数理解析研究所講究録
第 757 巻 1991 年 175-182



# Morse Inequalities for R-constructible Sheaves

by

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This note aims at giving a generalization of classical Morse inequalities for Betti numbers of compact manifolds. In this paper, we deal with cohomologies groups with coefficients in **R**-constructible pure sheaves instead and encounter the tight relation between Morse theory and Microlocal Analysis of Sheaves. See Hellfer-Sjöstrand[H-Sj1,2] for another approach to the theory via microlocal analysis and also Goresky-MacPherson[G-McP] who introduced the "stratified Morse theory". The authors were attracted to this problem through understanding the beuatiful papers[K1,2] due to M. Kashiwara. In fact all ideas can be traced to the papers above. But the authors consider it worthy to write it down explicitly to attract many people to the microlocal point of view, which is now found not only in the classical microlocal analysis of partial differential equations.

## 1. Statement of the Main Theorem

Let $X$ be a real analytic manifold, $k$ a commutative field of characteristic 0, and let $D^b_{\mathbf{R}-c}(X)$ denote the derived category of the category of sheaves of $k$ vector spaces on $X$ with **R**-constructible cohomologies. (cf. [K3] )

Let $F \in ob(D^b_{\mathbf{R}-c}(X))$. Then we denote by $SS(F)$ its microsupport, which is a $\mathbf{R}_+ - conic$ closed subset in $T^*X$. Refer to [KS] for all about $SS(F)$. Since we assume that $F$ is **R**-constructible , $SS(F)$ is a Lagrangean subvariety in $T^*X$. We set

$$(1) \qquad \Lambda = SS(F).$$

Moreover let $\phi : X \longrightarrow \mathbf{R}$ be a real valued $C^2$ function on $X$, and put

$$(2) \qquad \Lambda_\phi = \{(x, d\phi(x)) \in T^*X;\ x \in X\}.$$

We suppose :

$$(3) \qquad \{x \in supp(F); \phi(x) \leq t\} \text{ is compact for any } t \in \mathbf{R},$$

$$(4) \qquad \Lambda_\phi \cap \Lambda = \Lambda_\phi \cap \Lambda_{reg} = \{p_1, \cdots, p_N\},$$

(5) $\Lambda_\phi$ and $\Lambda_{reg}$ intersect transversally at each point $p_i$ ,

(6)

$F$ is pure at each $p_i$ with multiplicity $m_i$ and shift $d_i$ along $\Lambda$ in the sense of Ch. 7 of [KS].▮

Recall that (6) is equivalent to

$$\mathbf{R}\Gamma_{\{\phi(x)\geq\phi(x_i)\}}(F)_{x_i} = k^{m_i}[\delta^i] \tag{7}$$

where $x_i = \pi(p_i), \pi : T^*X \longrightarrow X$ is a natural projection and

$$\delta^i = d_i - \frac{1}{2}dim\ X - \frac{1}{2}\tau(\lambda_0(p_i), \lambda_\Lambda(p_i), \lambda_\phi(p_i)). \tag{8}$$

See chapter 7 of [KS] for the definition of Maslov index $\tau(\cdot,\cdot,\cdot)$.

Let $Mod^f(k)$ denote the abelian category of finite dimensional vector spaces, and $D^b(Mod^f(k))$ its derived category with bounded cohomologies. For $G \in ob(D^b(Mod^f(k)))$,▮ we set

$$b_l(G) = dim\ H^l(G), \quad b^\#(G) = \{b_l(G)\}_{l\in\mathbf{Z}}, \tag{9}$$

$$b_l^*(G) = (-)^l\sum_{j\leq l}(-)^j b_j(G), \tag{10}$$

$$b_\infty^*(G) = \sum_j(-)^j b_j(G). \tag{11}$$

As is shown in [K1] and [KS], we have

$$\mathbf{R}\Gamma(X,F) \in ob(D^b(Mod^f(k))).$$

Thus we set

$$b_l(X,F) = b_l(\mathbf{R}\Gamma(X,F)) = dim\ H^l(X,F) < +\infty \tag{12}$$

and define $b_l^*(X,F)$ and $b_\infty^*(X,F)$ as in (10) and (11).

Moreover we set

(13) $$n_l = \sum_{\delta^i = l} m_i,\ n_l^* = (-)^l \sum_{j \le l} (-)^j n_j$$

and

(14) $$n_\infty^* = \sum_j (-)^j n_j.$$

Then we have

**Theorem 1.** (a generalized Morse inequality.)

*For any* $l \in \mathbf{Z}$ *, we have*

(15) $$b_l^*(X, F) \le n_l^*.$$

## 2. Proof of the main theorem

In order to prove the theorem, we note

**Lemma.** *Let* $G, G', G'' \in ob(D^b(Mod^f(k)))$*. Then we have*

*i.* $b^\#(G[j]) = b^\#(G)[j]$,

*ii.* $b^\#(G' \oplus G'') = b^\#(G') \oplus b^\#(G'')$.

*Moreover if we have a distinguished triangle*

$$\longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow,$$

*then*

*iii.* $b_\infty^*(G) = b_\infty^*(G') + b_\infty^*(G'')$,

*iv.* $b_l^*(G) \le b_l^*(G') + b_l^*(G'')$ *for any* $l \in \mathbf{Z}$.

*(proof)* i) and ii) are easy, and iii) is classical. Thus we prove only iv). We may assume that $G, G'$ and $G''$ are concentrated in degree $\ge 0$. Then we have a long exact sequence

$$\begin{array}{llll} 0 & \longrightarrow H^0(G') & \longrightarrow H^0(G) & \longrightarrow H^0(G'') \\ & \longrightarrow H^1(G') & \longrightarrow \ldots & \\ & \longrightarrow H^l(G') & \longrightarrow H^l(G) & \longrightarrow B^l(G'') \quad \longrightarrow \quad 0, \end{array}$$

where

$$B^l(G") = Im(H^l(G) \longrightarrow H^l(G")) \tag{16}$$

Then setting

$$\tilde{b}_l(G") = dim\, B^l(G")\ (j = l)$$

and

$$\tilde{b}_j(G") = b_j(G")\ (j < l),$$

we get:

$$b_l^*(G) = b_l^*(G') + (-)^l \sum_{j \leq l} (-)^j \tilde{b}_j(G"). \tag{17}$$

Since $\tilde{b}_l(G") \leq dim\ H^l(G")$, the proof follows. *(q.e.d.)*

*[proof of Theorem 1]* We set

$$\Omega_t = \{x;\ \phi(x) < t\} \text{ and } Z_t = \{x;\ \phi(x) \leq t\}.$$

We write

$$\phi(\{x_1, \dots, x_N\}) = \{t_1, \dots, t_L\}$$

with $-\infty = t_0 < t_1 < \dots < t_L < t_{L+1} = +\infty$. We also put $\Omega_j = \Omega_{t_j}$ and $Z_j = Z_{t_j}$. As is shown in 5 of [K1], we have the isomorphism

$$H^k(\Omega_{j+1}; F) \simeq H^k(\Omega_t, F)\ (t_j < t \leq t_{j+1}).$$

By taking the inductive limit of the right hand side, we derive

$$H^k(\Omega_{j+1}; F) \simeq H^k(Z_j, F).$$

Then we can see that

$$dim\ H^k(X, F) = \sum_{1 \leq j \leq L} \{dim\ H^k(Z_j, F) - dim\ H^k(\Omega_j, F)\},$$

which implies that

$$b_l^*(X,F) \;=\; \sum_{1\le j\le L} \{b_l^*(Z_j,F) \;-\; b_l^*(\Omega_j,F)\}.$$

Here we set

$$b_l^*(Z_j,F) \;=\; b_l^*(\mathbf{R}\Gamma(Z_j,F))$$

and

$$b_l^*(\Omega_j,F) \;=\; b_l^*(\mathbf{R}\Gamma(\Omega_j,F)).$$

On the other hand, we have a distinguished triangle

$$\longrightarrow \mathbf{R}\Gamma(Z_j\setminus\Omega_j,\mathbf{R}\Gamma_{X\setminus\Omega_j}(F)) \;\longrightarrow\; \mathbf{R}\Gamma(Z_j,F) \;\longrightarrow\; \mathbf{R}\Gamma(\Omega_j,F) \;\longrightarrow,$$

from which we get by the lemma above

$$b_l^*(Z_j,F) \;-\; b_l^*(\Omega_j,F) \;\le\; b_l^*(\mathbf{R}\Gamma(Z_j\setminus\Omega_j,\mathbf{R}\Gamma_{X\setminus\Omega_j}(F))).$$

Hence we have

$$(18)\qquad b_l^*(X,F) \;\le\; \sum_{1\le j\le L} b_l^*(\mathbf{R}\Gamma(Z_j\setminus\Omega_j,\mathbf{R}\Gamma_{X\setminus\Omega_j}(F))).$$

Since

$$\mathbf{R}\Gamma_{X\setminus\Omega_j}(F)\Big|_{Z_j\setminus\Omega_j} = \mathbf{R}\Gamma_{\{\phi(x)\ge t_j\}}(F)\Big|_{\phi^{-1}(t_j)},$$

we find by the definition of microsupport that

$$supp(\mathbf{R}\Gamma_{X\setminus\Omega_j}(F)\Big|_{Z_j\setminus\Omega_j}) \;\subset\; \pi(\Lambda_\phi\cap SS(F)).$$

This leads us to the quasi-isomorphism

$$(19)\qquad \mathbf{R}\Gamma(Z_j\setminus\Omega_j,\mathbf{R}\Gamma_{X\setminus\Omega_j}(F)\Big|_{Z_j\setminus\Omega_j}) \;=\; \bigoplus_{\{i;\ \phi(x_i)=t_j\}} \mathbf{R}\Gamma_{\{\phi(x)\ge t_j\}}(F)_{x_i}.$$

Hence we have the equalities

$$(20)\qquad \sum_{1\le j\le L} b_l(\mathbf{R}\Gamma(Z_j\setminus\Omega_j,\mathbf{R}\Gamma_{X\setminus\Omega_j}(F))) \;=\; \sum_{\delta^i=l} m_i \;=\; n_l.$$

This implies

$$b_l^*(X,F) \;\le n_l^*$$

if we see the inequlities (18).

### 3. Example

Let $X$ be $\mathbf{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$ and set

$$S = \{z \in X;\ \sum_{1\leq j\leq n} z_j^2 = 0\}.$$

We take $F \in ob(D^b_{\mathbf{R}-c}(X))$ satisfying that

$$\Lambda = SS(F) = T^*_{S_{reg}}X \cup T^*_{\{0\}}X \cup T^*_X X.$$

Moreover we put

$$\Lambda_0 = T^*_{S_{reg}}X,\ \Lambda_1 = T^*_{\{0\}}X,\ \Lambda_2 = T^*_X X$$

and assume that for any j

$F$ is pure along $\Lambda_j$ with multiplicity $m_j$ and shift $d_j$.

We set

$$\phi(z) = |z-a|^2 \text{ with } a = (1, 2\sqrt{-1}, 0, \ldots, 0).$$

Then we have

$$\Lambda_\phi \cap \Lambda_0 = \{p_{0,1} = (x_{0,1}; d\phi(x_{0,1})),\ p_{0,2} = (x_{0,2}; d\phi(x_{0,2}))\},$$

$$\Lambda_\phi \cap \Lambda_1 = \{p_1 = (0; d\phi(0))\},$$

$$\Lambda_\phi \cap \Lambda_2 = \{p_2 = (a; 0)\}.$$

Here

$$x_{0,1} = \left(\frac{-1}{2}, \frac{1}{2}\sqrt{-1}, 0, \ldots, 0\right) \text{ and } x_{0,2} = \left(\frac{3}{2}, \frac{3}{2}\sqrt{-1}, 0, \ldots, 0\right).$$

Moreover about the Maslov index, we can show that

$$\tau(\lambda_0(p_{0,1}), \lambda_{\Lambda_0}(p_{0,1}), \lambda_\phi(p_{0,1})) = 2,$$

$$\tau(\lambda_0(p_{0,2}), \lambda_{\Lambda_0}(p_{0,2}), \lambda_\phi(p_{0,2})) = 2n-2,$$

$$\tau(\lambda_0(p,1), \lambda_{\Lambda_1}(p_1), \lambda_\phi(p_1)) = 0,$$

$$\tau(\lambda_0(p_2), \lambda_{\Lambda_2}(p_2), \lambda_\phi(p_2)) = 2n.$$

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