The Ihara zeta functions of algebraic groups

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Introduction

Let $G$ be a connected and reductive algebraic group defined over $\mathbb{Q}$ of hermitian type, and $X$ the bounded symmetric domain induced from the identity component $G(\mathbb{R})_+$ of $G(\mathbb{R})$. Let $\Gamma_0$ be a congruence subgroup of $G(\mathbb{Z}) \cap G(\mathbb{R})_+$, and $M$ the Shimura model of $X/\Gamma_0$. Langlands’ program [10] to parametrize the set $M(\overline{\mathbb{F}}_p)$ ($p$ : a prime on which $M$ has good reduction) was partially achieved by Kottwitz [9] for the Siegel modular case. In this note, when $G$ has a similitude-symplectic embedding (for the classification of such groups, see Satake [16] and Deligne [4]), we shall construct, without detailed proofs, a canonical bijection of a certain subset of $X/\Gamma_0$ to an algebraically defined subset of $M(\overline{\mathbb{F}}_p)$. This result can be regarded as a generalization of the result of Ihara [8] on zeta functions of Selberg type (Ihara zeta functions) for congruence subgroups of $PSL_2(\mathbb{Z}[1/p])$.

Following Ihara’s idea, we take a congruence subgroup $\Gamma$ of $G(\mathbb{Z}[1/p]) \cap G(\mathbb{R})_+$ such that $\Gamma \cap G(\mathbb{Z}) = \Gamma_0$. We call $x \in X$ is a $p$-ordinary point if there exists a torsion-free stabilizer of $x$ in $\Gamma$ inducing a $p$-adic structure on a faithful representation space $V$ of $G$ which is compatible with the Hodge structure on $V$ induced from $x$ (this definition is independent of the choice of $V$). When $G$ is a similitude-symplectic group, we show that the reduction map induces a canonical bijection of $\{p$-ordinary points of $X\}/\Gamma_0$ to the ordinary locus of $M(\overline{\mathbb{F}}_p)$. This is nothing but a reformation of a result of Deligne [2] and the inverse map corresponds to canonical liftings of ordinary abelian varieties. When $G$ has a
similitude-symplectic embedding, we show that the image of this bijection is algebraically defined which follows from that canonical liftings of abelian varieties preserve their deformations.

1 Zeta functions

1.1. Let $G$ be a linear algebraic group defined over $\mathbb{Q}$ which is connected and reductive. For any field $K$ containing $\mathbb{Q}$, let $G(K)$ denote the group of $K$-rational points of $G$, and put $G_K = G \otimes_{\mathbb{Q}} K$. Let $G(\mathbb{R})_+$ denote the identity component of the Lie group $G(\mathbb{R})$, and put $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. We assume that there exists an $\mathbb{R}$-homomorphism $h : S = R_{C/R}(G_{m[C}) \rightarrow G_{\mathbb{R}}$ such that

$$X = \{ \text{R-homomorphisms } S \rightarrow G_{\mathbb{R}} \text{ conjugate to } h \text{ over } G(\mathbb{R})_+ \}$$

is a bounded symmetric domain. Let $V$ be a $\mathbb{Q}$-vector space of finite dimension, and $\phi : G \rightarrow GL(V)$ an injective representation defined over $\mathbb{Q}$. Let $L$ be a $\mathbb{Z}$-lattice of $V$, $p$ a prime number, and put $L[1/p] = L \otimes \mathbb{Z}[1/p]$ which is a $\mathbb{Z}[1/p]$-lattice of $V$. Let $\Gamma$ be a congruence subgroup of

$$\phi^{-1}(\text{Aut}(L[1/p]))_+ = \{ g \in G(\mathbb{Q})_+ | \phi(g) \in \text{Aut}(L[1/p]) \}.$$

One can show that if there exists an integer $n \geq 3$ prime to $p$ such that $\phi(\Gamma) \subset \{ g \in \text{Aut}(L[1/p]) | g \equiv 1(n) \}$, then $\Gamma$ is torsion-free. For each $x \in X$, put $\Gamma_x = \{ \gamma \in \Gamma \mid \gamma(x) = x \}$. Let $h_x : S \rightarrow G_{\mathbb{R}}$ denote the homomorphism corresponding to $x$. Then $\phi_{\mathbb{R}} \circ h_x$ induces a Hodge decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{i,j} V_x^{i,j}$$

such that for any $(z, z') \in S(\mathbb{C}) = \mathbb{C}^x \times \mathbb{C}^x$ and $v \in V_x^{i,j}$, $(\phi_{\mathbb{R}} \circ h_x)((z, z'))(v) = z^{i} \cdot z'^j \cdot v$. Then for any $\gamma \in \Gamma_x$, $V_x^{i,j}$ is stable under the action of $\phi(\gamma)_{\mathbb{C}} = \phi(\gamma) \otimes \mathbb{Q} \mathbb{C}$. Fix an isomorphism $\iota : \mathbb{C} \simarrow \overline{\mathbb{Q}}_p$, and let $\Gamma_x'$ be the set which consists of $\gamma \in \Gamma_x$ such that there exists a rational number $d(\gamma)$ satisfying $\text{ord}_p(\iota(\epsilon)) = d(\gamma) \cdot i$ for any eigenvalue $\epsilon$ of $\phi(\gamma)_{\mathbb{C}}$ on each $V_x^{i,j}$.
1.2. Proposition. For any \( x \in X \), \( \Gamma'_x \) is independent of \( \phi \), and for any \( x \in X \) and \( \gamma \in \Gamma'_x \), \( d(\gamma) \) is independent of \( \phi \).

1.3. Proposition. Let \( Z \) be the centralizer of \( h(S(R)) = h(C^x) \) in \( G(R) \), and assume that \( Z/h(R^x) \) is compact. Then for any \( x \in X \) and \( \gamma \in \Gamma'_x \), \( d(\gamma) \neq 0 \) if and only if \( \gamma \) is torsion-free.

1.4. Corollary. Assume that there exist a positive integer \( g \) and an injective \( \mathbb{Q} \)-homomorphism of \( G \) into the similitude-symplectic algebraic group of size \( 2g \) which induces a map of \( X \) into the Siegel upper half space of degree \( g \). Then for any \( \gamma \in \Gamma'_x \), \( d(\gamma) \neq 0 \) if and only if \( \gamma \) is torsion-free.

1.5. Proposition. Let \( X^{\text{ord}}(\Gamma) \) be the set consisting of \( x \in X \) such that there exists \( \gamma \in \Gamma'_x \) with \( d(\gamma) \neq 0 \). Then \( X^{\text{ord}}(\Gamma) \) depends only on the \( \mathbb{Q} \)-structure of \( G \), i.e., it is independent of the choice of \( \Gamma \).

1.6. Remark. Propositions 1.2 and 1.5 follow the fact that any representation \( G \to GL(W) \) is a direct summand of

\[
G \to GL(\oplus_i (V^{\otimes m_i} \otimes (V^*)^{\otimes n_i}))
\]

for some \( m_i \) and \( n_i \) ([5], Proposition 3.1). Proposition 1.3 follows from the product formula for eigenvalues of \( \phi(\gamma) \).

1.7. By Proposition 1.5, \( X^{\text{ord}}(\Gamma) \) is independent of \( \Gamma \). Then we put \( X^{\text{ord}} = X^{\text{ord}}(\Gamma) \), and call it the set of ordinary points of \( X \) with respect to \( \iota \). For any \( x \in X^{\text{ord}} \), let \( \Gamma'_x(L) \) be the set consisting of \( \gamma \in \Gamma'_x \) such that there exists a decomposition \( L \otimes_{\mathbb{Z}} \mathbb{Z}_p \) as \( \mathbb{Z}_p \)-lattices:

\[
L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \oplus_{i,j} L^{i,j}
\]

which satisfies \( \phi(\gamma)_{\mathbb{Q}_p}(L^{i,j}) = \iota(e) \cdot L^{i,j} \) for any eigenvalue \( e \) of \( \phi(\gamma)_G \) on each \( V_x^{i,j} \). Put

\[
\deg(x) = \begin{cases} 
\min\{d(\gamma)| \gamma \in \Gamma'_x(L) \text{ with } d(\gamma) > 0 \} & \text{if } \Gamma'_x(L) \neq \emptyset, \\
0 & \text{if } \Gamma'_x(L) = \emptyset.
\end{cases}
\]
Let $\Gamma_0$ be the subgroup of $\Gamma$ defined by

$$\Gamma_0 = \{ \gamma \in \Gamma | \phi(\gamma) \in \text{Aut}(L) \}.$$ 

Then $\deg(x)$ depends only on the $\Gamma_0$-equivalence class containing $x$. Hence $\deg : X \to \mathbb{R}$ induces the map of

$$\mathcal{P}(\Gamma) = \{ x \in X^{\text{ord}} | \deg(x) : \text{positive integer} \} / \Gamma_0$$

to $\mathbb{N}$, which we denote by the same symbol. Then we define the zeta function $Z(\Gamma, t)$ of $\Gamma$ as the following formal power series with variable $t$:

$$\exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right),$$

where $N_r$ is the cardinality of $\{ P \in \mathcal{P}(\Gamma) | \deg(P) \leq r \}$.

1.8. *Conjecture.* Let $x$ be any ordinary point of $X$. Then

(1.8.1) $x$ is a special point of $X$ in the sense of [3].

(1.8.2) $\deg(x)$ is a positive integer, and

$$\{ d(\gamma) | \gamma \in \Gamma_x'(L) \} = Z \cdot \deg(x).$$

(1.8.3) If $\Gamma$ is torsion-free, then $\Gamma_x'(L)$ is a cyclic group generated by an element $\gamma \in \Gamma_x'(L)$ with $d(\gamma) = \deg(x)$.

Assuming this conjecture, $Z(\Gamma, t)$ can be regarded as a generalization of Ihara’s zeta function for $PSL_2$.

1.9. By results of Satake [15] and Baily-Borel [1], the quotient complex manifold $X/\Gamma_0$ is algebraizable. By results of Shimura [17], Deligne [4] [5], and Milne [13], there exist canonically a number field $K(\Gamma)$ contained in $\mathbb{C}$ and an integral scheme $M_\Gamma$ of finite type defined over $K(\Gamma)$, called the canonical model of $X/\Gamma_0$, such that $M_\Gamma(\mathbb{C}) = X/\Gamma_0$ and the behavior of special point of $M_\Gamma$ under the action of $\text{Gal}(\overline{K(\Gamma)}/K(\Gamma))$ is described by the theory of complex multiplication.
If \( G = GSp(V) \), then \( M_{\Gamma} \) is the moduli scheme of abelian varieties with polarization and level structure. If \( G \) has a similitude-symplectic embedding, then \( M_{\Gamma} \) is the moduli scheme of these objects with certain absolute Hodge cycles.

1.10. **Conjecture.** Let \( k(\Gamma) \) be the residue field of \( K(\Gamma) \) with respect to \( \iota \), and \( p^{a} \) the order of \( k(\Gamma) \). Then there exists a separated scheme \( F \) of finite type defined over \( k(\Gamma) \) whose zeta function \( Z(F,t) \) satisfies

\[
Z(\Gamma,t) = Z(F,t^{a}).
\]

Moreover, if \( M \) has good reduction at \( \iota \), then \( F \) can be given as a locally closed subset of the special fiber of \( M \) with respect to \( \iota \).

Assuming this Conjecture, by a result of Dwork [6], one can see that \( Z(\Gamma,t) \) is a rational function of \( t \).

## 2 Symplectic case

2.1. Let \( g \) be a positive integer, \( V \) a \( \mathbb{Q} \)-vector space with basis \( \{v_{1}, \ldots, v_{2g}\} \), and \( \psi : V \times V \rightarrow \mathbb{Q} \) be the alternating \( \mathbb{Q} \)-bilinear form given by

\[
\psi(v_{i}, v_{j}) = \delta_{i,j-g} \quad (1 \leq i, j \leq 2g).
\]

Let \( G \) denote the similitude-symplectic algebraic subgroup \( GSp(V,\psi) \) of \( GL(V) \) defined over \( \mathbb{Q} \) with respect to \( \psi \), i.e., \( g \in \text{Aut}(V) \) belongs to \( G(\mathbb{Q}) \) if and only if there exists an element \( \nu(g) \in \mathbb{Q}^{\times} \) such that \( \psi(gv, gw) = \nu(g) \cdot \psi(v, w) \) for all \( v, w \in V \). Let \( h : S \rightarrow G_{\mathbb{R}} \) be the \( \mathbb{R} \)-homomorphism given by

\[
h(a + b\sqrt{-1})(v_{i}) = \begin{cases} aw + bw' & (1 \leq i \leq g), \\ -bw + aw' & (g + 1 \leq i \leq 2g), \end{cases}
\]

where \( (a, b) \in \mathbb{R}^{2} - \{(0,0)\} \) and \( w = v_{1} + \ldots + v_{g}, \quad w' = v_{g+1} + \ldots + v_{2g} \). Then \( X \) is the Siegel upper half space \( H_{g} \) of degree \( g \) which is the bounded symmetric domain induced from \( G(\mathbb{R})_{+} = \{ g \in G(\mathbb{R}) | \nu(g) > 0 \} \). Let \( L \) be a \( \mathbb{Z} \)-lattice of \( V \) such that \( \psi(L \times L) = \mathbb{Z} \), and let \( d_{L} \) be the index of \( L \) in \( \{ v \in V | \psi(v, w) \in \mathbb{Z} \text{ for any } w \in L \} \).
For each $x \in X$, let $A_x$ be the $g$-dimensional abelian variety defined over $\mathbb{C}$ such that $H^1(A_x, \mathbb{Z}) = L$ and the Hodge decomposition of $H^1(A_x, \mathbb{C}) = V_C$ is given by $h_x$, and $\theta_x$ the polarization of $A_x$ whose Riemann form is given by $\psi$. Then by the correspondence

$$X \ni x \mapsto (A_x, \theta_x, i_x = \text{id.} : H^1(A_x, \mathbb{Z}) \simarrow L),$$

$X$ becomes the moduli space of the isomorphism classes of triples

$$(A, \theta, i : H^1(A, \mathbb{Z}) \simarrow L),$$

where $A$ is a $g$-dimensional abelian variety defined over $\mathbb{C}$ and $\theta$ is a polarization of $A$ whose Riemann form is given by

$$H^1(A, \mathbb{Z}) \times H^1(A, \mathbb{Z}) \ni (u, v) \mapsto \psi(i(u), i(v)) \in \mathbb{Z}.$$

Let $p$ be a prime number, and $\Gamma$ a congruence subgroup of $G(\mathbb{Q})_+ \cap \text{Aut}(L[p])$. Then $\Gamma_0 = \Gamma \cap \text{Aut}(L)$ is a subgroup of $G(\mathbb{Q})_+ \cap \text{Aut}(L)$ defined by congruence conditions prime to $p$. Two triples $(A_1, \theta_1, i_1)$ and $(A_2, \theta_2, i_2)$ are said to be $\Gamma_0$-equivalent if there exists an element $\gamma \in \Gamma_0$ such that $(A_1, \theta_1, \gamma \circ i_1)$ and $(A_2, \theta_2, i_2)$ are isomorphic. For each $\Gamma_0$-equivalence class $(A, \theta, \sigma)$, $\sigma$ is called a level $\Gamma_0$-structure of $A$. For each $x \in X$, let $(A_x, \theta_x, \sigma_x)$ denote the $\Gamma_0$-equivalence class containing $(A_x, \theta_x, i_x)$. Let $M = M_{\Gamma}$ be the canonical model of $X/\Gamma_0$ defined over $K(\Gamma)$. Assume that $(p, d_L) = 1$. Then by a result of Mumford [14], $M$ has good reduction with respect to $\iota$. Let $M_0$ denote its special fiber with respect to $\iota$. Let $U$ be the ordinary locus of $M_0$, i.e., the open subscheme of $M_0$ defined over $k(\Gamma)$ consisting of all points of $M_0$ corresponding to ordinary abelian varieties.

2.2. Let $k$ be a perfect field of characteristic $p$, and $A_0$ an ordinary abelian variety defined over $k$ of dimension $g$. Then the $p$-divisible group $A_0(p)$ associated with $A_0$ is the product of a multiplicative $p$-divisible group and an étale $p$-divisible group. Let $W(k)$ denote the ring of Witt vectors over $k$, and $R$ a complete discrete valuation ring containing $W(k)$ with residue field $k$. Then by a result of Lubin-Tate-Serre [11], there exists a unique pair $(A, i)$ up to isomorphism of an abelian
scheme $A$ over $R$ and an isomorphism $i : A \otimes_{R} k \rightarrow A_{0}$ such that $A(p)$ is the product of a multiplicative $p$-divisible group and an étale $p$-divisible group. The pair $(A, i)$ is called the canonical lifting of $A_{0}$ to $R$. Moreover, it is known that for all ordinary abelian varieties $A_{0}$ and $B_{0}$ defined over $k$, the reduction map induces the isomorphism

$$(2.2.1) \quad \text{Hom}_{R}((A, i), (B, i)) \rightarrow \text{Hom}_{k}(A_{0}, B_{0}),$$

where $(A, i)$ and $(B, i)$ are the canonical liftings of $A_{0}$ and $B_{0}$ to $R$ respectively ([11]).

Let $k$ be a finite field $\mathbb{F}_{q}$, and $A_{0}$ any ordinary abelian variety defined over $k$. Then by a result of Messing [12], a lifting $(A, i)$ of $A_{0}$ to $R$ is the canonical lifting if and only if there exists an endomorphism $f$ of $A$ such that $f \otimes_{R} k$ is the $q$-th power Frobenius endomorphism of $A_{0}$. Let $(A, i)$ be the canonical lifting of $A_{0}$ to $R$. Since $A_{0}$ has complex multiplication ([18]), by (2.2.1), $A$ has also complex multiplication.

2.3. Proposition. For any $x \in X$, the following two conditions are equivalent.

(A) $x$ is an ordinary point of $X$.

(B) There exists an ordinary abelian variety $A_{0}$ defined over $\overline{\mathbb{F}}_{p}$ such that $A_{x}$ is the canonical lifting of $A_{0}$ with respect to $\iota$, i.e.,

$$A_{x} \otimes_{C, \iota} \overline{\mathbb{Q}}_{p} \cong A \otimes_{W(\overline{\mathbb{F}}_{p})} \overline{\mathbb{Q}}_{p},$$

where $A$ is the canonical lifting of $A_{0}$ to $W(\overline{\mathbb{F}}_{p})$.

2.4. Theorem. Assume that $(p, d_{L}) = 1$. Then Conjectures 1.8 and 1.10 hold for any congruence subgroup $\Gamma$ of $GSp(L[1/p], \psi)_{+}$, where $F$ is given as the ordinary locus $U$ of $M_{0}$.

2.5. Remark. The key point of the proof of Proposition 2.3 and Theorem 2.4 is that any element $\gamma \in \Gamma'_{x}$ with $d(\gamma) > 0$ is the unique lifting of a Frobenius
endomorphism on a certain ordinary abelian variety defined over a finite field to its canonical lifting. To show the existence of such an abelian variety, we use a result of Honda [7].

3 Classical case

3.1. Let $\phi : G \to GL(V)$, $X$, and $\Gamma$ be as in 1.1, and let $\psi : V \times V \to Q$ and $L$ be as in 2.1. In what follows, assume the following:

(3.1.1) The image of $\phi$ is contained in $GSp(V, \psi)$ and $\phi$ induces a map $h : X \to H_g$.

(3.1.2) There exists a positive integer $n \geq 3$ prime to $p$ such that

$$\phi(\Gamma) \subset \{g \in Aut(L[1/p]) | g \equiv 1(n)\}.$$

Then $h$ is known to be a holomorphic embedding, and by Proposition 1.15 of [3], there exists a unique congruence subgroup $\Gamma'$ of $GSp(L[1/p], \psi)_+$ such that

$$\Gamma = \Gamma' \cap G(Q)_+$$

and the map

$$X/(\Gamma \cap \phi^{-1}(Aut(L))) \to H_g/(\Gamma' \cap Aut(L))$$

induced from $h$ is injective. By (3.1.2),

$$\Gamma' \subset \{g \in Aut(L[1/p]) | g \equiv 1(n)\}.$$

Hence $\Gamma'$ and $\Gamma$ are torsion-free.

3.2. Let $M'$ be the canonical model of $H_g/(\Gamma' \cap Aut(L))$ defined over $K' = K(\Gamma')$. Assume that $(p, d_L) = 1$. Then $M'$ has good reduction with respect to $\iota$. Let $k'$ be the residue field of $K'$ with respect to $\iota$. Let $U$ be the ordinary locus of the reduction of $M'$ with respect to $\iota$. Then $U$ is defined over $k'$. Let $\alpha : U \to M'$ be the map corresponding to the canonical lifting of ordinary abelian varieties, i.e., if $x \in U$ and $X = \alpha(x)$, then $(A_X, \theta_X, \sigma_X)$ is the canonical lifting of $(A_x, \theta_x, \sigma_x)$ with respect to $\iota$. 

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3.3. Proposition. Let $L$ be any finite field extension of $\iota(K')$, and $\mathbb{F}_q$ its residue field. Then $\alpha : U \otimes_{K'} \mathbb{F}_q \to M' \otimes_{K', \iota} L$ is continuous map with respect to the Zariski topology, i.e., if $z \in U \otimes_{K'} \mathbb{F}_q$ is a specialization of $y \in U \otimes_{K'} \mathbb{F}_q$, then $\alpha(z)$ is a specialization of $\alpha(y)$ in $M' \otimes_{K', \iota} L$.

3.4. Corollary. Put $Z = \{x \in U | \alpha(x) \in M\}$. Then $Z$ is a closed subset of $U$ defined over $k(\Gamma)$.

3.5. Proposition. Under Conditions (3.1.1) and (3.1.2), for any $x \in X^{\text{ord}}$,

$$\phi(\Gamma'_x(L)) = \{\gamma \in (\Gamma_1)^\prime_{h(x)}(L) | k(\Gamma) \subset \mathbb{F}_{p^{d(\gamma)}}\}.$$

3.6. Theorem. Assume that $(p, d_L) = 1$. Then under Conditions (3.1.1) and (3.1.2), Conjectures 1.8 and 1.10 hold for $\Gamma$, where $Z$ is given in Corollary 3.4.

3.7. Remark. To show Proposition 3.3, by using Serre-Tate’s $q$-theory ([11], [12]), we construct an abelian scheme with a polarization and a level structure over a discrete valuation ring whose general and special fibers correspond to $\alpha(y)$ and $\alpha(z)$ respectively. The proof of Proposition 3.5 is straightforward. Theorem 3.6 follows from Theorem 2.4, Corollary 3.4 and Proposition 3.5.

References


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