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k-NETWORKS, AND COVERING PROPERTIES OF CW-COMPLEXES

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We assume that all spaces are $T_2$. First of all, we shall recall some definitions.

Let $X$ be a space, and let $C$ be a cover of $X$. Then $X$ is determined by $C$ [3] (or $X$ has the weak topology with respect to $C$ in the usual sense), if $F \subset X$ is closed in $X$ if and only if $F \cap C$ is closed in $C$ for every $C \in C$. Here, we can replace "closed" by "open". Every space is determined by an open cover. $X$ is dominated by $C$, if the union of any subcollection $C'$ of $C$ is closed in $X$, and the union is determined by $C'$.

Let $X$ be a space, and $\mathcal{P}$ be a cover of $X$. Then $\mathcal{P}$ is a $k$-network, if whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset U \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. If we replace "compact" by "single point" then such a cover is called "net (or network)". $k$-networks have played a role in $K_\sigma$-spaces (i.e., regular spaces with a countable $k$-network) and $K_\sigma$-spaces (i.e., regular spaces with a $\sigma$-locally finite $k$-network).

Let $\mathcal{A} = \{A_\alpha; \alpha \in A\}$ be a collection of subsets of a space $X$. Then $\mathcal{A}$ is closure-preserving if $\overline{\bigcup \{A_\alpha; \alpha \in B\}} = \bigcup \overline{\{A_\alpha; \alpha \in B\}}$ for any $B \subset A$. $\mathcal{A}$ is hereditarily closure-preserving if $\overline{\bigcup \{B_\alpha; \alpha \in B\}} = \bigcup \overline{\{B_\alpha; \alpha \in B\}}$ whenever $B \subset A$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in B$. Every space is dominated by a hereditarily closure-preserving closed cover.
A $\sigma$-hereditarily closure-preserving collection is the union of countably many hereditarily closure-preserving collections. We shall use " $\sigma$-CP (resp. $\sigma$-HCP) " instead of " $\sigma$-closure-preserving (resp. $\sigma$-hereditarily closure-preserving) ".

$\mathcal{A}$ is point-finite (resp. point-countable) if every $x \in X$ is in at most finitely (resp. countably) many element of $\mathcal{A}$.

The concept of CW-complexes due to J. H. Whitehead [5] is well-known. A space $X$ is a CW-complex if it is a complex with cells $\{e_\lambda; \lambda\}$ satisfying (a) and (b) below.

(a) Each cell $e_\lambda$ is contained in a finite subcomplex of $X$.

(b) $X$ is determined by the closed cover $\{\overline{e}_\lambda; \lambda\}$ of $X$.

We note that every $\overline{e}_\lambda$ is not a subcomplex.

As is well-known, every CW-complex $X$ is dominated by the cover of all finite subcomplexes of $X$, hence $X$ is dominated by a cover of compact metric subsets of $X$.

Let $\{e_\lambda; \lambda\}$ be the cells of a CW-complex $X$. We shall say that $\{e_\lambda; \lambda\}$ is $(\sigma)$-locally finite; $(\sigma)$-HCP, etc., if so is respectively the collection $\{e_\lambda; \lambda\}$ of subsets of $X$. We note that the collection $\{e_\lambda; \lambda\}$ is $(\sigma)$-locally finite; $(\sigma)$-CP; $(\sigma)$-HCP if and only if so is respectively $\{\overline{e}_\lambda; \lambda\}$. 

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Results. Let $X$ be a CW-complex with cells $\{e_\lambda; \lambda\}$. Then the following hold. (a) is well-known, and (b) is due to [2].

(a) $X$ is a paracompact, and $\sigma$-space (i.e., $X$ has a $\sigma$-locally finite net).

(b) $X$ is an $M_1$-space (in the sense of [2]), hence $X$ has a $\sigma$-CP k-network.

(c) $X$ has a point-countable k-network.

However, every CW-complex is not a metric space (not even a Fréchet space, nor an $\mathcal{H}$-space). We have the following characterizations of $X$. Recall that a space is Fréchet, if whenever $x \in \overline{A}$, there exists a sequence in $A$ converging to the point $x$. (A) is well-known, and (B) is due to [4].

(A) $X$ is a metric space if and only if $\{e_\lambda; \lambda\}$ is locally finite.

(B) $X$ is a Fréchet space if and only if $\{e_\lambda; \lambda\}$ is HCP.

(C) $X$ is an $\mathcal{H}$-space if and only if $\{e_\lambda; \lambda\}$ is $\sigma$-locally finite.

(D) $X$ has a $\sigma$-HCP k-network if and only if $\{e_\lambda; \lambda\}$ is $\sigma$-HCP.

(E) $X$ is a symmetric space (in the sense of [1]) if and only if $\{\overline{e}_\lambda; \lambda\}$ is point-finite.

(F) $X$ has a point-countable closed k-network if and only if $\{\overline{e}_\lambda; \lambda\}$ is point-countable.
Remark. Let $X$ be a CW-complex with cells $\{e_\lambda; \lambda\}$.

(1) The property " $\{\overline{e}_\lambda; \lambda\}$ is HCP " need not imply that $X$ has a point-countable closed k-network, and not imply that $\{\overline{e}_\lambda; \lambda\}$ is point-countable.

(2) The property " $\{e_\lambda; \lambda\}$ is CP " need not imply that $X$ has a CP or $\sigma$-HCP k-network, and not imply that $\{e_\lambda; \lambda\}$ is $\sigma$-HCP.

(3) The property " $X$ is a symmetric space with a $\sigma$-CP k-network " need not imply that $X$ has a $\sigma$-HCP k-network, and not imply that $\{e_\lambda; \lambda\}$ is $\sigma$-CP.

Question. Let $X$ be a CW-complex with cells $\{e_\lambda; \lambda\}$. Characterize " $\{e_\lambda; \lambda\}$ is CP (or $\sigma$-CP) " by means of a nice topological property of $X$.

Finally, concerning spaces dominated by compact metric subsets, similarly to CW-complexes the following analogue holds.

Let $X$ be a space dominated by a cover $\{X_\mu; \mu\}$ with each $E_\mu$ compact metric. Here, $E_\emptyset = X_\emptyset$, $E_\lambda = X_\lambda = \cup \{X_\mu; \mu < \lambda\}$. Then it is possible to replace $\{e_\lambda; \lambda\}$ (or $\{\overline{e}_\lambda; \lambda\}$) by $\{E_\lambda; \lambda\}$ (or $\{\overline{E}_\lambda; \lambda\}$) in (A) $\sim$ (F).
References


