

Propagation of singularities of solutions
for a degenerate hyperbolic equation

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In this note we give a short review on the Gevrey wave front sets and give a properties of the propagation of singularities for solutions of the Cauchy problem for a degenerate hyperbolic operator.

§1. Gevrey wave front sets. Let κ satisfy $\kappa > 1$. As in [5] we introduce a class of ultradistributions as

$$\mathcal{D}_{L^2}^{\{\kappa\}'} = \text{proj} \lim_{\varepsilon \downarrow 0} \mathcal{D}_{L^2, \varepsilon}^{\{\kappa\}'}$$

Here, for $\varepsilon > 0$, $\mathcal{D}_{L^2, \varepsilon}^{\{\kappa\}'}$ is the dual space of the Hilbert space $\mathcal{D}_{L^2, \varepsilon}^{\{\kappa\}} \equiv \{u(x) \in L^2; \exp(\varepsilon \langle \xi \rangle^{1/\kappa}) \hat{u}(\xi) \in L^2\}$ and $F[u](\xi) \equiv \hat{u}(\xi)$ denotes the Fourier transform of $u(x)$. For the symbol classes of pseudo-differential operators we employ

Definition 1.1. i) Let $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. We say that a symbol $p(x, \xi)$ belongs to a class $S_{\rho, \delta, G(\kappa)}^m$ if

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq & C M^{-|\alpha+\beta|} (\alpha!^\kappa + \alpha!^{\kappa\rho} \langle \xi \rangle^{(1-\rho)|\alpha|}) \\ & \times (\beta!^\kappa + \beta!^{\kappa(1-\delta)} \langle \xi \rangle^{\delta|\beta|}) \langle \xi \rangle^{m-|\alpha|} \\ & \text{for all } x \text{ and } \xi \end{aligned}$$

hold for constants C and M independent of α and β . We also denote $S_{G(\kappa)}^m = S_{1, 0, G(\kappa)}^m$.

ii) We say that a symbol $p(x, \xi) (\in S^{-\infty})$ belongs to a

class $\mathcal{R}_{G(\kappa)}$ if for any α there exists a constant C_α such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_\alpha M^{-|\beta|} |\beta|!^\kappa \exp(-\varepsilon \langle \xi \rangle^{1/\kappa})$$

hold with a positive constant ε independent of α and β .

The following lemma follows by the similar way to the proof of Theorem 1 in [13].

Lemma 1.2. i) The pseudo-differential operator $p(X, D_x)$ with a symbol $p(x, \xi)$ in $S_{\rho, \delta, G(\kappa)}^m$ maps $\mathcal{D}_{L^2}^{\{\kappa\}}$ to $\mathcal{D}_{L^2}^{\{\kappa\}}$.

ii) The pseudo-differential operator $p(X, D_x)$ with a symbol $p(x, \xi)$ in $\mathcal{R}_{G(\kappa)}$ maps $\mathcal{D}_{L^2}^{\{\kappa\}}$ to $\bigcup_{\varepsilon > 0} \mathcal{D}_{L^2, \varepsilon}^{\{\kappa\}}$.

From Lemma 1.2 we can call pseudo-differential operators $p(X, D_x)$ with symbols in $\mathcal{R}_{G(\kappa)}$ regularizers.

Next, we introduce Gevrey wave front sets $WF_{G(\kappa_1)}(u)$ of ultradistributions $u(x)$ as follows.

Definition 1.3. Let κ_1 satisfy $\kappa_1 \geq \kappa$ and let (x_0, ξ_0) be a point in $T^*(\mathbb{R}^n) \setminus \{0\}$. Then, (x_0, ξ_0) does not belong to a Gevrey wave front set $WF_{G(\kappa_1)}(u)$ of $u \in \mathcal{D}_{L^2}^{\{\kappa\}}$ if and only if there exists a symbol $a(x, \xi)$ in $S_{G(\kappa)}^0$ such that the following i) and ii) hold:

i) For a conic neighborhood Γ of (x_0, ξ_0) an inequality

$$|a(x, \xi)| \geq C \quad \text{for } (x, \xi) \in \Gamma$$

holds for a positive constant C .

ii) The function $a(X, D_x)u$ belongs to the Gevrey class $\gamma^{(\kappa_1)}$ of order κ_1 , that is, $f(x) \equiv a(X, D_x)u$ satisfies

$$|\partial_x^\alpha f(x)| \leq CM^{-|\alpha|} |\alpha|!^{\kappa_1} \quad \text{for any } x \in \mathbb{R}_x^n.$$

For a distribution $u(x) (\in \mathcal{E}')$ with compact support we have proved in [13]

Lemma 1.4 (cf. Theorem 3 of [13]). Let $u \in \mathcal{E}'$ and $\kappa_1 > 1$. Then, the point $(x_0, \xi_0) \in T^*(\mathbb{R}^n) \setminus \{0\}$ does not belong to $WF_{G(\kappa_1)}(u)$ if and only if

(*) There exist a function $\chi(x)$ in $\gamma^{(\kappa_1)}(\mathbb{R}_x^n)$ with $\chi(x_0) \neq 0$ and a conic neighborhood Γ_ξ of ξ_0 such that

$$|F[\chi u](\xi)| \leq C \exp(-\varepsilon \langle \xi \rangle^{1/\kappa_1}) \quad \text{for } \xi \in \Gamma_\xi$$

holds for a positive constants C and ε .

The (*) is the original definition of the Gevrey wave front sets for distributions given by Hörmander [3]. So, our definition is an extension of the original one. We also note that from our definition we can treat Gevrey wave front sets by the almost same way as the treatment of wave front sets for the C^∞ case (cf. [2], [7].)

2. A degenerate hyperbolic operator. Consider a degenerate hyperbolic operator

$$(2.1) \quad L = D_t^2 - t^{2\ell} g(x)^{2\ell'} \sum_{j,j'=1}^n a_{j,j'}(t,x) D_{x_j} D_{x_{j'}} + t^k g(x)^{k'} \sum_{j=1}^n a_j(t,x) D_{x_j} + c(t,x)$$

on $[0, T] \times \mathbb{R}_x^n$.

We first assume

$$(A-1) \quad \ell - 1 \geq k \geq 0, \quad \ell' \geq k' \geq 1 \quad \text{and} \quad \ell' \geq 2.$$

Define

$$(2.2) \quad \sigma = \max((\ell - k - 1)/(2\ell - k), (\ell' - k')/(2\ell' - k')) .$$

Then, σ satisfies $\sigma < 1/2$. Now, we assume

$$(A-2) \quad \kappa \geq 2 \quad \text{and} \quad \kappa\sigma < 1 \quad \text{with} \quad \sigma \quad \text{in} \quad (2.2).$$

(A-3) The function $g(x)$ belongs to a Gevrey class of order κ with a uniform estimate:

$$(2.3) \quad |D_x^\alpha g(x)| \leq C M^{-|\alpha|} \alpha!^\kappa \quad \text{for all} \quad x \in \mathbb{R}^n.$$

(A-4) The coefficients $a_{j,j'}(t,x)$, $a_j(t,x)$ and $c(t,x)$ are analytic in t and of a Gevrey class of order κ in x with a uniform estimate (2.3).

(A-5) $a_{j,j'}(t,x)$ are real-valued and there exists a positive constant C such that

$$\sum_{j,j'} a_{j,j'}(t,x) \xi_j \xi_{j'} \geq C |\xi|^2 \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbb{R}_x^n.$$

Denote the characteristic roots of L by

$$(2.4) \quad \lambda_\pm(t,x,\xi) = \pm t^{\ell} g(x)^{\ell'} \left\{ \sum_{j,j'} a_{j,j'}(t,x) \xi_j \xi_{j'} \right\}^{\frac{1}{2}}$$

and let $\phi_\pm(t,s;x,\xi)$ be the phase functions corresponding to the modified characteristic roots

$$(2.5) \quad \tilde{\lambda}_\pm(t,x,\xi) = \lambda(t,x,\xi)(1 - \chi(\xi))$$

of $\lambda_\pm(t,x,\xi)$, that is, $\phi_\pm(t,s;x,\xi)$ are solutions of

$$\begin{cases} \frac{d\phi_\pm}{dt} = \tilde{\lambda}_\pm(t,x,\nabla_x \phi_\pm), \\ \phi_\pm|_{t=s} = x \cdot \xi . \end{cases}$$

Here, in (2.5), $\chi(\xi)$ is the function in $\gamma^{(\kappa)}$ satisfying

$$\chi(\xi) = 1 \quad \text{for} \quad |\xi| \leq 1/2, \quad \chi(\xi) = 0 \quad \text{for} \quad |\xi| \geq 1.$$

Then, we have

Theorem 1 (cf. Shinkai-Taniguchi [12]). Assume (A-1)-(A-5).

Set $\rho = 1 - (1 - \sigma)/\ell'$. Then, for a small $T_0 (\leq T)$ the funda-

mental solution of the Cauchy problem

$$(2.6) \quad \begin{cases} Lu = 0 & \text{on } [s, T_0], \\ u(s) = 0, \quad \partial_t u(s) = u_0 \end{cases}$$

with $s \in [0, T_0)$ can be constructed in the form

$$(2.7) \quad E(t, s) = \sum_{\pm} I_{\phi_{\pm}}(t, s) E_{\pm}(t, s) + E_0(t, s) + E_{\infty}(t, s).$$

Here, $I_{\phi_{\pm}}(t, s)$ are Fourier integral operators with phase functions $\phi_{\pm}(t, s; x, \xi)$ and the symbol l ; and $E_j(t, s)$, $j = 0, \pm, \infty$, are pseudo-differential operators with symbols $e_j(t, s; x, \xi)$ satisfying

$$(2.8) \quad \begin{aligned} |\partial_{\xi}^{\alpha} \partial_x^{\beta} e_{\pm}(t, s; x, \xi)| \\ \leq CM^{-|\alpha+\beta|} ((\alpha+\beta)!^{\kappa} + (\alpha+\beta)!^{\kappa \rho} \langle \xi \rangle^{(1-\rho)|\alpha+\beta|}) \\ \times \langle \xi \rangle^{-|\alpha|} \exp(C_1 \langle \xi \rangle^{\sigma'}) \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} |\partial_{\xi}^{\alpha} \partial_x^{\beta} e_0(t, s; x, \xi)| \\ \leq CM^{-|\alpha+\beta|} ((\alpha+\beta)!^{\kappa} + (\alpha+\beta)!^{\kappa \rho} \langle \xi \rangle^{(1-\rho)|\alpha+\beta|}) \\ \times \langle \xi \rangle^{-|\alpha|} \exp(C_1 \langle \xi \rangle^{\sigma'} - \varepsilon_1 t^{\ell+1} |g(x)|^{\ell'} \langle \xi \rangle^{1-\sigma}) \end{aligned}$$

for a positive constant ε_1 and the constant σ' satisfying

$$(2.10) \quad \sigma < \sigma' < 1/\kappa, \quad \sigma' \geq (1 + (\ell' - 1)\sigma) / (\ell'\kappa - \ell' + 1).$$

Moreover, for any multi-index α there exists a constant C_{α} such that

$$(2.11) \quad |\partial_{\xi}^{\alpha} \partial_x^{\beta} e_{\infty}(t, s; x, \xi)| \leq C_{\alpha} M^{-|\beta|} |\beta|^{\kappa} \exp(-\varepsilon_2 \langle \xi \rangle)^{1/\kappa}$$

holds for a positive constant ε_2 .

We note that the factor $\exp(C_1 \langle \xi \rangle^{\sigma'})$ appearing in (2.8)-(2.9) makes us to consider the Cauchy problem (2.6) of (2.1) in the class of ultradistributions and the Gevrey wave front sets.

Next, we consider the propagation of Gevrey wave front sets for the solution of (2.6). For the characteristic roots $\lambda_{\pm}(t, x, \xi)$ of (2.4), let $\{q^{\pm}, p^{\pm}\}(t, s; y, \eta)$ be the solution of

$$\begin{cases} \frac{dq^{\pm}}{dt} = -\nabla_{\xi} \lambda_{\pm}(t, q^{\pm}, p^{\pm}), & \frac{dp^{\pm}}{dt} = \nabla_x \lambda_{\pm}(t, q^{\pm}, p^{\pm}) \\ & (s \leq t \leq T_0), \\ \{q^{\pm}, p^{\pm}\}|_{t=s} = (y, \eta) \end{cases}$$

and, for $s \leq 0 \leq t$, let $\{\tilde{q}^{\pm}, \tilde{p}^{\pm}\}(t, s; y, \eta)$ be the solution of

$$\begin{cases} \frac{d\tilde{q}^{\pm}}{dt} = -\nabla_{\xi} \lambda_{\pm}(t, \tilde{q}^{\pm}, \tilde{p}^{\pm}), & \frac{d\tilde{p}^{\pm}}{dt} = \nabla_x \lambda_{\pm}(t, \tilde{q}^{\pm}, \tilde{p}^{\pm}) \\ & (0 \leq t \leq T_0), \\ \{\tilde{q}^{\pm}, \tilde{p}^{\pm}\}|_{t=0} = \{q^{\mp}, p^{\mp}\}(0, s; y, \eta). \end{cases}$$

Then, combining Theorem 1 and the discussion in [9] we obtain

Theorem 2. Consider a Cauchy problem (2.6) with $s < 0$. Then we have, when $t > 0$, for a solution $u(t)$ of (2.6)

$$(2.12) \quad \text{WF}_{G(\kappa)}(u(t)) \subset \Gamma_+(t) \cup \Gamma_-(t) \cup \hat{\Gamma}_+(t) \cup \hat{\Gamma}_-(t) \cup \Gamma_0(t),$$

where

$$\begin{aligned} \Gamma_{\pm}(t) &= \{(q^{\pm}(t, s; y, \eta), p^{\pm}(t, s; y, \eta)) ; (y, \eta) \in \text{WF}_{G(\kappa)}(u_0), \\ &\quad |\eta| \gg 1\}, \\ \hat{\Gamma}_{\pm}(t) &= \{(\tilde{q}^{\pm}(t, s; y, \eta), \tilde{p}^{\pm}(t, s; y, \eta)) ; (y, \eta) \in \text{WF}_{G(\kappa)}(u_0), \\ &\quad |\eta| \gg 1\} \end{aligned}$$

and

$$\Gamma_0(t) = \{(y, \eta) ; (y, \eta) \in \text{WF}_{G(\kappa)}(u_0), g(y) = 0\}.$$

This theorem corresponds to the branching property for the C^{∞} case, that is, for the Cauchy problem of the operator

$$(2.13) \quad D_t^2 - t^{2\ell} D_x^2 + at^k D_x$$

with $k = \ell - 1$ (see [1], [14] and [9]). For the case $k < \ell - 1$ in (2.13), setting $\kappa = (2\ell - k)/(\ell - k - 1)$, Shinkai gave $WF_{G(\kappa)}(u(t))$ exactly by using the exact form of the fundamental solution for the operator (2.13) (see [10] and [11]). Finally, we note that in (A-1) we assumed $\ell' \geq 2$ and $k' \geq 1$. But, when we assume $\ell' = 1$ or $k' = 0$ we have $\sigma = 1/2$ and the problem for the operator (2.1) is reduced to the case $1 < \kappa < 2$, which is the case of free lower order problem. In that case, $1 < \kappa < 2$, the problem (2.6) for (2.1) is always $\gamma^{(\kappa)}$ -well-posed and the propagation of singularities (2.12) for solutions of (2.6) is obtained in [8]. So, we assumed $\ell' \geq 2$ and $k' \geq 1$.

§3. A higher order degenerate hyperbolic operator. Following [4] we consider a higher order degenerate hyperbolic operator

$$(3.1) \quad L = D_t^m + \sum_{\substack{|\alpha|+j=m \\ j < m}} t^{|\alpha|\ell} g(x)^{|\alpha|\ell'} a_{\alpha,j}(t,x) D_x^\alpha D_t^j \\ + \sum_{|\alpha|+j < m} t^{v(\alpha,j)} g(x)^{\mu(\alpha,j)} a_{\alpha,j}(t,x) D_x^\alpha D_t^j$$

with integers $\ell, \ell', v(\alpha, j)$ and $\mu(\alpha, j)$. For the operator L of (3.1) we define an irregularity θ (cf. [6]) as

$$\theta = \max_{|\alpha|+j < m} \left\{ \frac{m-j-\{v(\alpha,j)+m-j\}/(\ell+1)}{m-j-|\alpha|}, \frac{m-j-\mu(\alpha,j)/\ell'}{m-j-|\alpha|}, 1 \right\}$$

and set

$$(3.2) \quad \sigma = (\theta - 1)/\theta .$$

We assume

$$(A-1)' \quad \ell' \geq 2 \quad \text{and} \quad \theta < 2.$$

Then, σ satisfies $\sigma < 1/2$. We assume, furthermore, (A-2) and

(A-3) in Section 2 and

(A-4)' The coefficients $a_{\alpha,j}(t,x)$ of (3.1) are analytic in t and of a Gevrey class of order κ in x with a uniform estimate (2.3).

(A-5)' The equation

$$\lambda^m + \sum_{\substack{|\alpha|+j=m \\ j < m}} a_{\alpha,j}(t,x) \xi^\alpha \lambda^j = 0$$

has distinct roots $\lambda_j^0(t,x,\xi)$, $j = 1, \dots, m$, and they satisfy an uniform estimate:

$$|\lambda_j^0(t,x,\xi) - \lambda_{j'}^0(t,x,\xi)| \geq C|\xi|$$

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}_{x,\xi}^{2n}$

with a positive constant C .

From (A-5)' the characteristic roots of (3.1) are

$$\lambda_j(t,x,\xi) = t^{\ell} g(x)^{\ell'} \lambda_j^0(t,x,\xi), \quad j = 1, \dots, m.$$

Let $\phi_j(t,s;x,\xi)$, $j = 1, \dots, m$, be the phase functions corresponding to the characteristic roots $\lambda_j(t,x,\xi)$ and denote by $I_{\phi_j}(t,s)$ the Fourier integral operators with phase functions $\phi_j(t,s;x,\xi)$ and with the symbol 1. Moreover, let $\{q^j, p^j\}(t,s;y,\eta)$ be the bicharacteristics of $\lambda_j(t,x,\xi)$ and we denote, for $s < 0 < t$, the (broken) bicharacteristics as $\{\tilde{q}^{j,j'}, \tilde{p}^{j,j'}\}(t,s;y,\eta) = \{q^j, p^j\}(t,0; \{q^{j'}, p^{j'}\}(0,s;y,\eta))$. Then, by the same method of proving Theorems 1 and 2 we obtain the following two theorems.

Theorem 3. We assume (A-1)', (A-2), (A-3), (A-4)' and (A-5)'. Let s satisfy $0 \leq s \leq T_0$ for a small $T_0 (\leq T)$. Then, the fundamental solution $E(t,s)$ of the Cauchy problem

$$(3.3) \quad \begin{cases} Lu = 0 & \text{on } [s, T_0], \\ \partial_t^j u(s) = 0 \quad (j=0, \dots, m-2), \quad \partial_t^{m-1} u(s) = u_0 \end{cases}$$

for the operator L of (3.1) can be constructed in the form

$$E(t, s) = \sum_{j=1}^m I_{\phi_j}(t, s) E_j(t, s) + E_0(t, s) + E_\infty(t, s)$$

with pseudo-differential operators $E_j(t, s)$, $j = 0, 1, \dots, m, \infty$,

whose symbols $e_j(t, s; x, \xi) = \sigma(E_j(t, s))$ satisfy

$$\begin{aligned} & |\partial_\xi^\alpha \partial_x^\beta e_j(t, s; x, \xi)| \\ & \leq C M^{-|\alpha+\beta|} ((\alpha+\beta)!^\kappa + (\alpha+\beta)!^{\kappa\rho} \langle \xi \rangle^{(1-\rho)|\alpha+\beta|}) \\ & \quad \times \langle \xi \rangle^{-|\alpha|} \exp(C_1 \langle \xi \rangle^{\sigma'}) \quad (j = 1, \dots, m) \end{aligned}$$

and (2.9) and (2.11). Here, σ' is a constant satisfying (2.10).

Theorem 4. Consider a Cauchy problem (3.3) with $s < 0$.

Then, we have, when $t > 0$, for a solution $u(t)$ of (3.3)

$$WF_{G(\kappa)}(u(t)) \subset \bigcup_{j=1}^m \Gamma_j(t) \cup \bigcup_{j, j'=1}^m \hat{\Gamma}_{j, j'}(t) \cup \Gamma_0(t),$$

where

$$\Gamma_j(t) = \{(q^j(t, s; y, \eta), p^j(t, s; y, \eta)) ; (y, \eta) \in WF_{G(\kappa)}(u_0), |\eta| \gg 1\},$$

$$\hat{\Gamma}_{j, j'}(t) = \{(\tilde{q}^{j, j'}(t, s; y, \eta), \tilde{p}^{j, j'}(t, s; y, \eta)) ; (y, \eta) \in WF_{G(\kappa)}(u_0), |\eta| \gg 1\}$$

and

$$\Gamma_0(t) = \{(y, \eta) ; (y, \eta) \in WF_{G(\kappa)}(u_0), g(y) = 0\}.$$

References

- [1] S. Alinhac: Paramétrie et propagation des singularités

- pour un problème de Cauchy à multiplicité variable, Soc. Math. France Astérisque, 34-35 (1976), 3-36.
- [2] L. Hörmander: Fourier integral operators I, Acta Math., 127 (1971), 79-183.
- [3] L. Hörmander: Uniqueness theorems and wave front sets for solutions of linear partial differential equations with analytic coefficients, Comm. Pure Appl. Math., 24 (1971), 671-704.
- [4] S. Itoh and H. Uryu: Conditions for well-posedness in Gevrey classes of the Cauchy problems for Fuchsian hyperbolic operators II, Publ. RIMS, Kyoto Univ., 23 (1987), 215-241.
- [5] H. Komatsu: Ultradistributions, I, Structure theorems and a characterization, J. Fac. Sci., Univ. Tokyo, Sec. IA, 20 (1973), 25-105.
- [6] H. Komatsu: Irregularity of characteristic elements and hyperbolicity, Publ. RIMS, Kyoto Univ., 12 (1977), 233-245.
- [7] H. Kumano-go: Pseudo-differential operators, The MIT Press Cambridge and London, 1982.
- [8] Y. Morimoto and K. Taniguchi: Propagation of wave front sets of solutions of the Cauchy problem for hyperbolic equations in Gevrey classes, Osaka J. Math., 23 (1986), 765-814.
- [9] K. Shinkai: Branching of singularities for a degenerate hyperbolic system, Comm. Partial Differential Equations,

- 7 (1982), 581-607.
- [10] K. Shinkai: Gevrey wave front sets of solutions for a weakly hyperbolic operator, *Math. Japon.*, 30 (1985), 701-717.
- [11] K. Shinkai: Stokes multipliers and a weakly hyperbolic operator, *University of Minnesota, Mathematics Report*, #86-134, (1987).
- [12] K. Shinkai and K. Taniguchi: Fundamental solution for a degenerate hyperbolic operator in Gevrey classes, to appear.
- [13] K. Taniguchi: Pseudo-differential operators acting on ultradistributions, *Math. Japon.*, 30 (1985), 719-741.
- [14] K. Taniguchi and Y. Tozaki: A hyperbolic equation with double characteristics which has a solution with branching singularities, *Math. Japon.*, 25 (1980), 279-300.