数理解析研究所講究録
761巻 1991年 76-102

Two kinds of constructions of generalized Kac-Moody algebras as subalgebras of Kac-Moody algebras

Satoshi NAITO

Department of Mathematics, Kyoto University

Introduction

In [1], R. Borcherds introduced the notion of generalized Kac-Moody algebras (= GKM algebras for short), which is a good generalization of Kac-Moody algebras. The aim of this paper is to exhibit two kinds of constructions of GKM algebras as subalgebras of a symmetrizable Kac-Moody algebra.

In the first construction (§2), we get GKM algebras as what we call *regular subalgebras*. The regular subalgebra is a natural infinite dimensional analogue of a regular semi-simple subalgebra in the sense of Dynkin of a finite dimensional complex semi-simple Lie algebra. The latter plays an important role in the classification of semi-simple subalgebras (cf. [2]).

In the second construction (§3), we get GKM algebras as what we call *folding subalgebras*. This folding subalgebra is generated by certain sums, corresponding to a diagram automorphism, of the Chevalley generators of the Kac-Moody algebra. In the finite dimensional case, a folding subalgebra coincides with a fixed point subalgebra of a certain automorphism (cf. [4]). In the general case, folding subalgebras are subalgebras of fixed point subalgebras, but not necessarily coincide with them (cf. [5]).

Folding subalgebras have some good properties: one is the inheritance of a standard invariant form, and another is the complete reducibility when representations are regarded as those of folding subalgebras (§3.4 and 3.5). In the affine case, a certain class of folding subalgebras and their branching rules are studied in [3].

§1. Generalized Kac-Moody algebras

In this section, we explain the notion of generalized Kac-Moody algebras for later use. Here we adopt the definition in [4, Chap. 11] of generalized Kac-Moody algebras, which is a little different from the original one in [1].

1.1. Definitions and notations.

Definition 1.1 ([4]). A real n×n matrix $A = (a_{ij})_{i,j=1}^{n}$ is called a GGCM (= generalized GCM) if it satisfies the following:

(C1) either $a_{ii} = 2$ or $a_{ii} \leq 0$;
(C2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$;
(C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Note that when $a_{ii} = 2$ for every i, A is a generalized Cartan matrix (= GCM).

Definition 1.2 ([4]). A triple $(\mathfrak{h}, \Pi = \{\alpha_i\}_{i=1}^{n}, \Pi^{\vee} = \{\alpha_i^{\vee}\}_{i=1}^{n})$ is called a *realization* of the GGCM $A = (a_{ij})_{i,j=1}^{n}$ if it satisfies the following:

(R1) $\mathfrak{h}$ is a finite dimensional complex vector space, and $\dim_{\mathbb{C}} \mathfrak{h} = 2n - \operatorname{rank} A$;

(R2) $\Pi^{\vee} = \{\alpha_i^{\vee}\}_{i=1}^{n}$ is a linearly independent subset of $\mathfrak{h}$, and $\Pi = \{\alpha_i\}_{i=1}^{n}$ is a linearly independent subset of $\mathfrak{h}^*$ (the algebraic dual of $\mathfrak{h}$);

(R3) $<\alpha_j, \alpha_i^{\vee}> = a_{ij}$ $(1 \le i, j \le n)$.

Let $\tilde{\mathfrak{g}}(A)$ be a Lie algebra with generators e_i, f_i $(1 \le i \le n)$, and $\mathfrak{h}$, and the following defining relations:

$$\text{(F1)} \quad \begin{aligned} &[e_i, f_j] = \delta_{ij}\alpha_i^{\vee} \quad (1 \le i, j \le n), \\ &[h, h'] = 0 \quad (h, h' \in \mathfrak{h}), \\ &[h, e_i] = <\alpha_i, h>e_i, \ [h, f_i] = -<\alpha_i, h>f_i \quad (1 \le i \le n, \ h \in \mathfrak{h}). \end{aligned}$$

We define $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$, where $\mathfrak{r}$ is a unique maximal ideal among the ideals of $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially. This Lie algebra $\mathfrak{g}(A)$ is called a *generalized Kac-Moody algebra* (= GKM algebra) associated to A. (Especially when A is a GCM, $\mathfrak{g}(A)$ is called a *Kac-Moody algebra.*) The subalgebra $\mathfrak{h}$ is called the *Cartan subalgebra* of $\mathfrak{g}(A)$, and the elements e_i, f_i $(1 \le i \le n)$ are called the *Chevalley generators* of $\mathfrak{g}(A)$.

It is shown in [4, Chap. 11] that, when the GGCM A is symmerizable, the GKM algebra $\mathfrak{g}(A)$ is a Lie algrbra with the generators e_i, f_i $(1 \le i \le n)$, and $\mathfrak{h}$, and the defining relations (F1) and the following:

$$\text{(F2)}\quad \begin{aligned} &(\mathrm{ad}\ e_i)^{1-a_{ij}}\, e_j = 0,\ (\mathrm{ad}\ f_i)^{1-a_{ij}}\, f_j = 0,\ \text{if } a_{ii} = 2,\ j\neq i, \\ &[e_i,\ e_j] = 0,\ [f_i,\ f_j] = 0 \text{ if } a_{ij} = 0. \end{aligned}$$

Here the GGCM A is called *symmetrizable* if there exist an invertible diagonal matrix D such that $D^{-1}A$ is symmetric. (In this case, we put $B := D^{-1}A$ and call it a *symmetrization* of A. And $\mathfrak{g}(A)$ is called a *symmetrizable GKM algebra.*)

We often consider the derived subalgebra $[\mathfrak{g}(A), \mathfrak{g}(A)]$ of $\mathfrak{g}(A)$ instead of a GKM algebra $\mathfrak{g}(A)$, and also call it a GKM algebra. Note that $\mathfrak{g}(A) = [\mathfrak{g}(A), \mathfrak{g}(A)] + \mathfrak{h}$, and that $[\mathfrak{g}(A), \mathfrak{g}(A)] \cap \mathfrak{h} = \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee$. Further, a GKM algebra $[\mathfrak{g}(A), \mathfrak{g}(A)]$ is a Lie algebra with the generators e_i, f_i, $\alpha_i^\vee$ $(1\le i\le n)$, and the defining relations (F2) and the following:

$$\text{(F'1)}\quad \begin{aligned} &[e_i,\ f_j] = \delta_{ij}\alpha_i^\vee \quad (1\le i,j\le n), \\ &[\alpha_i^\vee,\ \alpha_j^\vee] = 0 \quad (1\le i,j\le n), \\ &[\alpha_i^\vee,\ e_j] = a_{ij}e_j,\ [\alpha_i^\vee,\ f_j] = -a_{ij}f_j \quad (1\le i,j\le n). \end{aligned}$$

1.2. Roots and invariant bilinear forms. Let $A = (a_{ij})_{i,j=1}^{n}$ be a GGCM, $\mathfrak{g}(A)$ be a GKM algebra associated to A, and $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}(A)$. Then, we have the root space decomposition of $\mathfrak{g}(A)$:

$\mathfrak{g}(A) = \mathfrak{h} \oplus \sum_{\alpha\in\Delta}^{\oplus} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x\in\mathfrak{g}(A);\ [h,\ x] = \langle\alpha,\ h\rangle x,$ for all $h \in \mathfrak{h}\}$ $(\alpha \in \mathfrak{h}^*)$, and $\Delta = \{\alpha\in\mathfrak{h}^*\setminus\{0\};\ \mathfrak{g}_\alpha \neq \{0\}\}$. We call $\mathfrak{g}_\alpha$ the root space attached to α, and Δ the root system of $\mathfrak{g}(A)$.

Note that $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ $(1 \le i \le m)$. Moreover, every root $\alpha \in \Delta$ is of the form $\alpha = \sum_{i=1}^{n} k_i \alpha_i$, $k_i \in \mathbb{Z}_{\ge 0}$ $(1 \le i \le n)$ or $k_i \in \mathbb{Z}_{\ge 0}$ $(1 \le i \le n)$. So we call $\Pi = \{\alpha_i\}_{i=1}^{n} \subset \Delta$ the simple root system, and call $\alpha \in \Delta$ positive root (resp. negative root) if the coefficients of α_i in the above expression are all non-negative (resp. non-positive). Denote by Δ_+ (resp. Δ_-) the set of all positive (resp. negative) roots, and by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the subalgebra of $\mathfrak{g}(A)$ generated by e_i, $1 \le i \le n$ (resp. f_i, $1 \le i \le n$). Note that $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+}^{\oplus} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \sum_{\alpha \in \Delta_+}^{\oplus} \mathfrak{g}_{-\alpha}$.

Now, suppose that $A = (a_{ij})_{i,j=1}^{n}$ is a symmetrizable GGCM. Then, we can take a real diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n)$ and a real symmetric matrix $B = (b_{ij})_{i,j=1}^{n}$ such that $A = DB$ and $\varepsilon_i > 0$ $(1 \le i \le n)$. So, we fix such a decomposition of A and a complementary subspace $\mathfrak{h}''$ to $\mathfrak{h}' := \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee$ in $\mathfrak{h}$. Then, there exists uniquely a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ such that:

(B1) $(\alpha_i^\vee | h) = \langle \alpha_i, h \rangle \varepsilon_i$ $(1 \le i \le n,\ h \in \mathfrak{h})$,

(B2) $(h | h') = 0$ $(h,\ h' \in \mathfrak{h})$.

This bilinear form is called a *standard invariant form* on $\mathfrak{g}(A)$. Note that the restriction $(\cdot|\cdot)|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. So, we can define an isomorphism $\nu;\ \mathfrak{h} \longrightarrow \mathfrak{h}^*$, determined by $\langle \nu(h), h' \rangle = (h|h')$ $(h,\ h' \in \mathfrak{h})$ as well as an induced bilinear form on $\mathfrak{h}^*$. We know the following equalities (cf. [4]):

(1.2.1) $\nu(\alpha_i^\vee) = \varepsilon_i\alpha_i \quad (1\le i\le n)$,

(1.2.2) $(\alpha_i|\alpha_j) = b_{ij} = a_{ij}/\varepsilon_i \quad (1\le i,j\le n)$,

(1.2.3) $\alpha_i^\vee = 2/(\alpha_i|\alpha_i)\cdot\nu^{-1}(\alpha_i) \quad (1\le i\le n)$,

(1.2.4) $(\alpha_i^\vee|\alpha_j^\vee) = b_{ij}\varepsilon_i\varepsilon_j \quad (1\le i,j\le n)$,

(1.2.5) $[x, y] = (x|y)\cdot\nu^{-1}(\alpha)$ for $x\in\mathfrak{g}_\alpha$, $y\in\mathfrak{g}_{-\alpha}$ $(\alpha \in \Delta)$.

§2. Regular subalgebras of a symmetrizable Kac-Moody algebra

For the detailed accounts of this section, see [7].

2.1. Construction of regular subalgebras. In this subsection, we assume that $A = (a_{ij})_{i,j=1}^n$ is a symmetrizable GCM. Other notations are the same as in §1. For each i $(1\le i\le n)$, we define a simple reflection r_i of the space $\mathfrak{h}^*$ by $r_i(\lambda) := \lambda - \langle\lambda, \alpha_i^\vee\rangle\alpha_i$ $(\lambda \in \mathfrak{h}^*)$. The subgroup W of $GL(\mathfrak{h}^*)$ generated by r_i $(1\le i\le n)$ is called the *Weyl group* of $\mathfrak{g}(A)$.

Definition 2.1. A subset $\bar{\Pi} = \{\beta_1, \cdots, \beta_p, \beta_{p+1}, \cdots, \beta_{p+q}\}$ of the root system Δ of $\mathfrak{g}(A)$ is called *fundamental* if it satisfies the following:

(1) $\bar{\Pi} = \{\beta_i\}_{i=1}^{p+q}$ is a linearly independent subset of $\mathfrak{h}^*$;

(2) $\beta_i - \beta_j \notin \Delta \cup \{0\}$ $(1\le i\ne j\le p+q)$;

(3) $\beta_i \in \Delta^{re}$ $(1\le i\le p)$ and $\beta_j \in \Delta_+ \cap \Delta^{im}$ $(p+1\le j\le p+q)$.

Here $\Delta^{re} := W\cdot\Pi$ is the set of all *real roots* of $\mathfrak{g}(A)$ and $\Delta^{im} := \Delta\setminus\Delta^{re}$ is the set of all *imaginary roots* of $\mathfrak{g}(A)$.

Remark 2.1. The above definition is a generalization of that (which is the case $q = 0$) by Morita [6].

Now, for each imaginary root β_j $(p+1 \leq j \leq p+q)$, we define $\beta_j^\vee := \nu^{-1}(\beta_j) \in \mathfrak{h}$. For real root $\beta_i = w(\alpha_{k_i})$ $(w \in W)$, $1 \leq i \leq p$, $\beta_i^\vee := w(\alpha_{k_i}^\vee) \in \mathfrak{h}$ has been defined as a dual real root of β_i, and we know $\beta_i^\vee = 2/(\beta_i | \beta_i) \cdot \nu^{-1}(\beta_i)$.

Proposition 2.1. Let $\bar{\Pi} = \{\beta_i\}_{i=1}^{p+q}$ be a fundamental subset of Δ, and put $\bar{A} = (\bar{a}_{ij})_{i,j=1}^{p+q}$, where $\bar{a}_{ij} = \langle \beta_j, \beta_i^\vee \rangle$. Then, $\bar{A}$ is a symmetrizable GGCM. Further, $\bar{A}$ is a GCM if and only if every β_i is a real root.

Remark 2.2. The symmetrizability of $\bar{A}$ is shown as follows: Put $\bar{B} := ((\beta_i | \beta_j))_{i,j=1}^{p+q}$ and $\bar{D} := \mathrm{diag}(2/(\beta_1 | \beta_1), \cdots, 2/(\beta_p | \beta_p), 1, \cdots, 1)$. Then we have $\bar{A} = \bar{D}\bar{B}$, ${}^t(\bar{A}) = \bar{A}$, and $\det \bar{D} \neq 0$. Note that the j-th diagonal element is 1, while $\bar{a}_{jj} = (\beta_j | \beta_j) \leq 0$ $(p+1 \leq j \leq p+q)$.

Proposition 2.2. There exists a vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$, such that the triple $(\mathfrak{h}_0, \{\beta_i|_{\mathfrak{h}_0}\}_{i=1}^{p+q}, \{\beta_i^\vee\}_{i=1}^{p+q})$ is a realization of the GGCM $\bar{A}$.

We take and fix non-zero vectors $E_i \in \mathfrak{g}_{\beta_i}$ and $F_i \in \mathfrak{g}_{-\beta_i}$ such that $[E_i, F_i] = \beta_i^\vee$ $(1 \leq i \leq p+q)$. Note that such vectors always exist (see (1.2.5)). Let $\bar{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by

E_i, F_i ($1 \le i \le p+q$), and a vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$ such that the triple $(\mathfrak{h}_0, \{\beta_i|_{\mathfrak{h}_0}\}_{i=1}^{p+q}, \{\beta_i^\vee\}_{i=1}^{p+q})$ is a realization of $\bar{A}$. We call this kind of subalgebra a *regular subalgebra* of $\mathfrak{g}(A)$.

Theorem 2.1. Any regular subalgebra of $\mathfrak{g}(A)$ is canonically isomorphic to a GKM algebra. In fact, the above regular subalgebra $\bar{\mathfrak{g}}$ is isomorphic to a GKM algebra $\mathfrak{g}(\bar{A})$.

2.2. An embedding of GKM algebras into a symmetrizable Kac-Moody algebra. In §2.1, we constructed a regular subalgebra $\bar{\mathfrak{g}}$ of a symmetrizable Kac-Moody algebra $\mathfrak{g}(A)$, and $\bar{\mathfrak{g}}$ is canonically isomorphic to a GKM algebra $\mathfrak{g}(\bar{A})$. This GGCM $\bar{A}$ has the following *strong symmetrizability* ($\bar{\mathrm{S}}\bar{\mathrm{S}}$) (cf. Remark 2.2) and *integrality* ($\bar{\mathrm{I}}$), if we normalize a standard invariant form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ in such a way that $(\alpha_i|\alpha_j) \in \mathbb{Z}$ ($1 \le i,j \le n$).

($\bar{\mathrm{S}}\bar{\mathrm{S}}$) There exist an invertible rational diagonal matrix $\bar{D}$ and a rational symmetric matrix $\bar{B} = (\bar{b}_{ij})_{i,j=1}^{p+q}$, such that $\bar{A} = \bar{D}\bar{B}$ and $\bar{D} = \mathrm{diag}(\bar{\varepsilon}_1, \cdots, \bar{\varepsilon}_p, 1, \cdots, 1)$,

($\bar{\mathrm{I}}$) $\bar{a}_{ij} \in \mathbb{Z}$ ($1 \le i,j \le p+q$).

Recall that the j-th diagonal element of $\bar{D}$ is 1, if $\bar{a}_{jj} \le 0$.

Conversely, we have the following theorem.

Theorem 2.2. Let $A = (-a_{ij})_{i,j=1}^{p+q}$ be a GGCM, such that

(G1) $-a_{ii} = 2$ ($1 \le i \le p$),

(G2) $-a_{jj} \le 0$ ($p+1 \le j \le p+q$).

And assume that A satisfies the following *strong symmetrizability condition* (SS) and *integrality condition* (I):

(SS) There exist an invertible rational diagonal matrix D and a rational symmetric matrix $B = (b_{ij})_{i,j=1}^{p+q}$, such that $A = DB$ and $D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_p, 1, \cdots, 1)$;

(I) $a_{ij} \in \mathbb{Z}$ $(1 \leq i, j \leq p+q)$.

Then, there exists a symmetrizable GCM $\bar{A} = (\bar{a}_{ij})_{i,j=1}^{2(p+q)}$ such that the GKM algebra $\mathfrak{g}(A)$ is canonically isomorphic to a regular subalgebra $\bar{\mathfrak{g}}$ of the Kac-Moody algebra $\mathfrak{g}(A)$.

PROOF. Note that we can assume that $\varepsilon_i > 0$ $(1 \leq i \leq p)$ without any loss of generality.

STEP 1. First, we put for j $(p+1 \leq j \leq p+q)$ as follows:

$$u_j := \begin{cases} a_{jj} & (a_{jj} \neq 0) \\ 1 & (a_{jj} = 0) \end{cases}, \qquad v_j := -(a_{jj} + 2),$$

$$X_j := \begin{cases} -2^{-1}(a_{jj} + \dfrac{2}{a_{jj}}) & (a_{jj} \neq 0) \\ -1 & (a_{jj} = 0) \end{cases},$$

$$Y_j := \begin{cases} a_{jj}^{-1} & (a_{jj} \neq 0) \\ 1 & (a_{jj} = 0) \end{cases}.$$

And for i $(1 \leq i \leq p)$, $Z_i := -\varepsilon_i^{-1}$.

Second, we define $2(p+q) \times 2(p+q)$ matrix $\bar{D}$ and $\bar{B}$ as follows:

$$\bar{D} := \mathrm{diag}(-Z_1^{-1}, -Z_1^{-1}, -Z_2^{-1}, -Z_2^{-1}, \cdots, -Z_p^{-1}, -Z_p^{-1}, 2u_{p+1}, 2u_{p+1}, 2u_{p+2}, 2u_{p+2}, \cdots, 2u_{p+q}, 2u_{p+q}),$$

$\bar{B} := (\bar{b}_{ij})_{i,j=1}^{2(p+q)}$, where

$$\begin{bmatrix} \bar{b}_{2k-1,2\ell-1} & \bar{b}_{2k-1,2\ell} \\ \bar{b}_{2k,2\ell-1} & \bar{b}_{2k,2\ell} \end{bmatrix} := \begin{bmatrix} Z_k \cdot a_{k\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad \left(\begin{array}{l} 1 \leq k \leq p, \\ 1 \leq \ell \leq p+q, \ \ell \neq k \end{array}\right),$$

$$\begin{bmatrix} \bar{b}_{2k-1,2k-1} & \bar{b}_{2k-1,2k} \\ \bar{b}_{2k,2k-1} & \bar{b}_{2k,2k} \end{bmatrix} := \begin{bmatrix} -2Z_k & Z_k \\ Z_k & -2Z_k \end{bmatrix} \quad (1 \leq k \leq p),$$

$$\begin{bmatrix} \bar{b}_{2k-1,2\ell-1} & \bar{b}_{2k-1,2\ell} \\ \bar{b}_{2k,2\ell-1} & \bar{b}_{2k,2\ell} \end{bmatrix} := \begin{bmatrix} -a_{k\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad \left(\begin{array}{l} p+1 \leq k \leq p+q \\ 1 \leq \ell \leq p+q, \ \ell \neq k \end{array}\right),$$

and
$$\begin{bmatrix} \bar{b}_{2k-1,2k-1} & \bar{b}_{2k-1,2k} \\ \bar{b}_{2k,2k-1} & \bar{b}_{2k,2k} \end{bmatrix} := \begin{bmatrix} Y_k & X_k \\ X_k & Y_k \end{bmatrix} \quad (p+1 \leq k \leq p+q).$$

As $(b_{ij})_{i,j=1}^{p+q} = B = D^{-1}A = \mathrm{diag}(-Z_1, \cdots, -Z_p, 1, \cdots, 1) \cdot (-a_{ij})_{i,j=1}^{p+q}$, we have

$$b_{ij} = \begin{cases} Z_i \cdot a_{ij} & (1 \leq i \leq p, \ 1 \leq j \leq p+q) \\ -a_{ij} & (p+1 \leq i \leq p+q, \ 1 \leq j \leq p+q) \end{cases}.$$

Therefore, $\bar{B}$ is clearly a symmetric matrix (see also Figure 1). Finally, we put $\bar{A} = \bar{D}\bar{B}$. Then, $\bar{A} = (\bar{a}_{ij})_{i,j=1}^{2(p+q)}$, where

$$\begin{bmatrix} \bar{a}_{2k-1,2\ell-1} & \bar{a}_{2k-1,2\ell} \\ \bar{a}_{2k,2\ell-1} & \bar{a}_{2k,2\ell} \end{bmatrix} := \begin{bmatrix} -a_{k\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad \left(\begin{array}{l} 1 \leq k \leq p \\ 1 \leq \ell \leq p+q, \ \ell \neq k \end{array}\right),$$

$$\begin{bmatrix} \bar{a}_{2k-1,2k-1} & \bar{a}_{2k-1,2\ell} \\ \bar{a}_{2k,2k-1} & \bar{a}_{2k,2k} \end{bmatrix} := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad (1 \leq k \leq p),$$

$$\begin{bmatrix} \bar{a}_{2k-1,2\ell-1} & \bar{a}_{2k-1,2\ell} \\ \bar{a}_{2k,2\ell-1} & \bar{a}_{2k,2\ell} \end{bmatrix} := \begin{bmatrix} -2a_{k\ell}\cdot u_k & 0 \\ 0 & 0 \end{bmatrix} \quad \left({p+1\le k\le p+q \atop 1\le \ell\le p+q,\ \ell\ne k}\right),$$

and $$\begin{bmatrix} \bar{a}_{2k-1,2k-1} & \bar{a}_{2k-1,2k} \\ \bar{a}_{2k,2k-1} & \bar{a}_{2k,2k} \end{bmatrix} := \begin{bmatrix} 2 & v_k \\ v_k & 2 \end{bmatrix} \quad (p+1\le k\le p+q).$$

Hence $\bar{A}$ is clearly a symmetrizable GCM (see also Figure 2).

STEP 2. Let $\mathfrak{g}(\bar{A})$ be a Kac-Moody algebra associated to the above GCM $\bar{A}$, $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}(\bar{A})$, and $\Pi = \{\alpha_i\}_{i=1}^{2(p+q)} \subset \mathfrak{h}^*$ be the simple root system of $\mathfrak{g}(\bar{A})$. And let $(\cdot|\cdot)$ be a standard invariant form on $\mathfrak{g}(\bar{A})$ corresponding to the decomposition: $\bar{A} = \bar{D}\bar{B}$. Now, we put $\beta_k := \alpha_{2k-1} + \alpha_{2k}$ $(1\le k\le p+q)$. Then, $\beta_k \in \Delta_+$ since $\bar{a}_{2k-1,2k} \le -1$ $(1\le k\le p+q)$. Obviously, $\beta_i - \beta_j \notin \Delta \cup \{0\}$ $(1\le i\ne j\le p+q)$ and $\bar{\Pi} := \{\beta_i\}_{i=1}^{p+q} \subset \mathfrak{h}^*$ is linearly independent. So, $\bar{\Pi}$ is a fundamental subset of the root system Δ of $\mathfrak{g}(\bar{A})$. Therefore, we see from Theorem 2.1 that there exists a regular subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}(\bar{A})$, which is canonically isomorphic to a GKM algebra $\mathfrak{g}(\tilde{A})$ associated to $\tilde{A} := (\langle\beta_j, \beta_i^\vee\rangle)_{i,j=1}^{p+q}$. Theorem 2.2 now follows from the following claim:

CLAIM. $\tilde{A} = A$.

PROOF of the claim. Since $\beta_k = \alpha_{2k-1} + \alpha_{2k}$ $(1\le k\le p+q)$, we have

$$(\beta_k|\beta_\ell) = \bar{b}_{2k-1,2\ell-1} + \bar{b}_{2k-1,2\ell} + \bar{b}_{2k,2\ell-1} + \bar{b}_{2k,2\ell} =$$

$$\begin{cases} Z_k \cdot a_{k\ell} & (1 \le k \le p,\ 1 \le \ell \le p+q) \\ -a_{k\ell} & (p+1 \le k \le p+q,\ 1 \le \ell \le p+q) \end{cases}.$$

Recall that for $\alpha \in \Delta$, $\alpha \in \Delta^{re}$ if and only if $(\alpha|\alpha) > 0$, and $\alpha \in \Delta^{im}$ if and only if $(\alpha|\alpha) \le 0$, where $(\cdot|\cdot)$ is a standard invariant form on $\mathfrak{g}(\bar{A})$. Hence $\beta_k \in \Delta^{re} \cap \Delta_+$ $(1 \le k \le p)$ and $\beta_k \in \Delta^{im} \cap \Delta_+$ $(p+1 \le k \le p+q)$ from the above equalities. Therefore, for k $(1 \le k \le p)$ we have $\tilde{a}_{k\ell} = \langle \beta_\ell, \beta_k^\vee \rangle = 2(\beta_k|\beta_\ell)/(\beta_k|\beta_k) = -a_{k\ell}$. And for k $(p+1 \le k \le p+q)$, we have $\tilde{a}_{k\ell} = \langle \beta_\ell, \beta_k^\vee \rangle = (\beta_k|\beta_\ell) = -a_{k\ell}$. In conclusion, $\tilde{A} = (\tilde{a}_{k\ell})_{k,\ell=1}^{p+q} = (-a_{k\ell})_{k,\ell=1}^{p+q} = A$. Thus, the claim has been proved. □

2.3. Sufficient conditions for the strong symmetrizability of GGCM. Let $A = (-a_{ij})_{i,j=1}^{p+q}$ be a GGCM satisfying the integrality condition (I), and reordered so that (G1) and (G2) hold. Here, we give sufficient conditions for the strong symmetrizability (SS) of the above GGCM A. Obviously, if A is symmetric, then A satisfies the condition (SS). For other sufficient conditions, we have the following.

Proposition 2.3. If the above GGCM A is symmetrizable, and satisfies the following two conditions, then A satisfies (SS).

(a) For every i $(1 \le i \le p)$, there exists j $(p+1 \le j \le p+q)$ such that $a_{ij} \ne 0$.

(b) The principal submatrix $A^{im} := (-a_{ij})_{i,j=p+1}^{p+q}$ is indecomposable and symmetric.

PROOF. Note that if A satisfies (SS), then A^{im} is necessarily symmetric.

For each i $(1\le i\le p)$, we put $Z_i := -a_{ji}\cdot a_{ij}^{-1}$ if $a_{ij} \ne 0$ $(p+1\le j\le p+q)$. Such a j exists from the condition (a), and Z_i does not depend on the choice of j from the condition (b). Then, we put $D := \mathrm{diag}(-Z_1^{-1}, -Z_2^{-1}, \cdots, -Z_p^{-1}, 1, 1, \cdots, 1)$ and $B := (b_{ij})_{i,j=1}^{p+q}$, where

$$b_{ij} := \begin{cases} Z_i\cdot a_{ij} & (1\le i\le p,\ 1\le j\le p+q) \\ -a_{ij} & (p+1\le i\le p+q,\ 1\le j\le p+q)\end{cases}.$$

Then, A = DB, and B is symmetric from the condition (b). Hence A satisfies (SS). □

Corollary 2.1. Let $A = (-a_{ij})_{i,j=1}^{p+q}$ be a symmetrizable GGCM with (G1), (G2), and satisfying the integrality condition (I). If $a_{ij} \ne 0$ $(1\le i\ne j\le p+q)$ and $A^{im} := (-a_{ij})_{i,j=p+1}^{p+q}$ is symmetric, then A satisfies the strong symmetrizability condition (SS).

§3. Folding subalgebras of a symmetrizable GKM algebra

3.1. Diagram automorphisms of a GGCM. Let $A = (a_{ij})_{i,j=1}^{n}$ be an indecomposable, symmetrizable GGCM, $\mathfrak{g}(A)$ be a GKM algebra associated to A, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(A)$. Fix a decomposition of A: A = DB, where $D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n)$ $(\varepsilon_i > 0,$ $1\le i\le n)$, $B = (b_{ij})_{i,j=1}^{n}$ $(b_{ij} = b_{ji} \in \mathbb{R},\ 1\le i,j\le n)$. And let $(\cdot|\cdot)$ be a standard invariant form on $\mathfrak{g}(A)$ corresponding to the above decomposition of A.

Definition 3.1. A permutation π on $I := \{1, 2, \cdots, n\}$ is called a *diagram automorphism* of a GGCM A if

(D1) $a_{\pi(i),\pi(j)} = a_{ij}$ for every i, j $(1\le i,j\le n)$.

Lemma 3.1. We have $\varepsilon_{\pi(i)} = \varepsilon_i$ for every i $(1\le i\le n)$.

PROOF. Since A = DB, we have $a_{ij} = \varepsilon_i b_{ij}$ and $a_{\pi(i),\pi(j)} = \varepsilon_{\pi(i)} b_{\pi(i),\pi(j)}$ $(1\le i,j\le n)$. So, we have $\varepsilon_i b_{ij} = \varepsilon_{\pi(i)} b_{\pi(i),\pi(j)}$ and $\varepsilon_j b_{ji} = \varepsilon_{\pi(j)} b_{\pi(j),\pi(i)}$ $(1\le i,j\le n)$ from (D1). Therefore, we get $\varepsilon_i \varepsilon_{\pi(j)} b_{ij} b_{\pi(j),\pi(i)} = \varepsilon_j \varepsilon_{\pi(i)} b_{ji} b_{\pi(j),\pi(i)}$ $(1\le i,j\le n)$. Note that if $a_{ij} = a_{\pi(i),\pi(j)} \neq 0$, then $b_{ij} \neq 0$ and $b_{\pi(i),\pi(j)} \neq 0$. Hence, we obtain that $\varepsilon_i \varepsilon_{\pi(j)} = \varepsilon_j \varepsilon_{\pi(i)}$ or $\varepsilon_i^{-1} \cdot \varepsilon_{\pi(i)} = \varepsilon_j^{-1} \cdot \varepsilon_{\pi(j)}$ if $a_{ij} \neq 0$. Therefore, there exists a positive constant M such that $\varepsilon_i^{-1} \cdot \varepsilon_{\pi(i)} = M$ for every i $(1\le i\le n)$, since the GGCM A is indecomposable. So, we have $\Pi_{i=1}^n \dfrac{\varepsilon_{\pi(i)}}{\varepsilon_i} = M^n$. On the other hand, the left hand side equals to $\dfrac{\Pi_{i=1}^n \varepsilon_{\pi(i)}}{\Pi_{i=1}^n \varepsilon_i} = 1$. Therefore, we have $M^n = 1$, so that $M = 1$. Thus the assertion has been proved. □

Now, since a diagram automorphism π is a permutation on I, we can express it uniquely as a commuting product of cyclic permutations: There exists uniquely a decomposition of I into its disjoint subsets I_j $(1\le j\le m)$, such that the restriction $\pi_j := \pi|_{I_j}$ of π to I_j is a cyclic permutation $(1\le j\le m)$.

Lemma 3.2. For every j_1, j_2 $(1 \le j_1, j_2 \le m)$ and i_1, $i_2 \in I_{j_1}$, we have $\sum_{k \in I_{j_2}} a_{k,i_1} = \sum_{k \in I_{j_2}} a_{k,i_2}$.

PROOF. Since $\pi_{j_1} = \pi|_{I_{j_1}}$ is a cyclic permutation, it is enough to assume that $i_2 = \pi(i_1)$. Then, we have $\sum_{k \in I_{j_2}} a_{k,i_2} =$ $\sum_{k \in I_{j_2}} a_{k,\pi(i_1)} = \sum_{k \in I_{j_2}} a_{\pi^{-1}(k),i_1}$ (by (D1)) $= \sum_{k \in I_{j_2}} a_{k,i_1}$ □

Now, we set $\bar{a}_{ij} := \sum_{k \in I_i} a_{k\ell}$ for $\ell \in I_j$ $(1 \le i, j \le m)$. From Lemma 3.2, the right hand side does not depend on the choice of $\ell \in I_j$.

Lemma 3.3. If $\bar{a}_{ii} = \sum_{k \in I_i} a_{k,\ell}$ $(\ell \in I_i)$ is a positive real number, then we have the following two cases:

CASE (A) $a_{k\ell} = 0$ for every k, $\ell \in I_i$ $(k \ne \ell)$, and $a_{kk} = 2$ for every $k \in I_i$,

CASE (B) $a_{kk} = 2$ for every $k \in I_i$, and there exists a decomposition of I_i into its disjoint subsets $I_i^{(p)}$ $(1 \le p \le t_i)$ such that $|I_i^{(p)}| = 2$ for every p $(1 \le p \le t_i)$, $a_{k\ell} = 0$ for every $k \in I_i^{(p)}$, $\ell \in I_i^{(q)}$ $(1 \le p \ne q \le t_i)$, and $a_{k\ell} = a_{\ell k} = -1$ for every k, $\ell \in I_i^{(p)}$ $(k \ne \ell)$, $1 \le p \le t_i$. Here $|S|$ denotes the number of elements of a set S.

PROOF. First, recall that $\sum_{k \in I_i} a_{k\ell}$ $(\ell \in I_i)$ does not depend on the choice of $\ell \in I_i$. Therefore, we have the following for every $\ell \in I_i$:

$\bar{a}_{ii} = \sum_{k\in I_i} a_{k\ell} > 0 \implies a_{\ell\ell} > 0$ (since $a_{k\ell} \leq 0$, for $k\neq\ell$)

$\implies a_{\ell\ell} = 2$ (since $A = (a_{ij})_{i,j=1}^{n}$ is a GGCM)

$\implies a_{\ell k} \in \mathbb{Z}_{\leq 0}$ for $k \in I_i$ $(k\neq\ell)$ (since A satisfies(C2)).

Therefore, we have $a_{\ell\ell} = 2$ $(\ell \in I_i)$, and $a_{k\ell} \in \mathbb{Z}_{\leq 0}$ $(k\neq\ell,\ k,\ \ell \in I_i)$. Hence we deduce that $\sum_{k\in I_i\setminus\{\ell\}} a_{k\ell} = 0$ or -1 $(\ell \in I_i)$ from the assumption. Moreover, $\sum_{k\in I_i\setminus\{\ell\}} a_{k\ell}$ does not depend on $\ell \in I_i$ since $a_{\ell\ell} = 2$ $(\ell \in I_i)$. Therefore, we have the following two cases:

CASE (A) $\sum_{k\in I_i\setminus\{\ell\}} a_{k\ell} = 0$, for every $\ell \in I_i$,

CASE (B) $\sum_{k\in I_i\setminus\{\ell\}} a_{k\ell} = -1$, for every $\ell \in I_i$.

In case (A), we have $a_{k\ell} = 0$ $(k \neq \ell,\ k,\ \ell \in I_i)$, since $a_{k\ell} \leq 0$ $(k\neq\ell)$. In case (B), for every $\ell \in I_i$, there exists exactly one $k_\ell \in I_i\setminus\{\ell\}$ such that $a_{k_\ell,\ell} = -1$ and $a_{k\ell} = 0$ $(k \in I_i\setminus\{\ell, k_\ell\})$, since $a_{k\ell} \in \mathbb{Z}_{\leq 0}$ $(k \neq \ell,\ k,\ \ell \in I_i)$. Therefore, for every $\ell \in I_i$, $a_{\ell,k_\ell} = -1$ since $a_{k\ell} = 0$ implies $a_{\ell k} = 0$. Thus the assertion is now proved. □

3.2. Construction of folding subalgebras. Notations are the same as in 3.1. We put for j $(1\leq j\leq m)$

$$E'_j := \sum_{k\in I_j} e_k,\quad F'_j := \sum_{k\in I_j} f_k,\quad H'_j := \sum_{k\in I_j} \alpha_k^\vee,\ \text{and}\ \beta_j := \sum_{k\in I_j} \alpha_k,$$

where e_i, f_i $(1\leq i\leq n)$ are the Chevalley generators, $\{\alpha_i\}_{i=1}^{n}$ is the set of all simple roots, and $\{\alpha_i^\vee\}_{i=1}^{n}$ is the set of all simple coroots of the GKM algebra $\mathfrak{g}(A)$.

Proposition 3.1. We have the following equations:

$$(3.2.1)\quad [H_i', E_j'] = \bar{a}_{ij}E_j' \qquad (1\le i,j\le m),$$

$$(3.2.2)\quad [H_i', F_j'] = -\bar{a}_{ij}F_j' \qquad (1\le i,j\le m),$$

$$(3.2.3)\quad [E_i', F_j'] = \delta_{ij}H_i' \qquad (1\le i,j\le m).$$

PROOF. PROOF for (3.2.1).

$[H_i', E_j'] = [\Sigma_{k\in I_i}\alpha_k^\vee, \Sigma_{\ell\in I_j}e_\ell] = \Sigma_{k,\ell}[\alpha_k^\vee, e_\ell] = \Sigma_{k,\ell}<\alpha_\ell, \alpha_k^\vee>e_\ell =$ $\Sigma_{k,\ell}\, a_{k\ell}e_\ell = \Sigma_{\ell\in I_j}(\Sigma_{k\in I_i}a_{k\ell})e_\ell = \Sigma_{\ell\in I_j}\bar{a}_{ij}e_\ell = \bar{a}_{ij}(\Sigma_{\ell\in I_j}e_\ell) =$ $\bar{a}_{ij}E_j'$.

The proof is similar for (3.2.2), and (3.2.3) is obvious. □

Now, we say, for i $(1\le i\le m)$, "CASE X(i)" if $\bar{a}_{ii} \le 0$, or if $\bar{a}_{ii} > 0$ and case (A) happens, and "CASE Y(i)" if $\bar{a}_{ii} > 0$ and case (B) happens. And we put $\hat{A} := (\hat{a}_{ij})_{i,j=1}^{m}$, where

$$\hat{a}_{ij} := \begin{cases} \bar{a}_{ij} & \text{if } X(i) \\ 2\bar{a}_{ij} & \text{if } Y(i) \end{cases}.$$

Moreover, we put for i $(1\le i\le m)$

$$H_i := \begin{cases} H_i' & \text{if } X(i) \\ 2H_i' & \text{if } Y(i) \end{cases},$$

$$E_i := \begin{cases} E_i' & \text{if } X(i) \\ \sqrt{2}\cdot E_i' & \text{if } Y(i) \end{cases}, \qquad F_i := \begin{cases} F_i' & \text{if } X(i) \\ \sqrt{2}\cdot F_i' & \text{if } Y(i) \end{cases}.$$

Then, we have the following propositions.

Proposition 3.2. $\hat{A}$ is a GGCM.

PROOF. We have to check (C1)-(C3) in Definition 1.1.

PROOF for (C1). In the case X(i), we have

$$\hat{a}_{ii} = \bar{a}_{ii} = \begin{cases} \leq 0 & \text{if } \bar{a}_{ii} \leq 0, \\ 2 & \text{if } \bar{a}_{ii} > 0 \text{ and in the case (A).} \end{cases}$$

In the case Y(i) (i.e., if $\bar{a}_{ii} > 0$ and in the case (B)), we have

$$\hat{a}_{ii} = 2\bar{a}_{ii} = 2\times(2 - 1) = 2.$$

PROOF for (C2). We have $\bar{a}_{ij} = \sum_{k\in I_i} a_{k\ell}$ $(\ell \in I_j) \leq 0$ $(1\leq i\neq j\leq m)$, since $I_i \cap I_j = \phi$. So, we have $\hat{a}_{ij} \leq 0$. Further, if $\hat{a}_{ii} = 2$, then $\bar{a}_{ii} > 0$, and so $a_{kk} = 2$ for every $k \in I_i$. Therefore, $a_{k\ell} \in \mathbb{Z}_{\leq 0}$ $(k \in I_i,\ \ell\neq k)$ since $A = (a_{ij})_{i,j=1}^{n}$ is a GGCM. Hence, $\bar{a}_{ij} = \sum_{k\in I_i} a_{k\ell} \in \mathbb{Z}_{\leq 0}$.

PROOF for (C3). For i, j $(1\leq i\neq j\leq m)$, we have $\bar{a}_{ij} = \sum_{k\in I_i} a_{k\ell}$ $(\ell \in I_j)$ and $a_{k\ell} \leq 0$ $(k \in I_i)$. So, $\bar{a}_{ij} = 0$ if and only if $a_{k\ell} = 0$ for every $k \in I_i$. Note that $\sum_{k\in I_i} a_{k\ell}$ $(\ell \in I_j)$ does not depend on $\ell \in I_j$. Therefore, $\bar{a}_{ij} = 0$ if and only if $a_{k\ell} = 0$ for every $k \in I_i$ and every $\ell \in I_j$. Now, recall that $a_{k\ell} = 0$ implies $a_{\ell k} = 0$ $(k \in I_i,\ \ell \in I_j)$ since A is a GGCM. Hence, $\bar{a}_{ij} = 0$ implies $\bar{a}_{ji} = 0$. In conclusion, $\hat{A} = (\hat{a}_{ij})_{i,j=1}^{m}$ is a GGCM. □

Remark 3.1. From the proof for (C3), we can deduce that the GGCM $\hat{A}$ is indecomposable.

Remark 3.2. Even if A is a GCM, $\hat{A}$ is not a GCM except for the case that for every i $(1 \le i \le m)$, case (A) or case (B) in Lemma 3.3 happens.

Proposition 3.3. The GGCM $\hat{A}$ is symmetrizable.

PROOF. First, note that $(\alpha_{\pi(i)}|\alpha_{\pi(j)}) = b_{\pi(i),\pi(j)} = \varepsilon_{\pi(i)}^{-1} \cdot a_{\pi(i),\pi(j)} = \varepsilon_i^{-1} \cdot a_{ij} = b_{ij} = (\alpha_i|\alpha_j)$ from Lemma 3.1. So, we get

$$(3.2.4) \quad (\alpha_{i_1}|\Sigma_{k \in I_{j_2}} \alpha_k) = (\alpha_{i_2}|\Sigma_{k \in I_{j_2}} \alpha_k)$$

for every j_1, j_2 $(1 \le j_1, j_2 \le m)$ and i_1, $i_2 \in I_{j_1}$, by the same way as Lemma 3.2. Now, we put $\varepsilon^{(j)} := \varepsilon_k$ $(k \in I_j)$ for j $(1 \le j \le m)$ (not depend on the choice of $k \in I_j$ by Lemma 3.1). Then, we have $\bar{a}_{ij} = \Sigma_{k \in I_i} a_{k\ell}$ $(\ell \in I_j)$ $= \Sigma_{k \in I_i} \varepsilon_k b_{k\ell} = \varepsilon^{(i)} \cdot \Sigma_{k \in I_i} (\alpha_k|\alpha_\ell) = \varepsilon^{(i)} \cdot (\Sigma_{k \in I_i} \alpha_k|\alpha_\ell) = \varepsilon^{(i)} \cdot |I_j|^{-1} (\Sigma_{k \in I_i} \alpha_k|\Sigma_{\ell \in I_j} \alpha_\ell)$ (by (3.2.4)) $= \varepsilon^{(i)} |I_j|^{-1} (\beta_i|\beta_j)$. We define $\hat{B} := (|I_i|^{-1} |I_j|^{-1} (\beta_i|\beta_j))_{i,j=1}^{m}$, and $\hat{D} := \mathrm{diag}(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \cdots, \hat{\varepsilon}_m)$, where

$$\hat{\varepsilon}_i := \begin{cases} \varepsilon^{(i)} |I_i| & \text{in case } X(i) \\ 2\varepsilon^{(i)} |I_i| & \text{in case } Y(i) \end{cases} .$$

Then, $\hat{A} = \hat{D}\hat{B}$, ${}^t(\hat{B}) = \hat{B}$, and $\det \hat{D} \neq 0$. Hence the GGCM $\hat{A}$ is symmetrizable. □

Let $\hat{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_i, F_i, and H_i $(1 \le i \le m)$.

Definition 3.2. We call the above subalgebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}(A)$ the *folding subalgebra* (of $\mathfrak{g}(A)$) corresponding to a diagram automorphism π of A.

3.3. Main result. In this subsection, we obtain the following theorem, which is the main result of §3.

Theorem 3.1. Any folding subalgebra of $\mathfrak{g}(A)$ is canonically isomorphic to the derived algebra of a GKM algebra. Let $\hat{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_i, F_i, and H_i $(1 \le i \le m)$. Then, the canonical isomorphism Φ of the derived algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ onto the folding subalgebra $\hat{\mathfrak{g}}$ is given as:

$$\Phi(\hat{e}_i) = E_i,\ \Phi(\hat{f}_i) = F_i,\ \text{and}\ \Phi(\hat{\alpha}_i^\vee) = H_i\ (1 \le i \le m).$$

Here $\hat{e}_i$, $\hat{f}_i$ $(1 \le i \le m)$ are the Chevalley generators, and $\{\hat{\alpha}_i^\vee\}_{i=1}^m$ is the set of all simple coroots of the GKM algebra $\mathfrak{g}(\hat{A})$.

PROOF. We have to check that E_i, F_i, and H_i $(1 \le i \le m)$ satisfy all the defining relations for the symmetrizable GKM algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$. However, the relations (F'1) in §1.1 are clear from Proposition 3.1. So, we have only to check relations (F2).

STEP 1. We first check $[E_i, E_j] = 0$ and $[F_i, F_j] = 0$ if $\hat{a}_{ii} \le 0$ and $\hat{a}_{ij} = 0$ $(1 \le i \ne j \le m)$. As shown in the proof of Proposition 3.2, $\hat{a}_{ij} = 0$ if and only if $a_{k\ell} = 0$ for every $k \in I_i$

and $\ell \in I_j$. And note that $a_{k\ell} = 0$ implies $[e_k, e_\ell] = 0$ since $\mathfrak{g}(A)$ is a GKM algebra (see §1). So, we have $[E_i', E_j'] = [\Sigma_{k\in I_i} e_k, \Sigma_{\ell\in I_j} e_\ell] = \Sigma_{k,\ell}[e_k, e_\ell] = 0$. Hence, we have $[E_i, E_j] = 0$. The proof is similar for the relation $[F_i, F_j] = 0$.

STEP 2. Next, we check that $(\mathrm{ad}\, E_i)^{1-\hat{a}_{ij}} E_j = 0$ and $(\mathrm{ad}\, F_i)^{1-\hat{a}_{ij}} F_j = 0$ if $\hat{a}_{ii} = 2$ and $j\neq i$. We only check the relation for E_i and E_j, since the proof is similar for F_i and F_j.

Obviously, $\mathbb{C}E_i + \mathbb{C}H_i + \mathbb{C}F_i$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, with standard basis $\{E_i, H_i, F_i\}$. Further, we have $[H_i, E_j] = \hat{a}_{ij}E_j$ and $[F_i, E_j] = 0$, as shown above. Therefore, the relation $(\mathrm{ad}\, E_i)^{1-\hat{a}_{ij}} E_j = 0$ follows from the local nilpotency of ad E_i on $\mathfrak{g}(A)$ and the following relation in the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ $(\cong \mathbb{C}E_i + \mathbb{C}H_i + \mathbb{C}F_i)$:

$$[F_i, E_i^k] = -k(k-1)E_i^{k-1} - kE_i^{k-1}H_i \quad (k\geq 1).$$

Therefore, we have only to show the local nilpotency of ad E_i on $\mathfrak{g}(A)$.

STEP 3. Now, $\hat{a}_{ii} = 2$ implies $\bar{a}_{ii} > 0$, and so either case (A) or case (B) happens by Lemma 3.3.

In CASE (A). We have $a_{k\ell} = 0$ for $k, \ell \in I_i$ $(k\neq\ell)$ and $a_{kk} = 2$ for $k \in I_i$. In this case, $E_i = E_i' = \Sigma_{k\in I_i} e_k$. Note that ad e_k $(k \in I_i)$ is locally nilpotent on $\mathfrak{g}(A)$ since $a_{kk} = 2$. Further, we have $[e_k, e_\ell] = 0$ $(k, \ell \in I_i, k\neq\ell)$ since $a_{k\ell} = 0$. Therefore, ad $E_i = \sum_{k \in I_i} (\text{ad } e_k)$ is locally nilpotent on $\mathfrak{g}(A)$.

In CASE (B). Also in this case, ad e_k $(k \in I_i)$ is locally nilpotent on $\mathfrak{g}(A)$ since $a_{kk} = 2$. Recall that $|I_i^{(p)}| = 2$, and $a_{k\ell} = a_{\ell k} = -1$ $(k, \ell \in I_i^{(p)}, k \neq \ell)$ for every p $(1 \le p \le t_i)$. Say $I_i^{(p)} = \{k, \ell\}$, and put $z := [e_k, e_\ell]$. Then, $[e_k, z] = [e_k, [e_k, e_\ell]] = (\text{ad } e_k)^2 e_\ell = 0$, since $a_{k\ell} = -1$ (see §1). Similarly, $[e_\ell, z] = 0$ since $a_{\ell k} = -1$. On the other hand, $r_\ell(\alpha_k) = \alpha_k - \langle\alpha_k, \alpha_\ell^\vee\rangle\alpha_\ell = \alpha_k - a_{\ell k}\alpha_k = \alpha_k + \alpha_\ell$. So, $\alpha_k + \alpha_\ell$ is a real root of the GKM algebra $\mathfrak{g}(A)$ since $a_{kk} = a_{\ell\ell} = 2$ (see [4, Chap. 11]). Therefore, ad $z = \text{ad}([e_k, e_\ell])$ is locally nilpotent on $\mathfrak{g}(A)$, since $[e_k, e_\ell] \in \mathfrak{g}_{\alpha_k + \alpha_\ell}$ (cf. [4, Chap. 3]). Now, we can easily deduce that

$\text{ad}(e_k + e_\ell)$ is locally nilpotent on $\mathfrak{g}(A)$, from the local nilpotency of ad e_k, ad e_ℓ, and ad z, and the following commutation relations:

$[e_k, z] = 0$, $[e_\ell, z] = 0$, $[e_k, e_\ell] = z$.

Hence, if we put $e_i^{(p)} := \sum_{k \in I_i^{(p)}} e_k$ $(1 \le p \le t_i)$, then ad $e_i^{(p)}$ is locally nilpotent on $\mathfrak{g}(A)$. Further, we have $[e_i^{(p)}, e_i^{(q)}] = [\sum_{k \in I_i^{(p)}} e_k, \sum_{\ell \in I_i^{(q)}} e_\ell] = \sum_{k,\ell} [e_k, e_\ell] = 0$ $(1 \le p \neq q \le t_i)$, since $a_{k\ell} = 0$ for $k \in I_i^{(p)}$, $\ell \in I_i^{(q)}$. Therefore, ad $E_i' = \sum_{k \in I_i} (\text{ad } e_k) = \sum_{p=1}^{t_i} (\text{ad } e_i^{(p)})$ is locally nilpotent on $\mathfrak{g}(A)$, and so is ad E_i.

STEP 4. Thus we have checked all the defining relations for the GKM algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$. Therefore, we get the surjective homomorphism $\Phi: [\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})] \longrightarrow \hat{\mathfrak{g}}$, such that $\Phi(\hat{e}_i) = E_i$, $\Phi(\hat{f}_i) = F_i$, and $\Phi(\hat{\alpha}_i^\vee) = H_i$ $(1 \le i \le m)$. Because H_i $(1 \le i \le m) \in \mathfrak{h}$ are linearly independent, we have (Ker Φ) $\cap \sum_{i=1}^{m} \mathbb{C}\hat{\alpha}_i^\vee = \{0\}$. On the other hand, it is easy to see that Ker Φ is a graded ideal of $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})] = (\sum_{i=1}^{m} \mathbb{C}\hat{\alpha}_i^\vee) \oplus \sum_{\hat{\alpha}\in\hat{\Delta}}^{\oplus} \hat{\mathfrak{g}}_{\hat{\alpha}}$, where $\hat{\Delta}$ is the root system of the GKM algebra $\mathfrak{g}(\hat{A})$ and $\hat{\mathfrak{g}}_{\hat{\alpha}}$ is the root space attached to $\hat{\alpha} \in \hat{\Delta}$. Hence Ker $\Phi = \{0\}$ (see [4, Chap. 1]). This completes the proof of the Theorem. □

Remark 3.3. From the above theorem, we see that, with respect to the operation of making folding subalgebras, the category of Kac-Moody algebras is not closed (see Remark 3.2), but the category of GKM algebras is closed.

3.4. The inheritance of a standard invariant form. Here we prove a certain inheritance of a standard invariant form to a folding subalgebra, as was proved in the case of a regular subalgebra in [7]. The notations are the same as in §3.1 - 3.3.

Proposition 3.4. Let $(\cdot|\cdot)_1$ be a standard invariant form on $\mathfrak{g}(\hat{A})$ corresponding to the decomposition of $\hat{A}$: $\hat{A} = \hat{D}\hat{B}$ in Proposition 3.3. Then, the restriction of the standard invariant form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ to the folding subalgebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}(A)$ can be identified with the restriction of $(\cdot|\cdot)_1$ to the derived algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ of $\mathfrak{g}(\hat{A})$ through the canonical isomorphism Φ: $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})] \longrightarrow \hat{\mathfrak{g}}$, except for the following case:
A is of type $A_{n-1}^{(1)}$ $(n\geq 2)$, π is a cyclic permutation on $\{1, 2, \cdots \cdots, n\}$, and $\hat{A} = 0_1$ (1×1 zero-matrix).

PROOF. First, recall that the GGCM $\hat{A}$ is indecomposable (see Remark 3.1). Second, it is easy to check that $(H_i | H_j) = (\hat{\alpha}_i^\vee | \hat{\alpha}_j^\vee)_1$ for i, j $(1 \le i, j \le m)$. Then, the proposition follows from the following fact (see [4, Chap. 2]):

FACT. If the GGCM $\hat{A}$ is indecomposable, any two invariant bilinear forms on the derived algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ of $\mathfrak{g}(\hat{A})$ are proportional. □

Remark 3.4. If A is of type $A_{n-1}^{(1)}$ $(n \ge 2)$ and π is a cyclic permutation on $\{1, 2, \cdots, n\}$, then $\hat{A} = O_1$ and the proportional constant is n.

3.5. The complete reducibility. For integrable highest weight modules of a Kac-Moody algebra, we have the following complete reducibility with repect to its folding subalgebra.

Proposition 3.5. Let $A = (a_{ij})_{i,j=1}^n$ be an indecomposable, symmetrizable GCM, $\Lambda \in \mathfrak{h}^*$ be a dominant integral weight, and $L(\Lambda)$ be an integrable highest weight module with highest weight Λ over the Kac-Moody algebra $\mathfrak{g}(A)$. Assume that $\hat{A}$ is again a GCM. Then, as $\hat{\mathfrak{g}}$-modules, $L(\Lambda)$ is isomorphic to a direct sum of $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$-modules $L(\lambda)$ such that $\lambda \in (\sum_{i=1}^m \mathbb{C}\hat{\alpha}_i^\vee)^*$, $\langle \lambda, \hat{\alpha}_i^\vee \rangle \in \mathbb{Z}_{\ge 0}$ $(1 \le i \le m)$, under the identification: $\hat{\mathfrak{g}} \cong [\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$.

PROOF. We use the Kac's complete reducibility theorem (see [4, Chap. 10]). And we can show the conditions of Kac's theorem in exactly the same way as in the step 3 of the proof of Theorem 3.1. □

References

[1] R. Borcherds, Generalized Kac-Moody algebras, J. Algebra, 115 (1988), 501-512.

[2] E. B. Dynkin, Semi-simple subalgebras of semi-simple Lie algebras, Amer. Math. Soc. Transl., 6 (1957), 111-244.

[3] M. Jimbo and T. Miwa, On a duality of branching rules for affine Lie algebras, Adv. Stud. Pure Math., 6 (1985), 17-65.

[4] V. G. Kac, Infinite dimensional Lie algebras (3rd ed.), Cambridge University Press, Cambridge-New York, 1990.

[5] F. Levstein, A classification of involutive automorphisms of an affine Kac-Moody Lie algebra, J. Algebra, 114 (1988), 489-518.

[6] J. Morita, Certain rank two subsystems of Kac-Moody root systems, in "Infinite dimensional Lie algebras and groups (ed. by V. G. Kac)", Adv. Ser. in Math. Phys. 7, World Scientific, Singapore-New Jersey, 1989, pp. 52-56.

[7] S. Naito, On regular subalgebras of Kac-Moody algebras and their associated invariant forms - Symmetrizable case (to appear).

Satoshi NAITO
Department of Mathematics
Faculty of Science
Kyoto University
Sakyo-ku, Kyoto 606
Japan.

$$\overline{B}=\begin{array}{c}\\ \\ 2p\\ \\ \\ \\ \\ \\ 2q\\ \\ \\ \\ \end{array}\left\{\begin{array}{cccccccccccccc}
\multicolumn{7}{c}{2p} & \multicolumn{7}{c}{2q}\\
-2Z_1 & Z_1 & Z_1a_{12} & 0 & \cdots & Z_1a_{1p} & 0 & Z_1a_{1,p+1} & 0 & Z_1a_{1,p+2} & 0 & \cdots & Z_1a_{1,p+q} & 0\\
Z_1 & -2Z_1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0\\
Z_2a_{21} & 0 & -2Z_2 & Z_2 & \cdots & Z_2a_{2p} & 0 & Z_2a_{2,p+1} & 0 & Z_2a_{2,p+2} & 0 & \cdots & Z_2a_{2,p+q} & 0\\
0 & 0 & Z_2 & -2Z_2 & & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0\\
 & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & & & & & \vdots\\
Z_pa_{p1} & 0 & Z_pa_{p2} & 0 & \cdots & -2Z_p & Z_p & Z_pa_{p,p+1} & 0 & Z_pa_{p,p+2} & 0 & \cdots & Z_pa_{p,p+q} & 0\\
0 & 0 & 0 & 0 & & Z_p & -2Z_p & 0 & 0 & 0 & 0 & & 0 & 0\\
-a_{p+1,1} & 0 & -a_{p+1,2} & 0 & \cdots & -a_{p+1,p} & 0 & Y_{p+1} & X_{p+1} & -a_{p+1,p+2} & 0 & \cdots & -a_{p+1,p+q} & 0\\
0 & 0 & 0 & 0 & & 0 & 0 & X_{p+1} & Y_{p+1} & 0 & 0 & & 0 & 0\\
-a_{p+2,1} & 0 & -a_{p+2,2} & 0 & \cdots & -a_{p+2,p} & 0 & -a_{p+2,p+1} & 0 & Y_{p+2} & X_{p+2} & \cdots & -a_{p+2,p+q} & 0\\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & X_{p+2} & Y_{p+2} & & 0 & 0\\
 & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & & & & & \vdots\\
-a_{p+q,1} & 0 & -a_{p+q,2} & 0 & \cdots & -a_{p+q,p} & 0 & -a_{p+q,p+1} & 0 & -a_{p+q,p+2} & 0 & \cdots & Y_{p+q} & X_{p+q}\\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & & X_{p+q} & Y_{p+q}
\end{array}\right\}$$

Figure 1

$$\overline{A}=\begin{array}{c|cccccccccccccc}
 & & & & 2p & & & & & & & 2q & & & \\
\hline
 & 2 & -1 & -a_{12} & 0 & \cdots & -a_{1p} & 0 & -a_{1,p+1} & 0 & -a_{1,p+2} & 0 & \cdots & -a_{1,p+q} & 0 \\
 & -1 & 2 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
 & -a_{21} & 0 & 2 & -1 & \cdots & -a_{2p} & 0 & -a_{2,p+1} & 0 & -a_{2,p+2} & 0 & \cdots & -a_{2,p+q} & 0 \\
2p & 0 & 0 & -1 & 2 & & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
 & & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & & & & \vdots & \\
 & -a_{p1} & 0 & -a_{p2} & 0 & \cdots & 2 & -1 & -a_{p,p+1} & 0 & -a_{p,p+2} & 0 & \cdots & -a_{p,p+q} & 0 \\
 & 0 & 0 & 0 & 0 & & -1 & 2 & 0 & 0 & 0 & 0 & & 0 & 0 \\
\hline
 & -2a_{p+1,1}\cdot u_{p+1} & 0 & -2a_{p+1,2}\cdot u_{p+1} & 0 & \cdots & -2a_{p+1,p}\cdot u_{p+1} & 0 & 2 & v_{p+1} & -2a_{p+1,p+2}\cdot u_{p+1} & 0 & \cdots & -2a_{p+1,p+q}\cdot u_{p+1} & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & v_{p+1} & 2 & 0 & 0 & & 0 & 0 \\
 & -2a_{p+2,1}\cdot u_{p+2} & 0 & -2a_{p+2,2}\cdot u_{p+2} & 0 & \cdots & -2a_{p+2,p}\cdot u_{p+2} & 0 & -2a_{p+2,p+1}\cdot u_{p+2} & 0 & 2 & v_{p+2} & \cdots & -2a_{p+2,p+q}\cdot u_{p+2} & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & v_{p+2} & 2 & & 0 & 0 \\
2q & & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & & & & \vdots & \\
 & -2a_{p+q,1}\cdot u_{p+q} & 0 & -2a_{p+q,2}\cdot u_{p+q} & 0 & \cdots & -2a_{p+q,p}\cdot u_{p+q} & 0 & -2a_{p+q,p+1}\cdot u_{p+q} & 0 & -2a_{p+q,p+2}\cdot u_{p+q} & 0 & \cdots & 2 & v_{p+q} \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & & v_{p+q} & 2
\end{array}$$

Figure 2