees the existence of the functions V and W of class $C^{2+\theta}_{loc}(\mathbf{R}^N;\mathbf{R}_+)$ with finite C^1 norms such that

$$\begin{cases} L\tilde{V} = -G(x), \quad L\tilde{W} = G(x) \quad \text{in } \mathbf{R}^N \\ 0 < \tilde{V} \le \tilde{W} \quad \text{in } \mathbf{R}^N \\ \lim_{|x|\to\infty}\tilde{V}(x) = \lim_{|x|\to\infty}\tilde{W}(x) = \tilde{\xi} > 0 \end{cases} \tag{12}$$

Put now $\lambda^* = J_0/\max\{\|\tilde{\mathbf{V}}\|_1, \|\tilde{\mathbf{W}}\|_1\}$, and let $0 \le \lambda < \lambda^*$, where $\tilde{\mathbf{V}} = \tilde{V}\mathbf{1}$, $\tilde{\mathbf{W}} = \tilde{W}\mathbf{1}$ in $\mathbf{R}^N$, and

$$\|\tilde{\mathbf{V}}\|_1 = \sup\{|\tilde{V}(x)| + |D\tilde{V}(x)|: x \in \mathbf{R}^N\}.$$

By (12) and (F_2), the functions $\mathbf{V} = \xi\tilde{\mathbf{V}}$ and $\mathbf{W} = \xi\tilde{\mathbf{W}}$ with $\xi \in (\lambda, \lambda^*)$ satisfy (5) and (6), respectively. In fact

$$LV^k(x) = -\xi G(x) \le -\lambda G(x) \le -\lambda f^k(x,\sigma,DV^k(x))$$

for any $\sigma \in \mathbf{R}^M$ such that $V^j(x) \le \sigma^j \le W^j(x)$, $j \ne k$, $\sigma^k = V^k(x)$ at $x \in \mathbf{R}^N$. (6) is similarly satisfied by $\mathbf{W}$. Therefore, Lemma 2 guarantees the existence of a solution $\mathbf{u}$ of (1) satisfying $\mathbf{0} < \mathbf{V} \le \mathbf{u} \le \mathbf{W}$ in $\mathbf{R}^N$, and so $\lim_{|x|\to\infty}\mathbf{u}(x) = \lim_{|x|\to\infty}\mathbf{V}(x) = \lim_{|x|\to\infty}\mathbf{W}(x) = \xi\tilde{\xi} > \mathbf{0}$. Since $\xi \in (\lambda, \lambda^*)$ is arbitrary, infinitely many solutions of (1) exist.

Replacing $\mathbf{f}$ by $-\mathbf{f}$, we have the conclusion for negative λ. This completes the proof.

<u>Proof of Theorem</u> 2. Let $\tilde{W} \in C^{2+\theta}_{loc}(\mathbf{R}^N;\mathbf{R}_+)$ be a solution of

$$\begin{cases} L\tilde{W} = G(x) \quad \text{in } \mathbf{R}^N \\ \lim_{|x|\to\infty}\tilde{W}(X) = 0. \end{cases} \tag{13}$$

The existence of such $\tilde{W}$ is guaranteed by (F_2) and [5; Theorem 2.2]. Now we put $\lambda^* = \min\{J_0, J_1/\|\tilde{\mathbf{W}}\|_1\}$, where $\tilde{\mathbf{W}} = \tilde{W}\mathbf{1}$. As in the proof of Theorem 1, we see that the function $\mathbf{W} = \lambda\tilde{\mathbf{W}}$ satisfies (6) in which $\mathbf{V}(x)$ is replaced by $\mathbf{0}$ provided $0 < \lambda < \lambda^*$.

We next construct a function $\mathbf{V}$ satisfying (5). By (ii) of (F_3) we can choose a constant $\delta \in (0, J_1]$ such that

$$f^k(x,u^1,\dots,u^{k-1},t,u^{k+1},\dots,u^M,\mathbf{p}) \ge mt^\gamma/2 \tag{14}$$

provided $x \in \Omega_k$, $0 < u^j \le J_1$, $j \ne k$, $0 < t \le \delta$, and $|\mathbf{p}| \le J_1$. Take now a nonnegative function $G_0^k \in C_0^\theta(\mathbf{R}^N;\mathbf{R})$ such that $\phi \ne \operatorname{supp} G_0^k \subset\subset \Omega_k$ and

$$G_0^k(x) \le \min\{\lambda G(x), m/2\} \qquad \text{for } x \in \operatorname{supp} G_0^k. \tag{15}$$

By [5;Theorem 2.2], there exists a positive function $\tilde{V}^k \in C^{2+\theta}_{loc}(\mathbf{R}^N;\mathbf{R})$ satisfying

$$L\tilde{V}^k = G_0^k(x), \quad \text{in } \mathbf{R}^N, \text{ and } \lim_{|x|\to\infty}\tilde{V}^k(x) = 0. \tag{16}$$

From (13), (15), (16) and the maximum principle, we have $\tilde{\mathbf{V}} \le \mathbf{W}$ in $\mathbf{R}^N$. Put now $K = \min_{1\le k\le M}\min\{\tilde{V}^k(x) : x \in \operatorname{supp} G_0^k\}$ and $\mu = \min\{1, (\lambda K^\gamma)^{1/(1-\gamma)}, \delta/\|\tilde{\mathbf{V}}\|_1\}$, and define a function $\mathbf{V}$ by $\mathbf{V} = \mu\tilde{\mathbf{V}}$ in $\mathbf{R}^N$. Then by (14) and (i) of (F_3) we see

$$LV^k(x) = \mu G_0^k(x) \le \mu(m/2)\mu V^k(x))^{-\gamma}(\mu V^k(x))^{\gamma}$$
$$= \mu^{1-\gamma}(V^k(x))^{-\gamma}(m/2)(\mu V^k(x))^{\gamma} \le \lambda f^k(x,\sigma,DV^k(x))$$

for any $\sigma \in \mathbf{R}^M$ satisfying $0 < \sigma^j \le J_1$, $j \ne k$, $\sigma^k = V^k(x)$ at each $x \in \mathbf{R}^N$. Since $\mathbf{0} < \mathbf{V} \le \mathbf{W} \le J_1\mathbf{1}$ in $\mathbf{R}^N$, $\mathbf{V}$ and $\mathbf{W}$ satisfy the conditions in Lemma 2. Therefore, the existence of the desired solution of (1) follows from Lemma 2.

4. Some Corollaries and Examples

The following Corollary is a direct consequence of Theorems 1 and 2.

Corollary 3. *Assume that* (H_1)-(H_3), (F_1), (F_2) *and* (4) *hold. If moreover the set* $\mathfrak{K}$ *of indices* k *such that* f^k *satisfy condition* (F_3) *is not empty, then there exists* $\lambda^* > 0$ *such that, for every* $\lambda \in (0, \lambda^*)$, (1) *has infinitely many positive entire solutions* $\mathbf{u}$ *satisfying* $\lim_{|x|\to\infty}\mathbf{u}(x) = \xi = (\xi^1,\dots,\xi^M)$ *with* $\xi^k > 0$ *for* $k \notin \mathfrak{K}$ *and* $\xi^k \ge 0$ *for* $k \in \mathfrak{K}$.

The conclusions in Theorems 1 and 2 are not generally valid provided λ is large enough. We next give sufficient conditions for (1) to have positive solutions for any $\lambda \in \mathbf{R}$.

Corollary 4. *Assume that* (H_1)-(H_3) *and* (4) *hold. Let* $\mathbf{f}$ *be nonnegative and satisfy condition* (F_1). *Furthermore, suppose that there exists a function* $\mathbf{F} \in C^{\theta}_{loc}(\mathbf{R}^N\times\mathbf{R}^M\times\mathbf{R}^N;\mathbf{R}^M_+)$ *satisfying the following conditions* (i)-(iii):

(i) $\mathbf{F}$ *satisfies Nagumo's condition and* (F_2).

(ii) F^k *is nondecreasing with respect to* u^j *for* $j \neq k$, *and satisfies*

$$f^k(x,\mathbf{u},\mathbf{p}) \leq F^k(x,\mathbf{u},\mathbf{p}) \quad \text{for } (x,\mathbf{u},\mathbf{p}) \in \mathbf{R}^N \times \mathbf{R}_+^M \times \mathbf{R}^N.$$

(iii) *For any fixed* $\xi > 0$ *there corresponds* $\tau_0 > 0$ *such that*

$$\mathbf{F}(x,\tau\mathbf{u},\tau\mathbf{p}) \geq \tau\xi\mathbf{F}(x,\mathbf{u},\mathbf{p}), \quad (x,\mathbf{u},\mathbf{p}) \in \mathbf{R}^N \times \mathbf{R}_+^M \times \mathbf{R}^N$$

provided $\tau > \tau_0$ (*or provided* $\tau \in (0, \tau_0)$).

Then, for every $\lambda \in \mathbf{R}$ *there exist infinitely many positive entire solutions* $\mathbf{u}$ *of* (1) *tending to some positive constant vectors as* $|x| \to \infty$.

Proof. We only prove the assertion in the case where condition (iii) in (F_3) holds for $\tau > \tau_0$. It is enough to consider system (1) with $\lambda = \pm 1$.

From Theorem 1 and its proof we see the existence of $\tilde{\lambda} > 0$ and $\mathbf{w} \in C_{loc}^{2+\theta}(\mathbf{R}^N;\mathbf{R}_+^M)$ such that

$$\begin{cases} Lw^k = \tilde{\lambda}F^k(x,\mathbf{w},Dw^k) & \text{in } \mathbf{R}^N,\ 1 \leq k \leq M, \\ \lim_{|x|\to\infty}\mathbf{w}(x) = \xi \end{cases}$$

for some $\xi > \mathbf{0}$. By condition (iii), we can chose $\tau_0 > 0$ such that for any $\tau > \tau_0$

$$F^k(x,\tau\mathbf{u},\tau\mathbf{p}) \geq (\tau/\tilde{\lambda})F^k(x,\mathbf{u},\mathbf{p}), \ (x,\mathbf{u},\mathbf{p}) \in \mathbf{R}^N \times \mathbf{R}_+^M \times \mathbf{R}^N, \ 1 \leq k \leq M.$$

So, the function $W = \tau^{-1}\mathbf{w}$ satisfies

$$\begin{aligned} LW^k(x) &= \tau^{-1}\tilde{\lambda}F^k(x,\mathbf{w}(x),Dw^k(x)) = \tau^{-1}\tilde{\lambda}F^k(x,\tau W(x),\tau DW^k(x)) \\ &\geq F^k(x,W(x),DW^k(x)), \quad x \in \mathbf{R}^N,\ 1 \leq k \leq M. \end{aligned}$$

Since F^k is nondecreasing in u^j, $j \neq k$, we have by (ii)

$$LW^k(x) \geq F^k(x,\sigma,DW^k(x)) \geq f^k(x,\sigma,DW^k(x))$$

for any $\sigma \in \mathbf{R}^M$ satisfying $0 < \sigma^j \leq W^j(x)$, $j \neq k$, $\sigma^k = W^k(x)$ at x

$\in \mathbf{R}^N$.

On the other hand, by the proof of [5; Theorem 2.2] we see the existence of a function $\mathbf{V} \in C^{2+\theta}_{loc}(\mathbf{R}^N;\mathbf{R}^M_+)$ such that

$$LV^k = 0 \quad \text{in} \quad \mathbf{R}^N \quad \text{and} \quad \lim_{|x|\to\infty}\mathbf{V}(x) = \tau^{-1}\xi.$$

Furthermore, the maximum principle implies $\mathbf{V} \le \mathbf{W}$ in $\mathbf{R}^N$. Hence, by Lemma 2, (1) with $\lambda = 1$ possesses a positive entire solution $\mathbf{u}$ satisfying $\mathbf{V} \le \mathbf{u} \le \mathbf{W}$ in $\mathbf{R}^N$ and so $\lim_{|x|\to\infty}\mathbf{u}(x) = \tau^{-1}\xi > \mathbf{0}$. For $\lambda = -1$, the proof is simillar. This completes the proof.

Combining the proofs of Theorem 2 and Corollary 4, we also see the existence of a decaying positive entire solution of (1) for every $\lambda > 0$ as follows.

Corollary 5. Assume that (H_1)-(H_3), (F_1) and (F_3) hold and $\mathbf{f}$ is nonnegative in $\mathbf{R}^N$. If there exists a function $\mathbf{F}$ satisfying conditions (i)-(iii) in Corollary 2 in which (iii) holds for any $\tau \in (0, \tau_0)$, then for any $\lambda > 0$ (1) possesses a positive entire solution $\mathbf{u}$ tending to $\mathbf{0}$ as $|x| \to \infty$.

We now give some examples illustrating our theorems. In the examples below $\max_{|x|=r}|h(x)|$ is denoted by $h^*(r)$ for a continuous function h in $\mathbf{R}^N$.

Example 1. Consider the system

$$\begin{cases} -\Delta u + c(x)u = \lambda(\Phi_{11}(x)u^{\alpha} + \Phi_{12}(x)v^{\beta} + \Phi_{13}(x)|Du|^{\gamma}) \\ -\Delta v + c(x)v = \lambda(\Phi_{21}(x)u^{\rho} + \Phi_{22}(x)v^{\sigma} + \Phi_{23}(x)|Dv|^{\tau}) \end{cases} \tag{17}$$

$$\text{in } \mathbf{R}^N, \ N \ge 3,$$

where c and Φ_{ij} are bounded locally Hölder continuous functions in $\mathbf{R}^N$, α, β,. . ., and τ are nonnegative constants. Let $c(x) \geq 0$ in $\mathbf{R}^N$, $\gamma \leq 2$, $\tau \leq 2$ and assume

$$\int_1^\infty r\{c^*(r) + \Phi_{ij}^*(r)\}dr < \infty, \quad i = 1, 2, \ j = 1, 2, 3. \tag{18}$$

Then the following statements hold.

(i) By Theorem 1 there exists $\lambda^* > 0$ such that for every λ with $|\lambda| < \lambda^*$, (17) has infinitely many positive entire solutions (u,v) satisfying

$$\lim_{|x|\to\infty} u(x) = \xi \quad \text{and} \quad \lim_{|x|\to\infty} v(x) = \eta \tag{19}$$

with positive ξ and η.

(ii) Let all Φ_{ij} be nonnegative and let $\Phi_{ii} \not\equiv 0$, $i = 1, 2$, and $0 \leq \alpha$, $\sigma < 1$. Then, by Corollary 4, there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ (17) has infinity many positive entire solutions (u,v) satisfying (19) with nonnegative ξ and η.

(iii) Corollary 3 implies that if all of α, β,..., and τ are smaller than 1, then the same conclusion as in (ii) is valid for every $\lambda > 0$. In the case where all of α, β,..., and τ are greater than 1, the same conclusion with positive ξ and η remains valid.

Remark 2. System (17) is considered by Kusano and Swanson [11] in the case where $c = \Phi_{13} = \Phi_{23} = 0$, $\Phi_{12} \geq 0$ and $\Phi_{21} \geq 0$ in $\mathbf{R}^N$, and either $0 \leq \alpha, \beta, \rho, \sigma < 1$ or $\alpha, \beta, \rho, \sigma > 1$. However, in the general case their results donot cover system (17).

Example 2 Let Φ and Ψ be bounded locally Hölder continuous functions in $\mathbf{R}^N$, and consider the system

$$\begin{cases} -\Delta u = \lambda\Phi(x)u^{\alpha}v^{\beta}(1 + p_{11}u^{\gamma} + p_{12}v^{\delta}) \\ -\Delta v = \lambda\Psi(x)u^{\mu}v^{\nu}(1 + p_{21}u^{\sigma} + p_{22}v^{\tau}), \quad \text{in } \mathbf{R}^N,\ N \geq 3, \end{cases} \tag{20}$$

where α, β,..., and τ are nonnegative constants and p_{ij} are constants. If

$$\int_1^{\infty} r\{\Phi^*(r) + \Psi^*(r)\}dr < \infty, \tag{21}$$

then the following statements hold.

(i) By Theorem 1, there exists $\lambda^* > 0$ such that for every λ with $|\lambda| < \lambda^*$, regardless the sign of Φ, Ψ and p_{ij}, (20) has infinitely many positive entire solutions (u,v) satisfying (19) with positive ξ and η.

(ii) Let $\Phi > 0$ and $\Psi > 0$ in $\mathbf{R}^N$, $0 \leq \alpha$, $\nu < 1$, $\beta = \mu = 0$, and γ, δ, σ, τ be positive. Then Corollary 3 guarantees the existence of positive entire solutions (u,v) of (20) satisfying (19) with nonnegative ξ and η provided λ is positive and small enough.

Statement (ii) of Example 2 explains nothing about the existence of positive entire solutions decaying to $\mathbf{0}$ for (20) provided one of β and μ is positive. In the following theorem we attack this problem. For the purpose we need the following condition (F_4) istead of (F_3).

(F_4) There exists $J_1 > 0$ such that the following conditions (i)-(iii) hold:

(i) $\mathbf{f}(x,\mathbf{u},\mathbf{p}) \geq \mathbf{0}$ for $x \in \mathbf{R}^N$, $0 \leq \mathbf{u} \leq J_1\mathbf{1}$, $|\mathbf{p}| \leq J_1$.

(ii) There exists a domain $\Omega_0 \subset \mathbf{R}^N$ and a constant $\gamma \in (0,1)$

such that

$$\liminf_{t\to+0} \mathbf{f}(x,t\mathbf{1},\mathbf{p})/t^{\gamma} \geq m\mathbf{1}$$

for some $m > 0$ uniformly in $x \in \Omega_0$ and $\mathbf{p}$ with $|\mathbf{p}| \leq J_1$.

(iii) Each f^k is nondecreasing with respect to u^j in $(0,J_1]$ with $j \neq k$ for any fixed $x \in \mathbf{R}^N$, $u^k \in (0,J_1]$ and $|\mathbf{p}| \leq J_1$.

<u>Theorem</u> 3. <u>Assume</u> <u>that</u> (H_1)-(H_3), (F_1), (F_2) <u>and</u> (F_4) <u>hold</u>. <u>Then</u> <u>there</u> <u>exists</u> $\lambda^* > 0$ <u>such</u> <u>that</u> <u>for</u> <u>every</u> $\lambda \in (0, \lambda^*)$ (1) <u>has</u> <u>a</u> <u>positive</u> <u>entire</u> <u>solution</u> $\mathbf{u}$ <u>tending</u> <u>to</u> $\mathbf{0}$ <u>as</u> $|x| \to \infty$.

<u>Proof</u>. From (ii) of (F_4), we can choose a positive constant J_2 such that $J_2 \leq J_1$ and

$$f^k(x,t\mathbf{1},\mathbf{p}) \geq mt^{\gamma}/2 \quad \text{for } x \in \Omega_0,\ 0 < t \leq J_2,\ |\mathbf{p}| \leq J_1. \tag{22}$$

Let λ^* and $\mathbf{W}$ be as in the proof of Theorem 2. In the proof of Theorem 2, we may take the functions G_0^k and $\tilde{V}^k$ as $G_0^k = G_o$ and $\tilde{V}^k = \tilde{V}$, respectively, for all k. Let $\mu > 0$ be as in Theorem 2 and put $\mathbf{V} = \mu\tilde{\mathbf{V}}$. Then, from condition (iii) of (F_4), it follows that

$$LV(x) \leq \lambda f^k(x,\sigma,DV(x))$$

for any $\sigma \in \mathbf{R}^M$ satisfying $V(x) \leq \sigma^j \leq J_2$, $j \neq k$, $\sigma^k = V(x)$ at $x \in \mathbf{R}^N$, $k = 1,2,\ldots,M$. The assertion follows from Lemma 2 as in the proof of Theorem 2. This completes the proof.

<u>Example</u> 2(Continued). Consider system (20), and assume that Φ and Ψ are positive in $\mathbf{R}^N$ and that $0 \leq \alpha + \beta,\ \mu + \nu < 1$, and p_{ij} are nonnegative. If condition (21) holds, then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ (20) has a positive entire solution (u,v) satisfying (19) with $\xi = \eta = 0$.

In the case where $p_{ij} = 0$, $i, j = 1, 2$, for any $\lambda > 0$ the same conclusion as above is valid.

Remark 3. In the special case where $p_{ij} = 0$, $i, j = 1,2$, system (20) was treated in [4], [8]-[11]. However, in the general case previous results cannot directly apply system (20).

Remark 4. Although we only consider the examples for the operator $L = -\Delta$ in $\mathbf{R}^N$, $N \geq 3$, if for example, the coefficients of L satisfy the conditions

$$a_{ij} \in L^{\infty}(\mathbf{R}^N) \text{ and } \liminf_{|x|\to\infty} \sum_{i=1}^{N} b_i(x)x_i/|x| > 0, \qquad (23)$$

in addition to (H_1) and (H_2), then (H_1)-(H_3) hold for L (see [4]). Therefore, the assertions in Examples 1 and 2 remain valid for systems (18) and (20) replaced $-\Delta$ by L satisfying (23). In this case, we replace the integral condition for the coefficients, eg. (18) by

$$\int_0^{\infty} \{c^*(r) + \Phi_{ij}^*(r)\}dr < \infty,$$

where c^* and Φ_{ij}^* as in Example 1.

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