The asymptotic behaviour of singular solutions to the solutions

of linear partial differnential equatiions in the complex domain

Sunao ŌUCHI   (Sophia Univ. Tokyo)

(大内忠)

§ 1. Let $L(z, \partial_z)$ be a linear partial differential operator with the

order $m \geq 1$, whose coefficients are holomorphic functions in a

neighbourhood $\Omega = \{ z \in \mathbb{C}^{n+1}; \ |z| \leq R \}$ of $z=0$ in $\mathbb{C}^{n+1}$, where $z = (z_0, z_1, .., z_n)$

$= (z_0, z')$ and $|z| = \max_{0 \leq i \leq n} |z_i|$. Let K be a nonsingular hypersurface

through $z=0$. For the simplicity we choose the coordinate so that $K =$

$\{z_0 = 0\}$. In the following we consider the equation

(1.1) $L(z, \partial_z) u(z) = f(z)$,

where $u(z)$ may have singularities on K, and $f(z)$ is holomorphic in Ω.

We introduce some function spaces and the definitions. $\Omega(a,b)$ is

the set defined by $\Omega(a,b) = \{z; \ a < \arg z_0 < b; \ |z| \leq R\}$ and $\Omega' = \Omega \cap \{z_0 = 0\}$.

Firstly we define the function spaces:

$\mathcal{O}(\Omega) = \{f(z); \ f(z)$ is holomorphic in $\Omega\}$, $\mathcal{O}(\Omega') = \{f(z'); \ f(z')$ is

holomorphic in $\Omega'\}$ and $\widetilde{\mathcal{O}}(\Omega(a,b)) = \{f(z); \ f(z)$ is holomorphic in

$\Omega(a,b)\}$. If $b-a > 2\pi$, $\widetilde{\mathcal{O}}(\Omega(a,b))$ contains multi-valued functions. We

define other function spaces.

Definition 1.1. $\widetilde{\mathcal{O}}_{(\gamma)}(\Omega(a,b)) = \{f(z) \in \widetilde{\mathcal{O}}(\Omega(a,b)); $ for any a', b' with

$a < a' < b' < b$ and $\varepsilon > 0$, there is a $C_{\varepsilon, a', b'}$ such that

(1.2) $|f(z)| \leq C_{\varepsilon, a', b'} \exp(\varepsilon |z_0|^{-\gamma})$ in $\Omega(a', b')$.

Definition 1.2. $f(z) \in \widetilde{\mathcal{O}}(\Omega(a,b))$ is said to have the γ-asymptotic

expansion in $\Omega(a,b)$, if for any N

(1.3) $|f(z) - \sum_{k=0}^{N-1} a_k(z')(z_0)^k| \leq A B^N \Gamma(N/\gamma + 1) |z_0|^N$

holds in $\Omega(a', b')$ for any a', b' with $a < a' < b' < b$, where $a_k(z') \in \mathcal{O}(\Omega')$

and A and B are some constants. The totality of functions with the γ-asymptotic expansions is denoted by $\text{Asy}_{\{\gamma\}}(\Omega(a,b))$.

Secondly we define characteristic indices ([1],[2]): We write $L(z,\partial_z)$ in the following form.

$$(1.3) \qquad L(z,\partial_z) = \sum_{k=0}^{m} L_k(z,\partial_z),$$

$$L_k(z,\partial_z) = \sum_{\ell=s_k}^{k} A_{k,\ell}(z,\partial')(\partial_0)^{k-\ell}.$$

$L_k(z,\partial_z)$ is the homogenous part of the degree k. We expand $A_{k,\ell}(z,\partial')$ at $z_0=0$, $A_{k,\ell}(z,\partial')=(z_0)^j a_{k,\ell}(z,\partial')$, $j=j(k,\ell)$, where if $A_{k,\ell}(z,\partial')\not\equiv 0$, $a_{k,\ell}(0,z',\partial')\not\equiv 0$. Thus we have
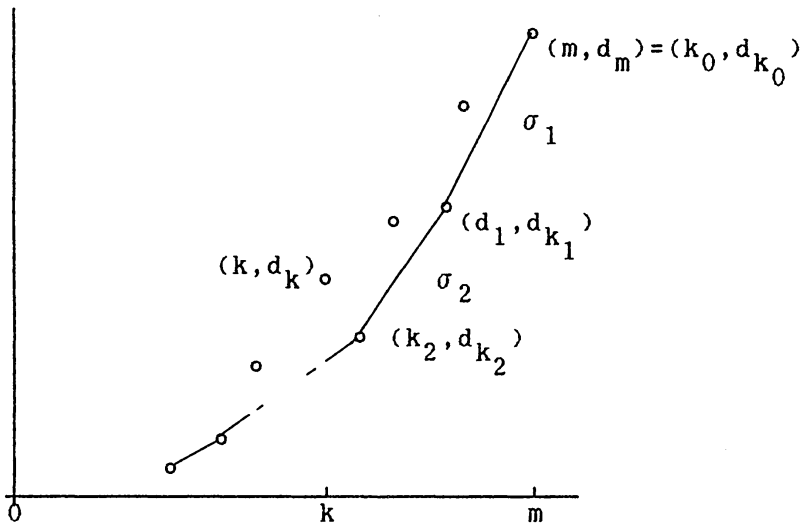
$$(1.4) \qquad L_k(z,\partial_z) = \sum_{\ell=s_k}^{k} (z_0)^j a_{k,\ell}(z,\partial')(\partial_0)^{k-\ell},$$

Define

$$(1.5) \qquad d_k = \min\{\ell+j(k,\ell); \ A_{k,\ell}(z,\partial')|_{z_0=0}\not\equiv 0\}.$$

Put $A=\{(k,d_k)\in\mathbb{R}^2 ; 0\leq k\leq m, \ d_k\neq+\infty\}$. Let $\hat{A}$ be the convex hull of A, Σ be the lower convex part of the boundary of $\hat{A}$ and Δ be the set of vertices of Σ. Δ consists of finite points: $\Delta=\{(k_i,d_{k_i});i=1,2,\ldots p'\}$, $m=k_0>k_1>\ldots>k_{p'}\geq 0$. We put

$$(1.6) \qquad \sigma_i = \max\{1, \ (d_{k_{i-1}}-d_{k_i})/(k_{i-1}-k_i)\}.$$

Then there is a $p \in \mathbb{N}$ such that $\sigma_1 > \sigma_2 > \ldots > \sigma_p = 1$. We call $\{\sigma_i\}$ $(1 \leq i \leq p)$ characteristic indices.

§2. By using the definition in §1 we can state Theorem:

Theorem. Assume

(a) $\sigma_1 > 1$, (b) $d_{k_{p-1}} = 0$ and (c) $d_{k_i} = s_{k_i}$ $(0 \leq i \leq p-2)$.

Let $u(z) \in \tilde{\mathcal{O}}(\Omega(a,b))$ $(b-a > \pi)$ be a solution of $L(z, \partial_z)u(z) = f(z) \in \mathcal{O}(\Omega)$. If $u(z) \in \tilde{\mathcal{O}}_{(\gamma)}(\Omega(a,b))$ $(\gamma = \sigma_{p-1} - 1)$, then $u(z) \in \text{Asy}_{(\gamma)}(\Omega(a,b))$, that is, $u(z)$ has the γ-asymptotic expansions in $\Omega(a,b)$.

This is a generalzation of the theorem in [3].

References

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