The Ihara zeta functions of algebraic groups

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Introduction

Let $G$ be a connected and reductive algebraic group defined over $\mathbb{Q}$ of hermitian type, and $X$ the bounded symmetric domain induced from the identity component $G(\mathbb{R})_+$ of $G(\mathbb{R})$. Let $\Gamma_0$ be a congruence subgroup of $G(\mathbb{Z}) \cap G(\mathbb{R})_+$, and $M$ the Shimura model of $X/\Gamma_0$. Langlands’ program [10] to parametrize the set $M(\overline{F}_p) \ (p: \text{a prime on which } M \text{ has good reduction})$ was partially achieved by Kottwitz [9] for the Siegel modular case. In this note, when $G$ has a similitude-symplectic embedding (for the classification of such groups, see Satake [16] and Deligne [4]), we shall construct, without detailed proofs, a canonical bijection of a certain subset of $X/\Gamma_0$ to an algebraically defined subset of $M(\overline{F}_p)$. This result can be regarded as a generalization of the result of Ihara [8] on zeta functions of Selberg type (Ihara zeta functions) for congruence subgroups of $PSL_2(\mathbb{Z}[1/p])$.

Following Ihara’s idea, we take a congruence subgroup $\Gamma$ of $G(\mathbb{Z}[1/p]) \cap G(\mathbb{R})_+$ such that $\Gamma \cap G(\mathbb{Z}) = \Gamma_0$. We call $x \in X$ is a $p$-ordinary point if there exists a torsion-free stabilizer of $x$ in $\Gamma$ inducing a $p$-adic structure on a faithful representation space $V$ of $G$ which is compatible with the Hodge structure on $V$ induced from $x$ (this definition is independent of the choice of $V$). When $G$ is a similitude-symplectic group, we show that the reduction map induces a canonical bijection of $\{p$-ordinary points of $X\}/\Gamma_0$ to the ordinary locus of $M(\overline{F}_p)$. This is nothing but a reformation of a result of Deligne [2] and the inverse map corresponds to canonical liftings of ordinary abelian varieties. When $G$ has a
similitude-symplectic embedding, we show that the image of this bijection is algebraically defined which follows from that canonical liftings of abelian varieties preserve their deformations.

1 Zeta functions

1.1. Let $G$ be a linear algebraic group defined over $\mathbb{Q}$ which is connected and reductive. For any field $K$ containing $\mathbb{Q}$, let $G(K)$ denote the group of $K$-rational points of $G$, and put $G_K = G \otimes_{\mathbb{Q}} K$. Let $G(\mathbb{R})_+$ denote the identity component of the Lie group $G(\mathbb{R})$, and put $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. We assume that there exists an $\mathbb{R}$-homomorphism $h : S = R_{\mathbb{C}/\mathbb{R}}(G_{m[\mathbb{C}}) \to G_{\mathbb{R}}$ such that

$$X = \{\text{R-homomorphisms } S \to G_{\mathbb{R}} \text{ conjugate to } h \text{ over } G(\mathbb{R})_+\}$$

is a bounded symmetric domain. Let $V$ be a $\mathbb{Q}$-vector space of finite dimension, and $\phi : G \to GL(V)$ an injective representation defined over $\mathbb{Q}$. Let $L$ be a $\mathbb{Z}$-lattice of $V$, $p$ a prime number, and put $L[1/p] = L \otimes \mathbb{Z}[1/p]$ which is a $\mathbb{Z}[1/p]$-lattice of $V$. Let $\Gamma$ be a congruence subgroup of

$$\phi^{-1}(\text{Aut}(L[1/p]))_+ = \{g \in G(\mathbb{Q})_+ | \phi(g) \in \text{Aut}(L[1/p])\}.$$

One can show that if there exists an integer $n \geq 3$ prime to $p$ such that $\phi(\Gamma) \subset \{g \in \text{Aut}(L[1/p]) | g \equiv 1(n)\}$, then $\Gamma$ is torsion-free. For each $x \in X$, put $\Gamma_x = \{\gamma \in \Gamma | \gamma(x) = x\}$. Let $h_x : S \to G_{\mathbb{R}}$ denote the homomorphism corresponding to $x$. Then $\phi_{\mathbb{R}} \circ h_x$ induces a Hodge decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{i,j} V_x^{i,j}$$

such that for any $(z, z') \in S(\mathbb{C}) = \mathbb{C}^x \times \mathbb{C}^x$ and $v \in V_x^{i,j}$, $(\phi_{\mathbb{R}} \circ h_x)((z, z'))(v) = z^i \cdot z'^j \cdot v$. Then for any $\gamma \in \Gamma_x$, $V_x^{i,j}$ is stable under the action of $\phi(\gamma)_\mathbb{C} = \phi(\gamma) \otimes_{\mathbb{Q}} \mathbb{C}$. Fix an isomorphism $\iota : \mathbb{C} \simarrow \overline{\mathbb{Q}}_{\mathbb{p}}$, and let $\Gamma'_x$ be the set which consists of $\gamma \in \Gamma_x$ such that there exists a rational number $d(\gamma)$ satisfying $\text{ord}_p(\iota(e)) = d(\gamma) \cdot i$ for any eigenvalue $e$ of $\phi(\gamma)_\mathbb{C}$ on each $V_x^{i,j}$. 

2
1.2. Proposition. For any $x \in X$, $\Gamma'_x$ is independent of $\phi$, and for any $x \in X$ and $\gamma \in \Gamma'_x$, $d(\gamma)$ is independent of $\phi$.

1.3. Proposition. Let $Z$ be the centralizer of $h(S(R)) = h(C^x)$ in $G(R)$, and assume that $Z/h(R^x)$ is compact. Then for any $x \in X$ and $\gamma \in \Gamma'_x$, $d(\gamma) \neq 0$ if and only if $\gamma$ is torsion-free.

1.4. Corollary. Assume that there exist a positive integer $g$ and an injective $\mathbb{Q}$-homomorphism of $G$ into the similitude-symplectic algebraic group of size $2g$ which induces a map of $X$ into the Siegel upper half space of degree $g$. Then for any $x \in X$ and $\gamma \in \Gamma'_x$, $d(\gamma)$ is independent of $\phi$.

1.5. Proposition. Let $X^{\text{ord}}(\Gamma)$ be the set consisting of $x \in X$ such that there exists $\gamma \in \Gamma'_x$ with $d(\gamma) \neq 0$. Then $X^{\text{ord}}(\Gamma)$ depends only on the $\mathbb{Q}$-structure of $G$, i.e., it is independent of the choice of $\Gamma$.

1.6. Remark. Propositions 1.2 and 1.5 follow the fact that any representation $G \to GL(W)$ is a direct summand of

$$G \to GL(\oplus_l (V^{\Phi m_l} \otimes (V^{*})^{\Phi n_l}))$$

for some $m_l$ and $n_l$ ([5], Proposition 3.1). Proposition 1.3 follows from the product formula for eigenvalues of $\phi(\gamma)$.

1.7. By Proposition 1.5, $X^{\text{ord}}(\Gamma)$ is independent of $\Gamma$. Then we put $X^{\text{ord}} = X^{\text{ord}}(\Gamma)$, and call it the set of ordinary points of $X$ with respect to $\iota$. For any $x \in X^{\text{ord}}$, let $\Gamma'_x(L)$ be the set consisting of $\gamma \in \Gamma'_x$ such that there exists a decomposition $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ as $\mathbb{Z}_p$-lattices:

$$L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \oplus_{i,j} L^{i,j}$$

which satisfies $\phi(\gamma)_{Q_p}(L^{i,j}) = \iota(e) \cdot L^{i,j}$ for any eigenvalue $e$ of $\phi(\gamma)_G$ on each $V_x^{i,j}$. Put

$$\deg(x) = \begin{cases} \min\{d(\gamma) | \gamma \in \Gamma'_x(L) \text{ with } d(\gamma) > 0\} & \text{if } \Gamma'_x(L) \neq \emptyset, \\ 0 & \text{if } \Gamma'_x(L) = \emptyset. \end{cases}$$
Let $\Gamma_0$ be the subgroup of $\Gamma$ defined by

$$\Gamma_0 = \{ \gamma \in \Gamma | \phi(\gamma) \in \text{Aut}(L) \}.$$ 

Then $\deg(x)$ depends only on the $\Gamma_0$-equivalence class containing $x$. Hence $\deg : X \to \mathbb{R}$ induces the map of

$$\mathbf{P}(\Gamma) = \{ x \in X^{\text{ord}} | \deg(x) : \text{positive integer} \} / \Gamma_0$$

to $\mathbb{N}$, which we denote by the same symbol. Then we define the zeta function $Z(\Gamma, t)$ of $\Gamma$ as the following formal power series with variable $t$:

$$\exp\left( \sum_{r=1}^{\infty} N_r \frac{t^r}{r} \right),$$

where $N_r$ is the cardinality of $\{ P \in \mathbf{P}(\Gamma) | \deg(P) \leq r \}$.

1.8. Conjecture. Let $x$ be any ordinary point of $X$. Then

(1.8.1) $x$ is a special point of $X$ in the sense of [3].

(1.8.2) $\deg(x)$ is a positive integer, and

$$\{ d(\gamma) | \gamma \in \Gamma_x'(L) \} = Z \cdot \deg(x).$$

(1.8.3) If $\Gamma$ is torsion-free, then $\Gamma_x'(L)$ is a cyclic group generated by an element $\gamma \in \Gamma_x'(L)$ with $d(\gamma) = \deg(x)$.

Assuming this conjecture, $Z(\Gamma, t)$ can be regarded as a generalization of Ihara's zeta function for $PSL_2$.

1.9. By results of Satake [15] and Baily-Borel [1], the quotient complex manifold $X/\Gamma_0$ is algebraizable. By results of Shimura [17], Deligne [4] [5], and Milne [13], there exist canonically a number field $K(\Gamma)$ contained in $\mathbb{C}$ and an integral scheme $M_{\Gamma}$ of finite type defined over $K(\Gamma)$, called the canonical model of $X/\Gamma_0$, such that $M_{\Gamma}(\overline{K(\Gamma)}) = X/\Gamma_0$ and the behavior of special point of $M_{\Gamma}$ under the action of $\text{Gal}(\overline{K(\Gamma)}/K(\Gamma))$ is described by the theory of complex multiplication.
If $G = GSp(V)$, then $M_\Gamma$ is the moduli scheme of abelian varieties with polarization and level structure. If $G$ has a similitude-symplectic embedding, then $M_\Gamma$ is the moduli scheme of these objects with certain absolute Hodge cycles.

1.10. *Conjecture.* Let $k(\Gamma)$ be the residue field of $K(\Gamma)$ with respect to $\iota$, and $p^a$ the order of $k(\Gamma)$. Then there exists a separated scheme $F$ of finite type defined over $k(\Gamma)$ whose zeta function $Z(F, t)$ satisfies

$$Z(\Gamma, t) = Z(F, t^a).$$

Moreover, if $M$ has good reduction at $\iota$, then $F$ can be given as a locally closed subset of the special fiber of $M$ with respect to $\iota$.

Assuming this Conjecture, by a result of Dwork [6], one can see that $Z(\Gamma, t)$ is a rational function of $t$.

# 2 Symplectic case

2.1. Let $g$ be a positive integer, $V$ a $\mathbb{Q}$-vector space with basis $\{v_1, \ldots, v_{2g}\}$, and $\psi : V \times V \to \mathbb{Q}$ be the alternating $\mathbb{Q}$-bilinear form given by

$$\psi(v_i, v_j) = \delta_{i,j-g} \ (1 \leq i, j \leq 2g).$$

Let $G$ denote the similitude-symplectic algebraic subgroup $GSp(V, \psi)$ of $GL(V)$ defined over $\mathbb{Q}$ with respect to $\psi$, i.e., $g \in \text{Aut}(V)$ belongs to $G(\mathbb{Q})$ if and only if there exists an element $\nu(g) \in \mathbb{Q}^\times$ such that $\psi(gv, gw) = \nu(g) \cdot \psi(v, w)$ for all $v, w \in V$. Let $h : S \to G_\mathbb{R}$ be the $\mathbb{R}$-homomorphism given by

$$h(a + b\sqrt{-1})(v_i) = \begin{cases} aw + bw' & (1 \leq i \leq g), \\ -bw + aw' & (g + 1 \leq i \leq 2g), \end{cases}$$

where $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$ and $w = v_1 + \ldots + v_g$, $w' = v_{g+1} + \ldots + v_{2g}$. Then $X$ is the Siegel upper half space $H_g$ of degree $g$ which is the bounded symmetric domain induced from $G(\mathbb{R})_+ = \{g \in G(\mathbb{R}) | \nu(g) > 0\}$. Let $L$ be a $\mathbb{Z}$-lattice of $V$ such that $\psi(L \times L) = \mathbb{Z}$, and let $d_L$ be the index of $L$ in $\{v \in V | \psi(v, w) \in \mathbb{Z}$ for any $w \in L\}$. 5
For each $x \in X$, let $A_x$ be the $g$-dimensional abelian variety defined over $C$ such that $H^1(A_x, \mathbb{Z}) = L$ and the Hodge decomposition of $H^1(A_x, C) = V_C$ is given by $h_x$, and $\theta_x$ the polarization of $A_x$ whose Riemann form is given by $\psi$. Then by the correspondence

$$X \ni x \mapsto (A_x, \theta_x, i_x = \text{id.} : H^1(A_x, \mathbb{Z}) \sim L),$$

$X$ becomes the moduli space of the isomorphism classes of triples

$$(A, \theta, i : H^1(A, \mathbb{Z}) \sim L),$$

where $A$ is a $g$-dimensional abelian variety defined over $C$ and $\theta$ is a polarization of $A$ whose Riemann form is given by

$$H^1(A, \mathbb{Z}) \times H^1(A, \mathbb{Z}) \ni (u, v) \mapsto \psi(i(u), i(v)) \in \mathbb{Z}.$$
scheme $A$ over $R$ and an isomorphism $i : A \otimes_R k \rightarrow A_0$ such that $A(p)$ is the product of a multiplicative $p$-divisible group and an étale $p$-divisible group. The pair $(A, i)$ is called the canonical lifting of $A_0$ to $R$. Moreover, it is known that for all ordinary abelian varieties $A_0$ and $B_0$ defined over $k$, the reduction map induces the isomorphism

$$(2.2.1) \quad \text{Hom}_R((A, i), (B, i)) \cong \text{Hom}_k(A_0, B_0),$$

where $(A, i)$ and $(B, i)$ are the canonical liftings of $A_0$ and $B_0$ to $R$ respectively ([11]).

Let $k$ be a finite field $\mathbb{F}_q$, and $A_0$ any ordinary abelian variety defined over $k$. Then by a result of Messing [12], a lifting $(A, i)$ of $A_0$ to $R$ is the canonical lifting if and only if there exists an endomorphism $f$ of $A$ such that $f \otimes_R k$ is the $q$-th power Frobenius endomorphism of $A_0$. Let $(A, i)$ be the canonical lifting of $A_0$ to $R$. Since $A_0$ has complex multiplication ([18]), by (2.2.1), $A$ has also complex multiplication.

2.3. Proposition. For any $x \in X$, the following two conditions are equivalent.

(A) $x$ is an ordinary point of $X$.

(B) There exists an ordinary abelian variety $A_0$ defined over $\overline{\mathbb{F}}_p$ such that $A_x$ is the canonical lifting of $A_0$ with respect to $\iota$, i.e.,

$$A_x \otimes_{C, \iota} \overline{\mathbb{Q}}_p \cong A \otimes_{W(\overline{\mathbb{F}}_p)} \overline{\mathbb{Q}}_p,$$

where $A$ is the canonical lifting of $A_0$ to $W(\overline{\mathbb{F}}_p)$.

2.4. Theorem. Assume that $(p, d_L) = 1$. Then Conjectures 1.8 and 1.10 hold for any congruence subgroup $\Gamma$ of $GSp(L[1/p], \psi)_+$, where $F$ is given as the ordinary locus $U$ of $M_0$.

2.5. Remark. The key point of the proof of Proposition 2.3 and Theorem 2.4 is that any element $\gamma \in \Gamma_x'$ with $d(\gamma) > 0$ is the unique lifting of a Frobenius
endomorphism on a certain ordinary abelian variety defined over a finite field to its canonical lifting. To show the existence of such an abelian variety, we use a result of Honda [7].

3 Classical case

3.1. Let $\phi : G \to GL(V)$, $X$, and $\Gamma$ be as in 1.1, and let $\psi : V \times V \to Q$ and $L$ be as in 2.1. In what follows, assume the following:

(3.1.1) The image of $\phi$ is contained in $GSp(V, \psi)$ and $\phi$ induces a map $h : X \to H_g$.

(3.1.2) There exists a positive integer $n \geq 3$ prime to $p$ such that

$$\phi(\Gamma) \subset \{g \in \text{Aut}(L[1/p]) | g \equiv 1(n)\}.$$ 

Then $h$ is known to be a holomorphic embedding, and by Proposition 1.15 of [3], there exists a unique congruence subgroup $\Gamma'$ of $GSp(L[1/p], \psi)_+$ such that $\Gamma = \Gamma' \cap G(Q)_+$ and the map

$$X/(\Gamma \cap \phi^{-1}(\text{Aut}(L))) \to H_g/(\Gamma' \cap \text{Aut}(L))$$

induced from $h$ is injective. By (3.1.2),

$$\Gamma' \subset \{g \in \text{Aut}(L[1/p]) | g \equiv 1(n)\}.$$ 

Hence $\Gamma'$ and $\Gamma$ are torsion-free.

3.2. Let $M'$ be the canonical model of $H_g/(\Gamma' \cap \text{Aut}(L))$ defined over $K' = K(\Gamma')$. Assume that $(p, d_L) = 1$. Then $M'$ has good reduction with respect to $\iota$. Let $k'$ be the residue field of $K'$ with respect to $\iota$. Let $U$ be the ordinary locus of the reduction of $M'$ with respect to $\iota$. Then $U$ is defined over $k'$. Let $\alpha : U \to M'$ be the map corresponding to the canonical lifting of ordinary abelian varieties, i.e., if $x \in U$ and $X = \alpha(x)$, then $(A_X, \theta_X, \sigma_X)$ is the canonical lifting of $(A_x, \theta_x, \sigma_x)$ with respect to $\iota$. 

8
3.3. **Proposition.** Let $L$ be any finite field extension of $\iota(K')$, and $F_q$ its residue field. Then $\alpha : U \otimes_{K'} F_q \rightarrow M' \otimes_{K', \iota} L$ is a continuous map with respect to the Zariski topology, i.e., if $z \in U \otimes_{K'} F_q$ is a specialization of $y \in U \otimes_{K'} F_q$, then $\alpha(z)$ is a specialization of $\alpha(y)$ in $M' \otimes_{K', \iota} L$.

3.4. **Corollary.** Put $Z = \{ x \in U | \alpha(x) \in M \}$. Then $Z$ is a closed subset of $U$ defined over $k(\Gamma)$.

3.5. **Proposition.** Under Conditions (3.1.1) and (3.1.2), for any $x \in X^{\text{ord}}$,

$$\phi(\Gamma'_x(L)) = \{ \gamma \in (\Gamma_1)^{\text{h}(x)}_k(L) | k(\Gamma) \subset F_{p^{d(\gamma)}} \}.$$ 

3.6. **Theorem.** Assume that $(p, d_L) = 1$. Then under Conditions (3.1.1) and (3.1.2), Conjectures 1.8 and 1.10 hold for $\Gamma$, where $Z$ is given in Corollary 3.4.

3.7. **Remark.** To show Proposition 3.3, by using Serre-Tate's $q$-theory ([11], [12]), we construct an abelian scheme with a polarization and a level structure over a discrete valuation ring whose general and special fibers correspond to $\alpha(y)$ and $\alpha(z)$ respectively. The proof of Proposition 3.5 is straightforward. Theorem 3.6 follows from Theorem 2.4, Corollary 3.4 and Proposition 3.5.

**References**


18. J. Tate, Endomorphisms of abelian varieties over finite fields, Invent.
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