<table>
<thead>
<tr>
<th>Field</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>k-NETWORKS, AND COVERING PROPERTIES OF CW-COMPLEXES (General Topology and around it)</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Tanaka, Yoshio</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1991), 758: 44-48</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82175">http://hdl.handle.net/2433/82175</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
k-NETWORKS, AND COVERING PROPERTIES OF CW-COMPLEXES

東京学芸大学  田中祥雄 (Yoshio Tanaka)

We assume that all spaces are $T_2$. First of all, we shall recall some definitions.

Let $X$ be a space, and let $C$ be a cover of $X$. Then $X$ is determined by $C$ [3] (or $X$ has the weak topology with respect to $C$ in the usual sense), if $F \subseteq X$ is closed in $X$ if and only if $F \cap C$ is closed in $C$ for every $C \in C$. Here, we can replace "closed" by "open". Every space is determined by an open cover. $X$ is dominated by $C$, if the union of any subcollection $C'$ of $C$ is closed in $X$, and the union is determined by $C'$.

Let $X$ be a space, and $P$ be a cover of $X$. Then $P$ is a k-network, if whenever $K \subseteq \cup$ with $K$ compact and $\cup$ open in $X$, then $K \subseteq \cup P' \subseteq \cup$ for some finite $P' \subseteq P$. If we replace " compact " by " single point " then such a cover is called " net (or network) ". k-networks have played a role in $K_0$-spaces (i.e., regular spaces with a countable k-network) and $K$-spaces (i.e., regular spaces with a $\sigma$-locally finite k-network).

Let $A = \{A_\alpha; \alpha \in A\}$ be a collection of subsets of a space $X$. Then $A$ is closure-preserving if $\cup \{A_\alpha; \alpha \in B\} = \cup \{\overline{A_\alpha}; \alpha \in B\}$ for any $B \subseteq A$. $A$ is hereditarily closure-preserving if $\overline{\cup \{B_\alpha; \alpha \in B\}} = \cup \{\overline{B_\alpha}; \alpha \in B\}$ whenever $B \subseteq A$ and $B_\alpha \subseteq A_\alpha$ for each $\alpha \in B$. Every space is dominated by a hereditarily closure-preserving closed cover.
A \( \sigma \)-hereditarily closure-preserving collection is the union of countably many hereditarily closure-preserving collections. We shall use " \( \sigma \)-CP (resp. \( \sigma \)-HCP) " instead of " \( \sigma \)-closure-preserving (resp. \( \sigma \)-hereditarily closure-preserving) ".

\( \mathfrak{A} \) is point-finite (resp. point-countable) if every \( x \in X \) is in at most finitely (resp. countably) many element of \( \mathfrak{A} \).

The concept of CW-complexes due to J. H. Whitehead [5] is well-known. A space \( X \) is a CW-complex if it is a complex with cells \( \{ e_\lambda ; \lambda \} \) satisfying (a) and (b) below.

(a) Each cell \( e_\lambda \) is contained in a finite subcomplex of \( X \).

(b) \( X \) is determined by the closed cover \( \{ \overline{e}_\lambda ; \lambda \} \) of \( X \).

We note that every \( \overline{e}_\lambda \) is not a subcomplex.

As is well-known, every CW-complex \( X \) is dominated by the cover of all finite subcomplexes of \( X \), hence \( X \) is dominated by a cover of compact metric subsets of \( X \).

Let \( \{ e_\lambda ; \lambda \} \) be the cells of a CW-complex \( X \). We shall say that \( \{ e_\lambda ; \lambda \} \) is (\( \sigma \)-) locally finite; (\( \sigma \)-) HCP, etc., if so is respectively the collection \( \{ e_\lambda ; \lambda \} \) of subsets of \( X \). We note that the collection \( \{ e_\lambda ; \lambda \} \) is (\( \sigma \)-) locally finite; (\( \sigma \)-) CP; (\( \sigma \)-) HCP if and only if so is respectively \( \{ \overline{e}_\lambda ; \lambda \} \).
Results. Let \( X \) be a CW-complex with cells \( \{e_{\lambda}; \lambda \} \). Then the following hold. (a) is well-known, and (b) is due to [2].

(a) \( X \) is a paracompact, and \( \sigma \)-space (i.e., \( X \) has a \( \sigma \)-locally finite net).

(b) \( X \) is an \( M_1 \)-space (in the sense of [2]), hence \( X \) has a \( \sigma \)-CP k-network.

(c) \( X \) has a point-countable k-network.

However, every CW-complex is not a metric space (not even a Fréchet space, nor an \( \mathcal{K} \)-space). We have the following characterizations of \( X \). Recall that a space is Fréchet, if whenever \( x \in \overline{A} \), there exists a sequence in \( A \) converging to the point \( x \). (A) is well-known, and (B) is due to [4].

(A) \( X \) is a metric space if and only if \( \{e_{\lambda}; \lambda \} \) is locally finite.

(B) \( X \) is a Fréchet space if and only if \( \{e_{\lambda}; \lambda \} \) is HCP.

(C) \( X \) is an \( \mathcal{K} \)-space if and only if \( \{e_{\lambda}; \lambda \} \) is \( \sigma \)-locally finite.

(D) \( X \) has a \( \sigma \)-HCP k-network if and only if \( \{e_{\lambda}; \lambda \} \) is \( \sigma \)-HCP.

(E) \( X \) is a symmetric space (in the sense of [1]) if and only if \( \{e_{\lambda}; \lambda \} \) is point-finite.

(F) \( X \) has a point-countable closed k-network if and only if \( \{e_{\lambda}; \lambda \} \) is point-countable.
Remark. Let $X$ be a CW-complex with cells $\{e_\lambda; \lambda \}$. 

(1) The property " $\{\overline{e}_\lambda; \lambda \}$ is HCP " need not imply that $X$ has a point-countable closed $k$-network, and not imply that $\{\overline{e}_\lambda; \lambda \}$ is point-countable.

(2) The property " $\{e_\lambda; \lambda \}$ is CP " need not imply that $X$ has a CP or $\sigma$-HCP $k$-network, and not imply that $\{e_\lambda; \lambda \}$ is $\sigma$-HCP.

(3) The property " $X$ is a symmetric space with a $\sigma$-CP $k$-network " need not imply that $X$ has a $\sigma$-HCP $k$-network, and not imply that $\{e_\lambda; \lambda \}$ is $\sigma$-CP.

Question. Let $X$ be a CW-complex with cells $\{e_\lambda; \lambda \}$.
Characterize " $\{e_\lambda; \lambda \}$ is CP (or $\sigma$-CP) " by means of a nice topological property of $X$.

Finally, concerning spaces dominated by compact metric subsets, similarly to CW-complexes the following analogue holds.

Let $X$ be a space dominated by a cover $\{X_\mu; \mu \}$ with each $E_\lambda$ compact metric. Here, $E_\emptyset = X_\emptyset, E_\lambda = X_\lambda \cup \{X_\mu; \mu < \lambda \}$. Then it is possible to replace $\{e_\lambda; \lambda \}$ (or $\{\overline{e}_\lambda; \lambda \}$) by $\{E_\lambda; \lambda \}$ (or $\{\overline{E}_\lambda; \lambda \}$) in (A) $\sim$ (F).
References


