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Comparison Principle and Convexity Preserving Properties for Singular Degenerate Parabolic Equations on Unbounded Domains

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Introduction. We prove comparison theorem for viscosity solutions of singular degenerate parabolic equations of general form in a domain not necessarily bounded. We concider the degenerate parabolic equations of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega \tag{0.1}$$

or more general equations

$$u_t + F(t, x, u, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega, \tag{0.2}$$

where Ω is a domain in $\mathbf{R}^n$ and $T > 0$. The equations are allowed to be singular in the sense that F has a singulality at $\nabla u = 0$. The unknown u will always be a real valued function on Q; $\partial_t u, \nabla u$ and $\nabla^2 u$ denote respectively the time derivative of u,the gradient of u and the Hessian of u in space variables. We also prove that the concavity of solutions is preserved as time develop under addional assumptions. Both results are applied to various equations including the mean curvature flow equation where every level set of solutions is moved by its man curvature.

§1. Comparison principle. Let Ω be a domain in $\mathbf{R}^n$ not necessarily bounded and let T be a positive number. We consider a degenerate parabolic equation of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega. \tag{1.1}$$

We first list assumptions on $F = F(p, X)$.

(F1) $F : (\mathbf{R}^n \backslash \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is continuous, where $\mathbf{S}^n$ denotes the space of real $n \times n$ symmetric matrices.

(F2) F is degenerate elliptic, i.e., $F(p, X + Y) \leq F(p, X)$ for all $Y \geq 0$.

(F3) $-\infty < F_*(0, O) = F^*(0, O) < \infty$ where F_* and F^* are the lower and upper semicontinuous relaxation (envelope) of F on $\mathbf{R}^n \times \mathbf{S}^n$, respectively, i.e.,

$$F_*(p, X) = \lim_{\varepsilon \downarrow 0} \inf\{F(q, Y); q \neq 0, |p - q| \leq \varepsilon, |X - Y| \leq \varepsilon\}$$

and $F^* = -(-F)_*$. Here $|X|$ denotes the operator norm of X as a self adjoint operator on $\mathbf{R}^n$.

(F4) For every $R > 0$

$$c_R = \sup\{|F(p, X)|; |p| \leq R, |X| \leq R, p \neq 0\} \quad \text{is finite.}$$

The assumption (F1) allows the possibility that (1.1) is singular at $\nabla u = 0$. The equation (1.1) is called degenerate parabolic if (F2) holds.

We next recall one of equivalent definitions of viscosity sub-and supersolutions of (1.1) (cf. [19]). A function $u : Q \to \mathbf{R}$ is called a *viscosity sub-(super) solution* of (1.1) in Q if $u^* < \infty$ (resp. $u_* > -\infty$) on $\bar{Q}$ and

$$\begin{aligned} &\tau + F_*(p, X) \leq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}_Q^{2,+} u^*(t, x), (t, x) \in Q \\ &(\text{resp.} \quad \tau + F^*(p, X) \geq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}_Q^{2,-} u_*(t, x), (t, x) \in Q). \end{aligned}$$

Here $\mathcal{P}_Q^{2,+}$ denotes the *parabolic super 2-jet* in Q, i.e., $\mathcal{P}_Q^{2,+} u(t, x)$ is the set of $(\tau, p, X) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n$ such that

$$\begin{aligned} u(s, y) \leq & u(t, x) + \tau(s - t) + \langle p, y - x \rangle + \frac{1}{2}\langle X(y - x), y - x \rangle \\ & + o(|s - t| + |y - x|^2) \quad \text{as} \quad (s, y) \to (t, x) \quad \text{in} \quad Q \end{aligned}$$

where $\langle , \rangle$ denotes the Euclidean inner product; similarly, $\mathcal{P}_Q^{2,-} u = -\mathcal{P}_Q^{2,+}(-u)$. In this paper we call a continuous function $m : [0, \infty) \to [0, \infty)$ a *modulus* if $m(0) = 0$ and it is nondecreasing. For $U = (0,T] \times D$, the set

$$\partial_p U = \{0\} \times D \cup [0,T] \times \partial D$$

is often called the *parabolic boundary* of U. We are now in position to state our main comparison theorem.

Theorem 1.1. *Suppose that F satisfies* (F1)-(F4). *Let u and v be, respectively, sub-and supersolutions of* (1.1) *in Q. Assume that*

(A1) $u(t,x) \leq K(|x|+1)$, $v(t,x) \geq -K(|x|+1)$ *for some $K > 0$ independent of $(t,x) \in Q$;*

(A2) *there is a modulus m_T such that*

$$u^*(t,x) - v_*(t,y) \leq m_T(|x-y|) \quad \text{for all} \quad (t,x,y) \in \partial_p U,$$

where $U = (0,T] \times D$ and $D = \Omega \times \Omega$;

(A3) $u^*(t,x) - v_*(t,y) \leq K(|x-y|+1)$ *on $\partial_p U$ for some $K > 0$ independent of $(t,x,y) \in \partial_p U$.*

Then there is a modulus m such that

$$u^*(t,x) - v_*(t,y) \leq m(|x-y|) \quad \text{on} \quad U. \tag{1.2}$$

In particular $u^* \leq v_*$ on Q.

We will prove Theorem 1.1 in several steps.

We begin by deriving a rough growth estimate for $u(t,x) - v(t,y)$ on U.

Proposition 1.2 *Suppose that F satisfies* (F1) *and* (F4). *Let u and v be, respectively, viscosity sub-and supersolutions of* (1.1) *in Q. Assume that u and v satisfy* (A1) *and* (A3) *and that u and $-v$ are upper semicontinuous in Q. Then for $K' > K$ there is a constant $M = M(K', F) > 0$ such that*

$$u(t,x) - v(t,y) \leq K'|x-y| + M(1+t) \quad \text{on} \quad U. \tag{1.3}$$

Proof. We set

$$w(s,t,x,y) = u(t,x) - v(s,y)$$
$$\psi(t,x,y) = K'(|x-y|^2+1)^{1/2} + M(1+t).$$

We will prove

$$w(t,t,x,y) \le \psi(t,x,y) \quad \text{for} \quad (t,x,y) \in U \tag{1.4}$$

by choosing M large. Let $\{g_R\}_{R>0}$ be a family of C^2 functions satisfying

(1.5a) $\quad g_R(x) = 0 \quad \text{for} \quad |x| < R$

(1.5b) $\quad g_R(x)/|x| \to 1 \quad \text{as} \quad |x| \to \infty$

(1.5c) $\quad G = \sup\{|\nabla g_R(x)| + |\nabla^2 g_R(x)|; \quad x \in \mathbf{R}^n, R > 0\}$ is finite.

Using this barrier g_R, we set $\phi = \psi + 2K' g_R$. By (A1) and (1.5b) we observe that for sufficiently large R_1 it holds

$$w(s,t,x,y) - \phi(t,x,y) < 0 \quad \text{if} \quad |x|^2 + |y|^2 \ge R_1^2 \quad \text{and} \quad 0 \le t,s \le T. \tag{1.6}$$

By (A3) if $M > K$, we see

$$w(t,t,x,y) - \phi(t,x,y) < 0 \quad \text{for} \quad (t,x,y) \in \partial_p U. \tag{1.7}$$

Since w is upper semicontinuous, (1.6) and (1.7) yield

$$w(s,t,x,y) - \frac{(t-s)^2}{\delta} - \phi(t,x,y) < 0 \quad \text{for} \quad (s,x,y) \in \partial_p U \quad \text{or} \quad (t,x,y) \in \partial_p U \tag{1.8}$$

with sufficiently small δ (independent of t, s, x, y). Suppose that (1.4) were false. Then from (1.5a) it would follow that

$$\sup_{V}(w - \Psi) > 0 \tag{1.9}$$

with $\Psi = (t-s)^2/\delta + \phi$ and $V = (0,T] \times U$ if R is sufficiently large. By (1.6)-(1.9) we now observe that $w - \Psi$ attains a maximum over $\bar{V}$ at a point $(\hat{s}, \hat{t}, \hat{x}, \hat{y}) \in V$. This implies that

$$(\partial_t \Psi, \nabla_x \Psi, \nabla_x^2 \Psi)(\hat{s}, \hat{t}, \hat{x}, \hat{y}) \in \mathcal{P}_Q^{2,+} u(\hat{t}, \hat{x})$$
$$(-\partial_s \Psi, -\nabla_y \Psi, -\nabla_y^2 \Psi)(\hat{s}, \hat{t}, \hat{x}, \hat{y}) \in \mathcal{P}_Q^{2,-} v(\hat{s}, \hat{y}),$$

where ∇_x denotes spatial derivatives in x variables. Since u and v are, respectively, viscosity, sub-and supersolutions of (1.1), we see

$$\partial_t \Psi + F_*(\nabla_x \phi, \nabla_x^2 \phi) \le 0, \tag{1.10a}$$

$$-\partial_s \Psi + F^*(-\nabla_y \phi, -\nabla_y^2 \phi) \ge 0 \quad \text{at} \quad (\hat{s}, \hat{t}, \hat{x}, \hat{y}). \tag{1.10b}$$

By (1.5c) and definition of ψ it holds

$$|\nabla \phi|, \quad |\nabla^2 \phi| \le N, \quad \nabla = (\nabla_x, \nabla_y)$$

with $N = N(K', G)$. Subtracting (1.10b) from (1.10a) and noting (F4) yield

$$\partial_t \Psi + \partial_s \Psi \le 2c_N.$$

Since $\partial_t (t-s)^2 = -\partial_s (t-s)^2$, this implies $M \le 2c_N$. If M is taken larger than $2c_N$ and K, we have a contradiction. We thus prove (1.4) for

$$M > \max(2c_N, K).$$

The estimate (1.3), with M replaced by $M + K'$, follows from (1.4). ∎

For ε, δ, $\gamma > 0$ we set

$$\begin{aligned} \Phi(t,x,y) &= w(t,x,y) - \Psi(t,x,y), \quad w(t,x,y) = u(t,x) - v(t,y), \\ \Psi(t,x,y) &= \frac{|x-y|^4}{4\varepsilon} + B(t,x,y), \quad B(t,x,y) = \delta(|x|^2 + |y|^2) + \frac{\gamma}{T-t}. \end{aligned} \tag{1.11}$$

The function B plays the role of a barrier for space infinity and $t = T$.

Proposition 1.3. *Suppose that u and v satisfy (1.3) and that*

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t,x,y); |x-y| < \theta, (t,x,y) \in \bar{U}\} > 0. \tag{1.12}$$

Then there are positive constants δ_0 and γ_0 such that

$$\sup_{\bar{U}} \Phi(t,x,y) > \frac{\alpha}{2} \tag{1.13}$$

holds for all $0<\delta<\delta_0$, $0<\gamma<\gamma_0$, $\varepsilon>0$.

Proposition 1.4. *Let u, v, δ_0, γ_0 be as in Proposition 1.3. Suppose that w is upper semicontinuous in $\bar{U}$.*

(i) *Φ attains a maximum over $\bar{U}$ at $(\hat{t},\hat{x},\hat{y}) \in \bar{U}$ with $\hat{t}<T$.*

(ii) *$|\hat{x}-\hat{y}|$ is bounded as a function of $0<\varepsilon<1$, $0<\delta<\delta_0$, $0<\gamma<\gamma_0$.*

(iii) *$\delta\hat{x}$ and $\delta\hat{y}$ tend to zero as $\delta \to 0$; the convergence is uniform in $0<\varepsilon<1$ and $0<\gamma<\gamma_0$. In particular, for fixed $\delta>0$, $\hat{x}$ and $\hat{y}$ are bounded on $0<\varepsilon<1$, $0<\gamma<\gamma_0$.*

(iv) *$|\hat{x}-\hat{y}|$ tends to zero as $\varepsilon \to 0$; the convergence is uniform in $0<\delta<\delta_0$ and $0<\gamma<\gamma_0$.*

Proposition 1.5. *Assume the hypotheses of Proposition 1.4. Suppose that (A2) holds for u and v. Then there is $\varepsilon_0>0$ such that Φ attains a maximum over $\bar{U}$ at an interior point $(\hat{t},\hat{x},\hat{y})$ of U, i.e,. $(\hat{t},\hat{x},\hat{y}) \in (0,T)\times\Omega\times\Omega$ for all $0<\varepsilon<\varepsilon_0$, $0<\delta<\delta_0$ and $0<\gamma<\gamma_0$.*

Lemma 1.6 ([3]). *Let u_i be an upper semicontinuous function with $u_i<\infty$ in $(0,T)\times\mathbf{R}^{N_i}$ for $i=1,2,\cdots,k$. Let w be a function in $(0,T)\times\mathbf{R}^N$ given by*

$$w(t,x) = u_1(t,x_1)+\cdots+u_k(t,x_k) \quad \text{for} \quad x=(x_1,\cdots,x_k)\in\mathbf{R}^N,$$

where $N=N_1+\cdots+N_k$. For $s\in(0,T)$, $z\in\mathbf{R}^N$ suppose that

$$(\tau,p,A)\in\mathcal{P}^{2,+}w(s,z)\subset\mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^N.$$

Assume that there is an $\omega>0$ such that for every $M>0$

$$\begin{aligned} &\sigma_i \le C \quad \textit{whenever} \quad (\sigma_i,q_i,Y_i)\in\mathcal{P}^{2,+}u(t,x_i), \\ &|x_i-z_i|+|s-t|<\omega \quad \textit{and} \quad |u_i(t,x_i)|+|q_i|+|Y_i|\le M \quad (i=1,\cdots,k), \end{aligned} \tag{1.14}$$

with some $C = C(M)$. Then for each $\lambda > 0$ there exists $(\tau_i, X_i) \in \mathbf{R} \times \mathbf{S}^{N_i}$ such that

$$(\tau_i, p_i, X_i) \in \bar{\mathcal{P}}^{2,+} u_i(s, z_i) \quad \textit{for} \quad i = 1, \cdots k$$

and

$$-\left(\frac{1}{\lambda} + |A|\right) I \leq \begin{pmatrix} X_1 & \cdots & O \\ \vdots & & \vdots \\ O & \cdots & X_k \end{pmatrix} \leq A + \lambda A^2 \quad \textit{and} \quad \tau_1 + \cdots + \tau_k = \tau,$$

where I denotes the identity matrix and $p = (p_1, \cdots, p_k)$.

Remark 1.7. This lemma is Theorem 6 in [3]. Here and hereafter the subscript of $\mathcal{P}^{2,+}$ is suppressed. The bar over $\mathcal{P}^{2,+}$ means the closure. Although the domain considered here is $\mathbf{R}^{N_i}$, it is easily seen that the result is local and that $\mathbf{R}^{N_i}$ may be replaced by a neighborhood of $z_i \in \mathbf{R}^{N_i}$.

Proof of Theorem 1.1. We may assume that u and v are, respectively, upper and lower semicontinuous so that

$$w(t, x, y) = u(t, x) - v(t, y)$$

is upper semicontinuous in $\bar{U}$. Suppose that (1.2) were false. Then we would have (1.12), i.e.,

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t, x, y); |x - y| < \theta, \quad (t, x, y) \in \bar{U}\} > 0.$$

By Proposition 1.2 and (1.12) we see all conclusions in Propositions 1.3-1.5 would hold for Φ defined in (1.11). Proposition 1.5 says that Φ attains a maximum over $\bar{U}$ at $(\hat{t}, \hat{x}, \hat{y}) \in (0, T) \times \Omega \times \Omega$ for small ε, δ, γ. In particular

$$w(t, x, y) \leq w(\hat{t}, \hat{x}, \hat{y}) + \Psi(t, x, y) - \Psi(\hat{t}, \hat{x}, \hat{y}) \quad \text{in} \quad U.$$

Expanding Ψ at $(\hat{t}, \hat{x}, \hat{y})$ yields

$$(1.15) \qquad (\hat{\Psi}_t, \hat{\Psi}_{x,y}, A)(\hat{t}, \hat{x}, \hat{y}) \in \mathcal{P}^{2,+} w(\hat{t}, \hat{x}, \hat{y}) \quad \text{with} \quad \nabla^2 \Psi(\hat{t}, \hat{x}, \hat{y}) \leq A$$

where $\hat{\Psi}_t = \partial_t \Psi(\hat{t}, \hat{x}, \hat{y})$, $\hat{\Psi}_{x,y} = \nabla \Psi(\hat{t}, \hat{x}, \hat{y})$ and $\nabla = (\nabla_x, \nabla_y)$.

We will apply Lemma 1.6 with $k = 2$, $u_1 = u$, $u_2 = -v$, $s = \hat{t}$, $z = (\hat{x}, \hat{y})$. Since u and v are, respectively, sub-and supersolution of (1.1) with F satisfying (F4), we easily see the assumption (1.14) holds. Since $(\hat{t}, \hat{x}, \hat{y})$ is an interior point of U, by Remark 1.7 we now apply Lemma 1.6 and conclude that for each $\lambda > 0$ there are (τ_1, X) and $(\tau_2, Y) \in \mathbf{R} \times \mathbf{S}^n$ such that

$$(1.16) \qquad (\tau_1, \hat{\Psi}_x, X) \in \bar{\mathcal{P}}^{2,+} u(\hat{t}, \hat{x}), \quad (-\tau_2, -\hat{\Psi}_y, -Y) \in \bar{\mathcal{P}}^{2,-} v(\hat{t}, \hat{y}), \quad \hat{\Psi}_t = \tau_1 + \tau_2$$

$$(1.17) \qquad -\left(\frac{1}{\lambda} + |A|\right) I \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le A + \lambda A^2,$$

where $\hat{\Psi}_t = \partial_t \Psi(\hat{t}, \hat{x}, \hat{y})$, $\hat{\Psi}_x = \nabla_x \Psi(\hat{t}, \hat{x}, \hat{y})$, etc. Since u and v are, respectively, sub-and supersolution of (1.1) it follows from (1.16) that

$$\tau_1 + F_*(\hat{\Psi}_x, X) \le 0, \quad -\tau_2 + F^*(-\hat{\Psi}_y, -Y) \ge 0,$$

which yields

$$(1.18) \qquad 0 \ge \hat{\Psi}_t + F_*(\hat{\Psi}_x, X) - F^*(-\hat{\Psi}_y, -Y).$$

We next take a special A. Differentiating Ψ in (1.11) yields

$$(1.19) \qquad \hat{\Psi}_x = |\eta|^2 \eta / \varepsilon + 2\delta \hat{x}, \quad \hat{\Psi}_y = -|\eta|^2 \eta / \varepsilon + 2\delta \hat{y}, \qquad (\eta = \hat{x} - \hat{y})$$

$$\begin{aligned} \begin{pmatrix} \hat{\Psi}_{xx} & \hat{\Psi}_{xy} \\ \hat{\Psi}_{yx} & \hat{\Psi}_{yy} \end{pmatrix} &= \frac{1}{\varepsilon}(|\eta|^2 + 2\eta \otimes \eta) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\delta \begin{pmatrix} I & O \\ O & I \end{pmatrix} \\ &\le \frac{3}{\varepsilon}|\eta|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\delta \begin{pmatrix} I & O \\ O & I \end{pmatrix} = A. \end{aligned}$$

With this A the estimate (1.17) becomes

$$(1.20) \qquad -\mu \begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le \nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega \begin{pmatrix} I & O \\ O & I \end{pmatrix},$$

$$\mu = \lambda^{-1} + 6|\eta|^2/\varepsilon + 2\delta, \quad \nu = (18|\eta|^2\lambda + 3\varepsilon + 12\delta\varepsilon\lambda)|\eta|^2/\varepsilon^2,$$

$$\omega = 4\delta^2\lambda + 2\delta.$$

We will study (1.18). We take $\lambda = 1$ in (1.20) and fix ε, γ such that $0 < \varepsilon < \varepsilon_0$, $0 < \gamma < \gamma_0$, where ε_0 and γ_0 are as in Propositions 1.5 and 1.3. We let $\delta \to 0$ in (1.18). We divide the situation in two cases depending on the behavior of $\eta = \hat{x} - \hat{y}$ as $\delta \to 0$.

Case 1. $\eta = \hat{x} - \hat{y} \to 0$ *as* $\delta \to 0$. From (1.20) it follows that

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$
$$\leq \theta \begin{pmatrix} I & O \\ O & I \end{pmatrix} \quad \text{with} \quad \theta = 2\nu + \omega.$$

This implies $X \leq \theta I$ and $-Y \geq -\theta I$. By the degenerate ellipticity (F2) we have

$$F_*(\hat{\Psi}_x, X) \geq F_*(\hat{\Psi}_x, \theta I), \quad F^*(-\hat{\Psi}_y, -Y) \leq F^*(-\hat{\Psi}_y, -\theta I), \tag{1.21}$$

where $\hat{\Psi}_x$, $\hat{\Psi}_y$ is defined by (1.19). If $\delta \to 0$, we see $\hat{\Psi}_x$ and $\hat{\Psi}_y$ converge to zero since $\eta \to 0$ and $\delta\hat{x}$, $\delta\hat{y} \to 0$ by Proposition 1.4. Letting $\delta \to 0$ in (1.21) yields

$$\varliminf_{\delta \to 0} F_*(\hat{\Psi}_x, X) \geq F_*(0, O), \quad \varlimsup_{\delta \to 0} F^*(-\hat{\Psi}_y, -Y) \leq F^*(0, O)$$

since $\theta \to 0$. Applying this estimate to (1.18) and noting that

$$\Psi_t = \gamma(T-t)^{-2} \geq \gamma T^{-2},$$

we obtain

$$0 \geq \gamma T^{-2} + F_*(0, O) - F^*(0, O).$$

By (F3) this yields $0 \geq \gamma T^{-2}$, which contradicts $\gamma > 0$.

Case 2. $\hat{x} - \hat{y} \to a \neq 0$ *for some subsequence* $\delta_j \to 0$. Since the singularity of F is not important in this case our argument is essentially the same as in [10]. From (1.20) it follows that

$$\langle Xp, p \rangle + \langle Yq, q \rangle \leq \nu(|p|^2 + |q|^2) - 2\nu\langle p, q \rangle + \omega(|p|^2 + |q|^2).$$

Taking $p = q$ yields

$$X + Y \leq 2\omega I.$$

By (F2) we see

$$F^*(-\hat{\Psi}_y, -Y) \leq F^*(-\hat{\Psi}_y, X - 2\omega I) \tag{1.22}$$

Since X and Y are bounded as $\delta \to 0$ by (1.20) there are a subsequence $X_j = X(\delta_j)$ and $\bar{X} \in \mathbf{S}^n$ such that $X_j \to \bar{X}$ as $\delta_j \to 0$ (see e.g. [10, Lemma 5.3]). Applying (1.22) to (1.18) and letting $\delta_j \to 0$ now yield

$$0 \geq \gamma T^{-2} + F_*(|a|^2 a/\varepsilon, \bar{X}) - F^*(|a|^2 a/\varepsilon, \bar{X}).$$

Since F is continuous at $(|a|^2 a/\varepsilon, \bar{X})$ for $a \neq 0$, this again contradicts $\gamma > 0$. We thus prove (1.2). ∎

Remark 1.8. The assumption (F4) in Theorem 1.1 is unnecessary if we assume that u and v satisfy the rough growth estimate (1.3). In particular, if u and v are bounded (F4) is unnecessary. Indeed, other than in Proposition 1.2 we use (F4) only to prove (1.14) in Lemma 1.6 so that we derive (1.16)-(1.18). However to carry out the proof of Theorem 1.1 we only need (1.17) and (1.18). Without showing (1.16) one can circumvent (F4) to derive (1.17) and (1.18) by applying the following lemma, which can be proved similarly as Lemma 1.6.

§2. Convexity preserving. We consider the Cauchy problem

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \mathbf{R}^n \tag{2.1}$$

$$u(0,x) = u_0(x). \tag{2.2}$$

We will show that the concavity of u in x is preserved as time develops provided that $F(p,X)$ is convex in X and that u grows at most linearly near space infinity. For this purpose we apply Lemma 1.6 to

$$w(t,\xi) = u(t,x) + u(t,y) - 2u(t,z), \quad \xi = (x,y,z).$$

and conclude that

$$w(t,\xi) \le L|x+y-2z|$$

with some constant L. Similar technique is found in [11], where it is applied to the semi-concavity of solutions of Bellman equations.

Theorem 2.1. *Suppose that F satisfies* (F1)-(F4) *and*

$$\text{(F5)} \qquad X \mapsto F(p,X) \text{ is convex on } \mathbf{S}^n \text{ for all } p \in \mathbf{R}^n\setminus\{0\}.$$

Let u be a viscosity solution of (2.1) *with* (2.2). *Assume that u is continuous in* $[0,T]\times\mathbf{R}^n$ *and that*

$$\text{(2.3)} \qquad |u(t,x)| \le K(|x|+1) \quad \text{with} \quad K \quad \text{independent of} \quad (t,x) \in Q.$$

If the initial data u_0 is concave and globally Lipschitz with constant L, then it holds

$$\text{(2.4)} \qquad u(t,x)+u(t,y)-2u(t,z) \le L|x+y-2z|, \quad x,y,z \in \mathbf{R}^n, \quad 0 \le t \le T.$$

In particular $x \mapsto u(t,x)$ is concave for all $t \in [0,T]$.

We will state lemma and proposition to prove Theorem 2.1.

Lemma 2.2. *Suppose that u_0 is concave and globally Lipschitz with constant L in $\mathbf{R}^n$. Then it holds*

$$\text{(2.5)} \qquad u_0(x)+u_0(y)-2u_0(z) \le L|x+y-2z| \quad \text{for all} \quad x,y,z \in \mathbf{R}^n.$$

Proof. Since u_0 is concave, it follows that

$$\begin{aligned} &u_0(x)+u_0(y)-2u_0(z) \\ =&u_0(x)+u_0(y)-2u_0((x+y)/2)+2(u_0((x+y)/2)-u_0(z)) \\ \le&2(u_0((x+y)/2)-u_0(z)). \end{aligned}$$

The right hand side is dominated by $2L|(x+y)/2-z|$ so (2.5) follows. ∎

Proposition 2.3. *Suppose that F satisfies* (F1) *and* (F4). *Assume that the hypotheses of Theorem 2.1 concerning u hold. Then for $K' > L$ there is a constant $M = M(K', F) > 0$ such that*

$$u(t,x)+u(t,y)-2u(t,z) \le K'|x+y-2z| + M(1+t) \tag{2.6}$$

for all $\xi = (x,y,z) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n, \quad 0 \le t \le T$.

§3 General comparison theorem This section extends the comparison priceple in §1 to a general equation of form

$$u_t + F(t,x,u,\nabla u,\nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega, \tag{3.1}$$

where $T > 0$ and Ω is a domain in $\mathbf{R}^n$. Our approach is basically the same as in §1. However, since F depends on x, we are forced to let $\varepsilon \to 0$ in our test function Ψ of (2.11) at the end of the proof. The crucial step is to establish that $|\hat{x}-\hat{y}|^4/4\varepsilon$ converges to zero as $\varepsilon \to 0$ after we let $\delta \to 0$, $\gamma \to 0$.

We consider F satisfying

(F1) $F : J_0 = Q \times \mathbf{R} \times (\mathbf{R}^n \backslash \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is continuous.

We continue to assume (F2) and (F3) i.e.,

(F2) F is degenerate elliptic,i.e., $F(t,x,r,p,X+Y) \le F(t,x,r,p,X)$ in J_0 if $Y \ge 0$.

(F3) $-\infty < F_*(t,x,r,0,O) = F^*(t,x,r,0,O) < \infty$ for all $(t,x,r) \in Q \times \mathbf{R}$.

For boundedness of F we also impose uniformity in t, x and r.

(F4) For every $R > 0$, $c_R = \sup\{|F(t,x,r,p,X)|; |p|,|X| \le R, (t,x,r,p,X) \in J_0\} < \infty$;

this, of course, is the same as (F4) in §1 when F is independent of t, x and r. We assume a kind of monotonicity in r.

(F5) For every $H > 0$, there is a constant $c_0 = c_0(n,T,H)$ such that $r \mapsto F(t,x,r,p,X) + c_0 r$ is nondecreasing for all $(t,x,r,p,X) \in J_0$ with $|r| \le H$.

Outside singularities we assume uniform continuity in (p, X).

(F6) For every $R > \rho > 0$ there is a modulus $\sigma = \sigma_{R\rho}$ such that

$$|F(t,x,r,p,X) - F(t,x,r,q,Y)| \leq \sigma_{R\rho}(|p-q| + |X-Y|)$$

for all $(t,x,r) \in Q \times \mathbf{R}$, $\rho \leq |p|$, $|q| \leq R$, $|X|$, $|Y| \leq R$.

The behavior near $(p, X) = (0, O)$ is assumed to be uniform in t, x and r.

(F7) There are $\rho_0 > 0$ and a modulus σ_1 such that

$$F^*(t,x,r,p,X) - F^*(t,x,r,0,O) \leq \sigma_1(|p| + |X|)$$
$$F_*(t,x,r,p,X) - F_*(t,x,r,0,O) \geq -\sigma_1(|p| + |X|)$$

provided that $(t,x,r) \in Q \times \mathbf{R}$ and $|p|$, $|X| \leq \rho_0$.

We further assume some equicontinuity in x.

(F8) There is a modulus σ_2 such that

$$|F(t,x,r,p,X) - F(t,y,r,p,X)| \leq \sigma_2(|x-y|(|p|+1))$$

for $y \in \Omega$, $(t,x,r,p,X) \in J_0$.

Theorem 3.1. *Suppose that F satisfies* (F1)-(F8). *Let u and v be, respectively, sub-and supersolutions of* (3.1) *in Q. Assume that* (A1)-(A3) *holds for u and v. Then there is a modulus m such that*

(3.2) $$u^*(t,x) - v_*(t,y) \leq m(|x-y|) \quad \textit{on} \quad U.$$

The assumption (F8) has a disadvantage because it excludes variable coefficients in second order terms, even if the equation is linear. We will prove (3.2) under weaker assumptions.

(F6') For every $R > \rho > 0$ there is a modulus $\sigma = \sigma_{R\rho}$ such that

$$|F(t,x,r,p,X) - F(t,x,r,q,X)| \leq \sigma_{R\rho}(|p-q|)$$

for all $(t,x,r,p,X) \in J_0$, $\rho \leq |p|$, $|q| \leq R$, $|X| \leq R$.

(F9) There is a modulus σ_2 such that

$$F_*(t,x,r,0,O) - F^*(t,y,r,0,O) \geq -\sigma_2(|x-y|)$$

for all $(t,x,r) \in Q \times \mathbf{R}$, $y \in \Omega$.

(F10) Suppose that

$$(3.3) \qquad -\mu \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

with μ, ν, $\omega \geq 0$. Let R be taken so that $R \geq \max(\mu, \theta) + 2\omega$ with $\theta = 2\nu + \omega$. Let ρ be a positive number. Then it holds

$$\begin{aligned} &F_*(t,x,r,p,X) - F^*(t,y,r,p,-Y) \\ &\geq -\bar{\sigma}(|x-y|(|p|+1) + \nu|x-y|^2) - \bar{\sigma}(2\omega) \quad \text{for} \quad \rho \leq |p| \leq R. \end{aligned}$$

with some modulus $\bar{\sigma} = \bar{\sigma}_{R\rho}$ independent of t, x, y, r, X, Y, μ, ν, ω.

Theorem 3.2. *Suppose that F satisfies* (F1), (F3)-(F5), (F6'), (F7), (F9), (F10). *Let u and v be respectively, sub-and supersolutions of* (3.1) *in Q. Assume that* (A1)-(A3) *holds for u and v. Then there is a modulus m such that* (3.2) *holds.* The following proposition shows that Theorem 3.1 is the special case of Theorem 3.2.

Proposition 3.3. (i) *The assumptions* (F3) *and* (F8) *imply* (F9).
(ii) *Assumptions* (F2), (F6), (F8) *imply* (F10).

Proof. (i) We suppress t and r to simplify notations. By (F8) we observe

$$\varliminf_{\substack{p \to 0 \\ X \to O}} (F(x,p,X) - F(y,p,X)) \geq -\sigma_2(|x-y|).$$

The left hand side is dominated from above by

$$\lim_{\varepsilon\to 0}\left(\inf_{|p|+|X|\le\varepsilon} F(x,p,X) - \inf_{|p|+|X|\le\varepsilon} F(y,p,X)\right) = F_*(x,0,O) - F_*(y,0,O).$$

The condition (F3) now yields (F9).

(ii) As is observed in Case 2 of the proof of Theorem 1.1, (3.3) yields $X + Y \le 2\omega I$. From (F2) it follows that

$$\begin{aligned} &F(x,p,X) - F(y,p,-Y) \\ \ge &F(x,p,X) - F(y,p,X-2\omega I) \\ \ge &-\bar{\sigma}_{R\rho}(2\omega I) + F(x,p,X) - F(y,p,X) \quad \text{for} \quad \rho \le |p| \le R \quad \text{by(F6)} \end{aligned}$$

since (3.3) yields $|X|$, $|Y| \le \max(\mu,\theta)$ so that $|X|$, $|X - 2\omega I| \le R$. From (F8) it now follows (F10). ∎

Proposition 1.2'. *Suppose that F satisfies* (F1) *and* (F4). *Let u and v be, respectively, viscosity sub-and supersolutions of* (3.1) *in Q and that u and $-v$ are upper semicontinuous in Q. Then for $K' > K$ there is a constant $M = M(K', F) > 0$ such that* (1.3) *holds.*

We now recall Φ and Ψ of (1.11) and let $(\hat{t},\hat{x},\hat{y})$ be a point attaining a maximum of Φ over $\bar{U}$ defined in Propositions 1.4 and 1.5. To carry out the proof of Theorem 3.2 we need to study $|\hat{x}-\hat{y}|^4/\varepsilon$ as $\varepsilon \to 0$.

Proposition 3.4. *Suppose that u and v satisfies* (1.2) *and that* (1.12) *holds. Let $(\hat{t},\hat{x},\hat{y})$ be as in Proposition* 1.4. *It holds*

$$\lim_{\varepsilon\downarrow 0}\ \overline{\lim_{\delta,\gamma\downarrow 0}}\ \frac{|\hat{x}-\hat{y}|^4}{\varepsilon} = 0. \tag{3.4}$$

Remark 3.5. When Ω is bounded, (F6), (F6'), (F7) and (A1), (A3) are unnecessary, because we may assume that u and v are bounded; (A2) may be replaced by $u^* \le v_*$ on $\partial_p Q$. Moreover, we may take $\delta = 0$ in the definition of Φ in (1.11). If δ is taken as zero, we may take $\omega = 0$ in (F10). Since Theorems 3.1 and 3.2 are new for F depending on x even if Ω is bounded, we restate them for bounded Ω.

Theorem 3.6. *Let Ω be a bounded domain in $\mathbf{R}^n$. Suppose that F satisfies* (F1)-(F3), (F5), (F8) *or* (F1), (F3), (F5), (F9), (F10) *with $\omega = 0$. Let u and v be, respectively, sub-and supersolutions of* (3.1) *in Q. Assume that $u^* \leq v_*$ on $\partial_p Q$. Then $u^* \leq v_*$ on Q.*

Remark 3.7. By Theorem 3.6 all results in [1, §6, §7] extend to F depending on x. We state one of typical results on global existence of solutions.

Theorem 3.8. *Let $\Omega = \mathbf{R}^n$ and $\beta \in \mathbf{R}$. Assume the hypotheses of Theorem* 3.6 *concerning F. Suppose that F is geometric, i.e., F is independent of r and*

$$F(t, x, \lambda p, \lambda X + \sigma p \otimes p) = \lambda F(t, x, p, X)$$

for all $\lambda > 0$, $\sigma \in \mathbf{R}$, $(t, x) \in Q$, $(p, X) \in (\mathbf{R}^n \backslash \{0\}) \times \mathbf{S}^n$ and that

$$F_*(t, x, p, -I) \leq c(|p|), \quad F^*(t, x, p, I) \geq -c(|p|)$$

for some $c(q) \in C^1[0, \infty)$ and $c(q) \geq c_0 > 0$ with some constant c_0. Then for $a \in C_\beta(\mathbf{R}^n)$ there is a unique viscosity solution $u_a \in C_\beta([0, T] \times \mathbf{R}^n)$ of (3.1) *with $u_a(0, x) = a(x)$.Here $C_\beta(K)$ denotes the space of continuous function u such that $u - \beta$ is compactly supported in K.*

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