数理解析研究所講究録
761巻 1991年 52-75

THE MULTIPLE FOURIER SERIES

R. R. ASHUROV

DEPARTMENT OF MATHEMATICS
TASHKENT STATE UNIVERSITY
TASHKENT USSR 700095

ABSTRACT

Let $A(\xi)$ be a homogeneous elliptic polynomial and $f \in L_p(-\pi,\pi)^N$. Consider the Riesz means of partial sums of the multiple Furier series:

$$E_\lambda^s f(x) = \sum_{A(n)<\lambda} f_n e^{inx} \left(1 - \frac{A(n)}{\lambda}\right)^s, \quad s \geq 0.$$

In this paper we give a survey of some recent results on uniform convertgence and convergence almost every-where of $E_\lambda^s f(x)$. Special attention will be payed to the study of the influence of the geometry of the surface $\partial\Omega_A = \{\xi \in R^N; A(\xi) = 1\}$ on the precise conditions of convergence of $E_\lambda^s f(x)$.

1. Statement of the Results

1.1 Definitions. Let $f(x)=f(x_1,\dots,x_N)$ be a Lebesque integrable function defined on the fundamental cube $T^N=(-\pi,\pi)^N$ in Euclidean space $\mathbb{R}^N$. Consider the multiple Fourier series of f:

$$\sum_n f_n e^{inx} \qquad (1.1)$$

where $n=(n_1,\dots,n_N)\in\mathbb{Z}^N$ -the set of vectors with integer components, $nx=n_1x_1+\dots+n_Nx_N$ and

$$f_n=(2\pi)^{-N}\int_{T^N} f(x)e^{-inx}dx.$$

To sum the series (1.1) we must define partial sums of (1.1). One can consider different kinds of partial sums (rectangular, square, ...). In this paper we shall deal with partial sums defined by means of elliptic polynomials.

Let $A(\xi)=\sum_{|\alpha|=m} a_\alpha \xi^\alpha$ be a homogeneous polynomial on $\xi\in\mathbb{R}^N$ with constant coefficients, where α -multi-index, i.e. $\alpha=(\alpha_1,\dots,\alpha_N)$, α_j-nonnegative integers, $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_N$ and $\xi^\alpha=\xi_1^{\alpha_1}\dots\xi_N^{\alpha_N}$. Suppose that $A(\xi)$ is elliptic, i.e. $A(\xi)>0$ for all nonzero $\xi\in\mathbb{R}^N$. Each such polynomial determinens partial sums of the series (1.1) in the following way

$$E_\lambda f(x)=\sum_{A(n)<\lambda} f_n e^{inx}. \qquad (1.2)$$

In particular, if $A(\xi)=|\xi|^2$ then $E_\lambda f(x)$ coinsides with the spherical partial sums of (1.1).

In this paper as in most of the literature on eigen -

function expansions only Riesz means will be considered.Thus we introduce for all s with $Re\, s \geq 0$ the Riesz means of $E_\lambda f(x)$:

$$E_\lambda^s f(x) = \sum_{A(n)<\lambda} \left(1 - \frac{A(n)}{\lambda}\right)^s f_n e^{inx} \qquad (1.3)$$

Note $E_\lambda^0 f(x) = E_\lambda f(x)$. We allow s to be complex since this is required in the interpolation method of Stien [1] but does not cause any significant difficulties.

1.2.<u>Spectral Resolutions of Elliptic Differential Operators</u>.Let $C^\infty(T^N)$ be the set of all infinitly differentiable and 2π- periodic on each argument functions.In $L_2(T^N)$ we consider a homogeneous differential operator $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ with constant coefficients and domain of definition $C^\infty(T^N)$, where $D^\alpha = D_1^{\alpha_1} \cdot D_2^{\alpha_2} \cdots D_N^{\alpha_N}$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$.

The operator $A(D)$ is said to be elliptic if corresponding polynomial $A(\xi)$ is elliptic.Obviously an elliptic operator $A(D)$ is symmetric and non-negative:

$$(Au, v) = (u, Av) \;, \quad (Au, u) \geq 0, \quad u, v \in C^\infty(T^N).$$

Therefore,according to Friedrichs theorem the operator $A(D)$ has a selfadjoint extension $\hat{A}$ .It is not hard to see that this selfadjoint extension is unique and it coinsides with the closure of $A(D)$.

The operator $\hat{A}$ has a complete orthonormal in $L_2(T^N)$ system of eigenfunctions

$$\left\{ (2\pi)^{-\frac{N}{2}} e^{inx} \right\}, \quad n \in \mathbb{Z}^N$$

corresponding to eigenvalues $\{A(n)\}$, $n \in \mathbb{Z}^N$ .According to the spectral theorem of von Neumann we have

$$\hat{A} = \int_0^\infty \lambda \, dE_\lambda$$

where E_λ is a resolution of identity.The expression $E_\lambda f(x)$ is called the spectral resolution of an element $f \in L_2(T^N)$ and it can be verified that

$$E_\lambda f(x) = \sum_{A(n)<\lambda} f_n e^{inx},$$

i.e. the spectral resolution coinsides with partial sums of the multiple Fourier series (1.2).

We also will have an importent spectral resolution if we consider an operator $A(\mathcal{D})$ in $\mathbb{R}^N$ .Let $C_0^\infty(\mathbb{R}^N)$ - the class of all infinitly differentiable functions with compact support, be the domain of definition of an elliptic operator $A(\mathcal{D})$ acting in $L_2(\mathbb{R}^N)$ .Again the closure of $A(\mathcal{D})$ is the unique selfadjoint extension in $L_2(R^N)$ of $A(\mathcal{D})$ and in this case we obtain the following spectral resolution of an element $f \in L_2(R^N)$:

$$\sigma_\lambda f(x) = (2\pi)^{-\frac{N}{2}} \int_{A(\xi)<\lambda} \hat{f}(\xi) e^{i\xi x} d\xi, \quad (1.4)$$

where $\hat{f}(\xi)$ is the Fourier-Plancherel transfomation of f:

$$\hat{f}(\xi) = (2\pi)^{-\frac{N}{2}} \lim_{R\to\infty} \int_{|x|<R} f(x) e^{-i\xi x} dx.$$

We note that (1.4) is, on the other hand, partial integrals of the multiple Fourier integrals. As in classical one-dimensional case the multiple Fourier series are closely connected with the multiple Fourier integrals.

1.3. Localization of the Multiple Fourier Series. We shall discuss the problem of conditions for localization of the Riesz means E_λ^S, that is, of conditions on a function at points far from the one under dicussion under which the convergence of E_λ^s depends only on the behaviour of the function in a small neighbourhood of the point in question. In what follows it is convenient to use the following definitions.

Let U be an arbitrary region of T^N. We say that localization principle for $E_\lambda^s f$ holds in the class $L_p(T^N)$ if it follows from the conditions

$$f \in L_p(T^N), \quad f(x)=0 \quad for \quad x \in U \tag{1.5}$$

that the following equality holds uniformly on every compact subset of U

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0 . \tag{1.6}$$

If for given p , S and some $x_0 \in U$ there exists a function f satisfying conditions (1.5) but $\overline{\lim\limits_{\lambda\to\infty}} |E_\lambda^s f(x_0)| > 0$,

we say that the localization principle for the resolution $E_\lambda^s f$ fails in the class L_p .

In one- dimensional case localization for $E_\lambda f$ holds in the class $L_1(-\pi,\pi)$ (the classical theorem of Riemann - Lebesque). But for multidimensional case the classes $L_p(T^N)$ are too wide to hold localization and hence we must consider either a class of smooth functions or some reqularization methods, in particular, the Riesz means E_λ^s

The first results on localization for multiple Fourier series are due to Bochner, Titchmarsh, Minakshisundaram, Chandrasekharam, Levitan, Stein, Ilin, Hörmander and other mathematicians. A detiled survey of this question can be found in [2], [3].

The question of localization of the Riesz means $E_\lambda^s f$ in the classes $L_p(T^N)$ for $p \geq 2$ has been completly soleved, i.e. if $s = \frac{N-1}{2}$ then for any elliptic polynomial $A(\xi)$ localization holds in the class $L_p(T^N)$, $p \geq 2$ [4]. If $s < \frac{N-1}{2}$, then localization fails even for the Laplace (i.e. $A(\xi) = |\xi|^2$) operator and even in the class $C(T^N)$ of continuous functions, hence fortiori in any $L_p(T^N)$, $p \geq 1$ [5].

Thus covergence or divergence of $E_\lambda^s f$ in (1.6) for functions $f \in L_p(T^N)$ does not depend on geometry of the set $\Omega_A = \{\xi \in R^N;\ A(\xi) < 1\}$ when $p \geq 2$, i.e. behaviour of all partial sums (1.2) is similar to spherical ones.

Unlike this case the precise conditions for localization of E_λ^s in $L_p(T^N)$ with $p < 2$ depend strongly on the geometry of the surface $\partial\Omega_A = \{\xi \in R^N;\ A(\xi) = 1\}$. For example, when $A(\xi) = |\xi|^2$ (in this case all the principal curvatures of the surface $\partial\Omega_A$ are equal to 1) Stein [1] has proved that the localization pricipal holds for $E_\lambda^s f$, $s = \frac{N-1}{2}$ in any $L_p(T^N)$, $p > 1$.

This result holds also for elliptic polynomials $A(\xi)$, if all the principal curvatures of the surface $\partial\Omega_A$ are different from zero at every point [4] (in this case we say that the set Ω_A is stictly convex). But if $A(\xi)$ is an arbitrary elliptic polynomial, then, as Hörmander has shown [4], localization for E_λ^s holds in $L_p(T^N)$ for $s \geq \frac{N-1}{p}$, $1 \leq p \leq 2$ (note, that in [4] this result was proved for the spectral resolutions associated with an arbitrary selfadjoint elliptic differential operator on an N-dimansional paracompact manifold).

The first question which arises here is the following: is it possible to improve the condition of localization $s \geq \frac{N-1}{p}$ for $p \in [1, 2)$?

To answer this question we consider an elliptic polynomial

$$M(\xi) = \xi_1^{2m} + \left(\sum_{j=2}^{N} \xi_j^2 \right)^m$$

of order $2m$ (a similar polynomial was first investigated by Peetre [6]).The corresponding set $\Omega_M = \{ \xi \in R;\ M(\xi) < 1 \}$ is convex but not strictly convex,since all principal curvatures are zero on points $(\pm 1, 0, \ldots, 0)$.For this reason we have the following statement.

Theorem 1. Let $s < \Delta_m(p) \equiv \frac{N-1}{p}\left(1 - \frac{1}{2m}\right)$ .Then the localization principle fails for $E_\lambda^s(M)$ in the classes $L_p(T^N)$, $1 \leq p \leq 2\left(1 - \frac{1}{2m}\right)$.

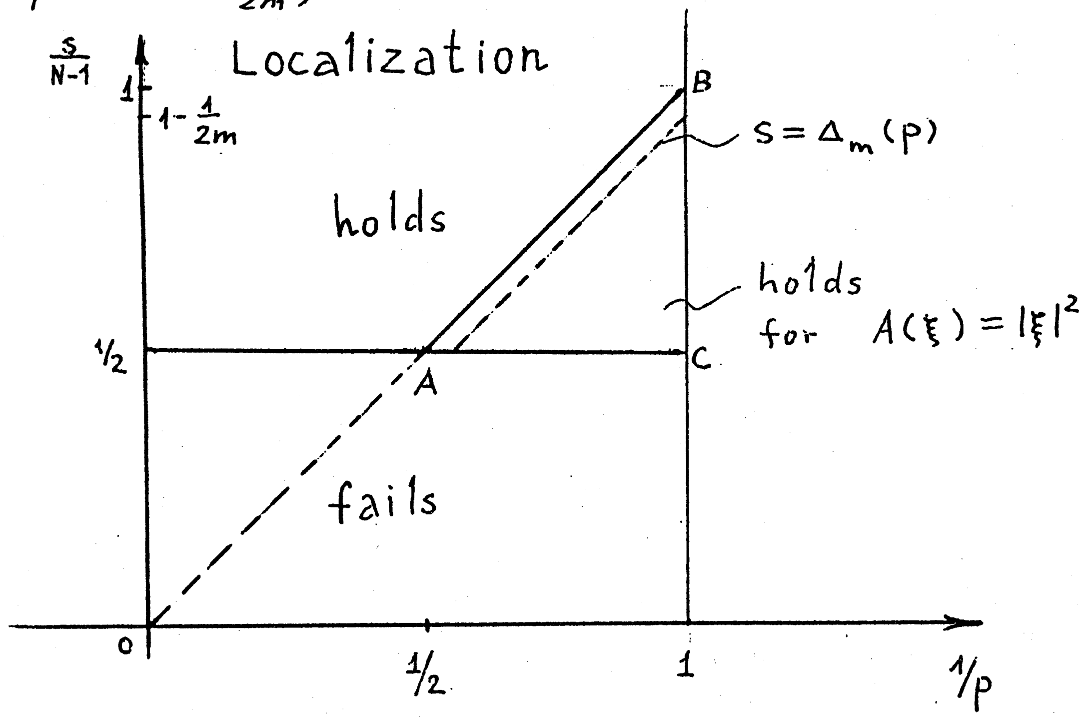

Fig. 1.

Since $\Delta_m(p) \to \frac{N-1}{p}$ as $m \to \infty$ it follows that the condition $s \geq \frac{N-1}{p}$ for localization is sharp in the class of all elliptic polynomials.

But is there an elliptic polynomial for which the condition $s \geq \frac{N-1}{p}$ can not be improved? The following theorem gives a negative answer to this question.

Theorem 2. Let $A(\xi)$ be an arbitrary elliptic polynomial. Then there exists a function $\varepsilon_A(p) \geq 0$, $\varepsilon_A(1) > 0$ such that localization for E_λ^s holds in $L_p(T^N)$, $1 \leq p \leq 2$ when $s \geq \max\left\{\frac{N-1}{2}, \frac{N-1}{p} - \varepsilon_A(p)\right\}$.

Note, Theorem 1 shows that $\inf\limits_A \varepsilon_A(p) = 0$ and Theorem 2 shows that there is no elliptic polynomial A for which $\varepsilon_A(p) \equiv 0$.

To understand what is happening on the triangle ABC in Fig.1 we introduce the following classes of elliptic polynomials.

Definition. We shall say that the elliptic polynomial $A(\xi)$ belongs to the class A_r, $r = 0, 1, \ldots, N-1$ if at every point of the surface $\partial\Omega_A$ at least r of the $N-1$ principal curvatures are different from zero.

Obviously

$$A_{N-1} \subset \cdots \subset A_1 \subset A_0$$

and A_0 coinsides with the class of all elliptic polynomials. The smollest class A_{N-1} is the class of all elliptic polynomials with a strictly convex set Ω_A .For example, $|\xi|^2 \in A_{N-1}$.

We shall use the notation $\delta_r(p) = \frac{N-1}{p} - r\left(\frac{1}{p} - \frac{1}{2}\right)$.

Theorem 3. Let $A(\xi) \in A_r$, $r = 0, 1, \ldots, N-1$ .Then for the Riesz means E_λ^s of order $s \geq \delta_r(p)$ localization holds in the classes $L_p(T^N)$, $1 \leq p \leq 2$.

We note when r changes from 0 to $N-1$ the lines $s = \delta_r(p)$ fill the triangle ABC in Fig.1.

When $r > 0$ there is no analogous to Theorem 2 in classes A_r. Nevertheless we have the following statement.

Theorem 4. Let $A(\xi) \in A_r$, $r > 0$, and the set Ω_A is convex. Then there exists a function $\varepsilon_A(p) > 0$ such that localization for E_λ^s holds in classes $L_p(T^N)$, $1 \leq p \leq 2$ when

$s \geq \max\left\{\frac{N-1}{2}, \delta_r(p) - \varepsilon_A(p)\right\}$.

To show that the conditions on s in Theorems 3 and 4 are optimal, consider the elliptic polynomial of order $2m+2$

$$L_{m,r}(\xi) = \left(\sum_{j=1}^{r+1} \xi_j^2\right)^{m+1} + \left(\sum_{j=r+2}^{N} \xi_j^2\right)\left(\sum_{j=1}^{N} \xi_j^2\right)$$

which belongs to the class A_r when $r < N-1$ and $L_{m,r} \notin A_{r+1}$. It is not hard to see that the set $\{\xi \in \mathbb{R}^N;\ L_{m,r}(\xi) < 1\}$ is convex.

Theorem 5. Localization holds for the Riesz means $E_\lambda^s(L_{m,r})$ in the classes $L_p(T^N)$, $1 \leq p \leq 2\left(1 - \frac{1}{2m}\right)$ iff

$s \geq \delta_{m,r}(p) \equiv (N-r-1)\left(1 - \frac{1}{2m}\right)\frac{1}{p} + \frac{r}{2}$.

Since $\delta_{m,r} \to \delta_r$ as $m \to \infty$ it follows that the conditions of Theorems 3 and 4 are sharp in the class A_r.

Next statement shows the influence of the geometry of the set Ω_A to uniform convergence of $E_\lambda^s f$.

Theorem 6. Let $A(\xi) \in A_r$, $r = 0, 1, \ldots, N-1$, $s > \delta_r(p)$, $1 \leq p \leq 2$ and a function $f \in L_p(T^N)$ is continuous on $U \subset T^N$. Then

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = f(x)$$

uniformly on any compact subset of U

1.4. Localization of Multiple Fourier Integrals. In case of multiple Fourier integrals one can obtain the precise conditions for localization in classes of smooth functions.

Here it is convenient to consider the Liouville classes $L_p^a(R^N)$, $a>0$. To give the definition of L_p^a we introduce the operators

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{N}{2}} \cdot \int_{R^N} f(x)\, e^{-ix\xi}\, dx,$$

$$\mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{N}{2}} \int_{R^N} f(\xi)\, e^{ix\xi}\, d\xi.$$

We shall say that a function $f \in L_p(R^N)$ belongs to the class $L_p^a(R^N)$, $a>0$ if the following norm is bounded

$$\|f\|_{L_p^a(R^N)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{a}{2}} \mathcal{F}f\|_{L_p(R^N)}.$$

When a is an integer then $L_p^a(R^N)$ coinsides with the usual Sobolev classes $W_p^a(R^N)$ -the set of functions $f \in L_p(R^N)$ for which all partial derivatives of order a belong to $L_p(R^N)$.

We shall also consider the Nikolskii classes H_p^a, $a = \ell + æ$, ℓ is integer, $0 < æ \leq 1$ -the set of functions $f \in L_p(R^N)$ for which all partial derivatives $D^\alpha f \in L_p(R^N)$, $|\alpha| = \ell$ and

$$\|D^\alpha f(x+h) - 2D^\alpha f(x) + D^\alpha f(x-h)\|_{L_p(R^N)} \leq c|h|^{æ}.$$

Theorem 7. Suppose that $A(\xi) \in A_r$, $r = 0, 1, \ldots, N-1$. Then for the Riesz means $G_\lambda^s f$ localization holds in the classes $L_p^a(R^N)$, $1 \leq p \leq \infty$ when $a + s \geq \max\{\frac{N-1}{2}, \delta_r(p)\}$.

For the partial integrals associated with the elliptic polynomials $L_{m,r}(\xi)$ we have the following assertion.

Theorem 8. Localization holds for the Riesz means $\sigma_\lambda^s(L_{m,r})$ in the classes $H_p^a(R^N)$, $1 \le p \le \infty$ iff $a+s \ge \max\{\frac{N-1}{2}, \delta_{m,r}(p)\}$.

By virtue of the imbedding $H_p^{a+\varepsilon} \to L_p^a$ for any $\varepsilon > 0$ (see [7], § 7.3), it follows from Theorem 8 that the conditions stated in Theorem 7 for localization in L_p^a are sharp in the class A_r (since $\delta_{m,r} \to \delta_r$ as $m \to \infty$).

1.5. Convergence Almost Everywhere. The question of convergence almost everywhere of onedimensional trigonometric Fourier series has been completely solved in classes $L_p(-\pi,\pi)$, i.e. if $f \in L_p(-\pi,\pi)$, $p>1$ then its Fourier series converge to $f(x)$ almost everywhere on $(-\pi,\pi)$ (Carleson- Hunt's theorem) and there exists a function $f \in L_1(-\pi,\pi)$ having Fourier series which diverge almost everywhere on $(-\pi,\pi)$ (Kolmogorov's theorem).

In multidimensional case there is no convergence almost everywhere of the partial sums $E_\lambda f$ in classes $L_p(T^N)$ at least when $1 \le p < 2$ (Nikishin's theorem, see [2], [3]). For that reason we must again regularizate $E_\lambda f$ by Riesz means. Convergence almost everywhere of the Riesz means E_λ^s was investigated by many authors (see for example the survay papers [2], [3]). Here we only remind some results.

For the elliptic polynomial $A(\xi) = |\xi|^2$ Stein [1] proved convergence almost everewhere on T^N of the Riesz means E_λ^s of the order $s > (N-1)(\frac{1}{p} - \frac{1}{2})$ of functions from $L_p(T^N)$, $1 \le p \le 2$. This result holds also for any elliptic polynomials $A(\xi) \in A_{N-1}$ [4]. If $A(\xi)$ is an arbitrary elliptic polynomial then the Riesz means of functions $f \in L_p(T^N)$, $1 \le p \le 2$ converge to $f(x)$ almost everywhere on T^N when $s >$

$2(N-1)(\frac{1}{p}-\frac{1}{2})$ .This result is due to Hörmander [4].

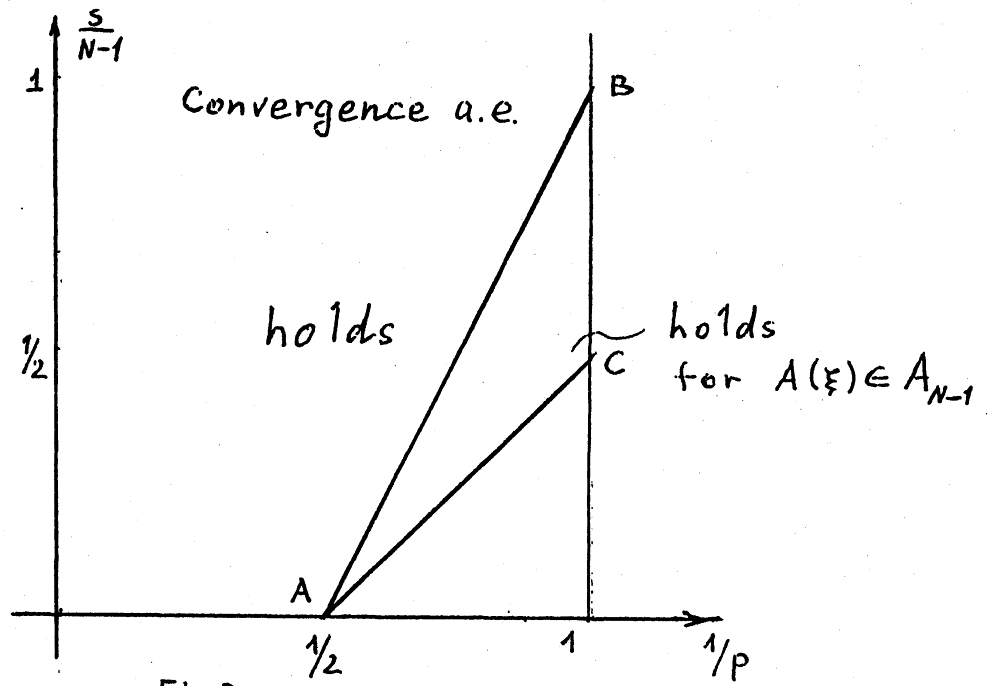

Fig.2

The first impression is that that on the triangle ABC in Fig.2 the precise conditions for convergence almost everewhere of E_λ^s, like localization pricipale,must depend on the number of nonzero curvatures of the surface $\partial\Omega_A$ (this was infact proved by N.Mahamedjanov,Candidetes Dissertation,Moscow State University,1973). The following statement shows that this is not the case.

Theorem 9. Let $A(\xi)$ be an arbitrary elliptic polynomial and $\Omega_A=\{\xi\in R^N; A(\xi)<1\}$ be a covex set.Then $\lim_{\lambda\to\infty} E_\lambda^s f(x)=f(x)$ for almost every $x\in T^N$ if $s>(N-1)(\frac{1}{p}-\frac{1}{2})$, $1\le p\le 2$.

The analogouse theorem is true for the multiple Fourier integrals $\sigma_\lambda^s f$

We note that convexsity of the set Ω_A simplifies the proof.But, as examples show,this condition is likely not necessary.

2. Proofs of the Theorems

In this chapter we present some main points of the proofs of the theorems formulated in chapter 1.

2.1. Spectral Functions. Consider the Riesz means $E_\lambda^s f$. Using the definition of coefficients f_n we have

$$E_\lambda^s f(x) = \int_{T^N} \Theta^s(x-y, \lambda) f(y) dy ,$$

where

$$\Theta^s(x,\lambda) = (2\pi)^{-N} \sum_{A(n)<\lambda} \left(1 - \frac{A(n)}{\lambda}\right)^s e^{inx}.$$

The partial integrals $\sigma_\lambda^s f(x)$ are also an integral operators with a kernal

$$e^s(x,\lambda) = (2\pi)^{-N} \int_{A(\xi)<\lambda} \left(1 - \frac{A(\xi)}{\lambda}\right)^s e^{i\xi x} d\xi.$$

In the spectral theory of differential operators the functions $\Theta(x,\lambda) = \Theta^0(x,\lambda)$ and $e(x,\lambda) = e^0(x,\lambda)$ are called a spectral function.

To study convergence of $E_\lambda^s f(x)$ $(\sigma_\lambda^s f(x))$ as $\lambda \to \infty$ we must investigate an asymptotic behaviour of the spectral function $\Theta^s(x,\lambda)$ $(e^s(x,\lambda))$ when $\lambda \to \infty$. One can investigate the asymptotics of $e^s(x,\lambda)$ by the method of stationary phase since

$$e^s(x,\lambda) = (2\pi)^{-N} \lambda^{\frac{N}{m}} \int_{A(\xi)<1} e^{i\lambda^{\frac{1}{m}}\xi x} (1 - A(\xi))^s d\xi \quad (2.1)$$

and therefore $e^s(x,\lambda)$ is an oscillatory integral. Note when $s=0$ the integral in (2.1) is the Fourier transformation of an indicator function of the set Ω_A .

A phase function $\varphi(\xi,x)=\xi x$ of the integral (2.1) does not have stationary points in the set Ω_A .Therefore it is clear that the basic contribution to the integral (2.1) for large λ is given by a neighbourhood of those critical (stationary) points of the surface $\partial\Omega_A$ where the exterior normal coinsides with $\omega=\frac{x}{|x|}$.

For example, if $A(\xi)\in A_{N-1}$ then for all directions $\omega\in S^{N-1}$ (unit sphere in R^N) the phase function of $e^s(x,\lambda)$ is the Morse type, hence it has a good estimate uniformly on $\omega\in S^{N-1}$:

$$|e^s(x,\lambda)|\le\frac{c\,\lambda^{\frac{N}{m}}}{(1+|x|\lambda^{\frac{1}{m}})^{\frac{N+1}{2}+Res}}\,, \qquad (2.2)$$

what is analogous to the simple case $A(\xi)=|\xi|^2$ .Since this estimate the Poisson summation formula allows us to prove the following equality for $Re\,s>\frac{N-1}{2}$

$$\Theta^s(x,\lambda)=\sum_{n\in Z^N}e^s(x+2\pi n,\lambda)\,. \qquad (2.3)$$

Now we can establish the necessary estimates for $\Theta^s(x,\lambda)$.

But if $A(\xi)\notin A_{N-1}$ then the series (2.3), in general ,does not converge, since the estimate (2.2) is true only for almost all directions $\omega\in S^{N-1}$.Nevertheless, if we integrate $|e^s(x,\lambda)|^2$ on the sphere S^{N-1} then we will have a similar to (2.2) estimate

$$\left(\int_{S^{N-1}}|e^s(r\theta,\lambda)|^2 d\theta\right)^{1/2}\le\frac{c\lambda^{\frac{N}{m}}}{(1+r\lambda^{1/m})^{\frac{N+1}{2}+Res}} \qquad (2.4)$$

(for $s=0$ this estimate was proved in [8]) .Using the estimate (2.4) one can prove the following assertion .

Let $f \in L_p(T^N)$, $p \geq 1$. By $F(x) = F(x_1, \dots, x_N)$ we denote a function which is 2π- periodic in each variable, x_j and $F(x) = f(x)$ for $x \in T^N$

Lemma 1. Let $Re\ s > \frac{N-1}{2}$. Then for each $x \in T^N$ we have the equality

$$\sum_{A(n) < \lambda} \left(1 - \frac{A(n)}{\lambda}\right)^s f_n e^{ixn} = \int_{R^N} e^s(y, \lambda) F(x-y) dy . \quad (2.5)$$

Both functions in (2.5) are 2π- periodic in each x_j and since the estimate (2.4) the right-hand side of (2.5) belongs to $L_p(T^N)$. Hence to prove lemma 1 it is enough to show that the Fourier coefficients by the system $\{e^{inx}\}$ of both functions in (2.5) are equal. But this can be verified by a simple calculations. Lemma 1 is proved.

The equality (2.5) will be our main tool in the study of $E_\lambda^s f$. By this Lemma we can reduce the investigation of the Riesz means $E_\lambda^s f$ to the investigation of the integral opertors $\mathcal{G}_\lambda^s f$ with more simple kernals.

2.2 Proof of the Theorems on Localization. As we have seen above to prove theorems on convergence of $E_\lambda^s f$ we must study an asymptotic behaviour of the oscillatory integral $e^s(x, \lambda)$.

Lemma 2. Let $A(\xi) \in A_r$, $r = 0, 1, \dots, N-1$.

Then the estimate

$$|e^s(x, \lambda)| \leq \frac{C \lambda^{\frac{N}{m}}}{\left(1 + \lambda^{\frac{1}{m}} |x|\right)^{\frac{r+2}{2} + Re\, s}} \quad (2.6)$$

holds uniformly with respect to $x \in R^N$, where C is a positiv constant.

Proof. Let $s=0$. Using the divergent formula we write the spectral function in the form

$$e(x,\lambda)=(2\pi)^{-N}\frac{\lambda^{\frac{N}{m}}}{i\lambda^{\frac{1}{m}}|x|}\int_{A(\xi)=1} e^{i\lambda^{\frac{1}{m}}x\xi}\cos(\omega,\hat{n}_\xi)\,d\sigma_\xi\,, \quad (2.7)$$

where n_ξ is the exterior normal at the point ξ, $d\sigma_\xi$ is the element of erea. By the localization principle in the stationary phase method, we have

$$I_\lambda(x)=\sum_{j=1}^{k}\int_{A(\xi)=1} e^{i\lambda^{\frac{1}{m}}x\xi}\cos(\omega,\hat{n}_\xi)\,\mathcal{X}_j(\xi)\,d\sigma_\xi+O(\lambda^{-\infty}), \quad (2.8)$$

where $I_\lambda(x)$ is the integral in (2.7) and $\mathcal{X}_j$ are the truncating functions of the neighbourhoods of the stationary points $\xi_j(\omega)$, corresponding to the vector $\omega=\frac{x}{|x|}$. We note that if the set Ω_A is convex then to each direction ω there correspond only two stationary points and if it is not convex then there correspond either finite stationary points or even n -dimensional surfaces, $n\le N-2$.

The integrals in the sum (2.8) are estimated in an entirely similar manner. Let $J_\lambda(x)$ be one of these integrals. Performing a change of variables in the integral $J_\lambda(x)$ we obtain

$$J_\lambda(x)=\int_Q e^{i\lambda^{\frac{1}{m}}|x|\varphi(\tilde{\xi})}f(\tilde{\xi})\,d\xi_1\cdots d\xi_{N-1}\,,$$

where $\tilde{\xi}=(\xi_1,\dots,\xi_{N-1})$, Q is a neighbourhood of zero in R^N and $f(\tilde{\xi})\in C_0^\infty(Q)$. We note that $\nabla\varphi|_{\tilde{\xi}=0}=0$ and $\operatorname{rank}\varphi_{\tilde{\xi}\tilde{\xi}}(0)=r$ (since $A(\xi)\in A_r$). According to a generalization of Morses lemma (see [9], Lemma 3.5.1) there exists a diffeomorphism $\tilde{\xi}=\psi(\eta)$ $(\psi(0)=0)$ such that in a small neighbourhood of the critical point $\tilde{\xi}=0$ we have

$$(\varphi \circ \Psi)(\eta) = const + \sum_{j=1}^{r} \pm \eta_j^2 + \varphi_1(\eta_{r+1}, \dots, \eta_{N-1});$$

where $\nabla \varphi_1(0) = 0$, $\dfrac{\partial^2 \varphi_1(0)}{\partial \eta_j \partial \eta_k} = 0$ for all $j, k = 1, 2, \dots, N-1$.

Using this assertion and estimating first r integrals we obtain the estimate (2.6) for $s=0$ .To obtain (2.6) for an integer $s>0$ we must first integrate by parts in (2.1).Applying, for example, a tauberian theorem of Hörmander (see theorem 2.4 of [4]) we establish (2.6) for general s .Lemma is proved.

Let $f \in L_p(T^N)$, $p>1$ and $f(x)=0$ for $x \in U \subset T^N$. If $Re\, s \geq N-1-\frac{r}{2}$ then application of Lemmas 1 and 2 to $E_\lambda^s f$ gives the uniform on $x \in K$ estimate

$$|E_\lambda^s f(x)| \leq C \|f\|_{L_p(T^N)}, \qquad (2.9)$$

where K is an arbitrary compact subset of U.

If $f \in L_2(T^N)$ then the estimate (2.9) holds for s with $Re\, s \geq \frac{N-1}{2}$. To see this we first note that since (2.4) we get

$$\int_{T^N \setminus U} |e^s(x-y, \lambda)|^2 dy \leq C_K \lambda^{(N-1-2Re\, s)\frac{1}{m}}, \quad x \in K. \qquad (2.10)$$

Now,using an estimate in L_2 of the difference $\Theta^s - e^s$, established by Bergendal [10] we obtain (2.10) for Θ^s, $Re\, s \geq 0$. Hence we have

$$|E_\lambda^s f(x)| \leq C \|f\|_{L_2(T^N)}, \quad x \in K, \qquad (2.11)$$

for all s with $Re\, s \geq \frac{N-1}{2}$.

Applying to the estimates (2.9) and (2.11) the interpolation theorem of Stein [11] for linear operators, depending analytically on a parameter, we get

$$|E_\lambda^s f(x)| \leq c \|f\|_{L_p(T^N)} \quad , \quad x \in K \, , \qquad (2.12)$$

where $1 < p \leq 2$ and $s \geq \delta_r(p)$ .Note to use this theorem we allow s to be complex.

Theorem 3 follows from the estimate (2.12) since $C^\infty(T^N)$ is dense in $L_p(T^N)$ and for $f \in C^\infty(T^N)$ the theorem is obviously true.

To prove Theorem 2 we use the following estimate

$$|e^s(x,\lambda)| \leq \frac{c\,\lambda^{\frac{N}{m}}}{(1+|x|\lambda^{\frac{1}{m}})^{1+\varepsilon}} \, , \quad \varepsilon = \varepsilon(A) > 0 ,$$

which is based on the estimate of a oscillatory integral established in [12].

2.3.<u>On the Absence of Localization</u>. The proofs of Theorem 1 and the part " and only if" of Theorems 5 and 8 are technicaly complicated to present here.For this reason,we give only the main ideas of the proofs of these Theorems.

To prove these theorems we actualy construct such a function $f_0(x)$ from a necessary class that $f_0(x)=0$ in a neighbourhood of the origin and

$$\overline{\lim_{\lambda\to\infty}} \; |E_\lambda^s f_0(0)| > 0 .$$

We note that if $E_\lambda^s f(x)$ converges then this is because of the oscillation of the spectral function $\theta^s(x,\lambda)$ as $\lambda \to \infty$. Therefore we first construct a sequence of functions $f_{\lambda,\delta}(x)$ which have a support in outside of the origin and the same asymptotic behaviour as $\Theta^s(x, \lambda_j)$ .Then we consider a function

$$f_0(x) = \sum_j v_j f_{\lambda_j}(x) .$$

It is obvious that $f_0(x)=0$ in a neighbourhood of the origin. We shall choose sequences of numbers $\{v_j\}_1^\infty$ and $\{\lambda_j\}_1^\infty$ such that the function $f_0(x)$ belongs to a necessary class and

$$\overline{\lim_{\lambda\to\infty}} \, | E_{\lambda_j}^s f_0(0) | > 0.$$

We obtain this inequality as follows: choosing the numbers λ_j seldom enough we make quantatives $E_{\lambda_j}^s\left(\sum_{k=1}^{j-1} v_k f_{\lambda_k}\right)(0)$ and $E_{\lambda_j}^s\left(\sum_{k=j+1}^{\infty} v_k f_{\lambda_k}\right)(0)$ to be much more smoller (since the oscillation of $\Theta^s(x, \lambda_j)$) in compering with $v_j \, E_{\lambda_j}^s f_{\lambda_j}(0)$. The latter does not tend to zero since the asymptotic behaviour of the functions $f_{\lambda_j}(x)$ and $\Theta(x, \lambda_j)$ coinsides as $\lambda_j \to \infty$.

To do all this we must know the asymptotic behaviour of the spectral function $\Theta^s(x, \lambda)$ as $\lambda \to \infty$. The elliptic polynomials $M(\xi)$ and $L_{m,r}(\xi)$ have been chosen such that one can easily study the asymtotics of the corresponding spectral functions $e^s(x, \lambda)$ by the stationary phase method.Having this done we prove the equality (2.3) in this case for $Re\, s > \frac{N-1}{2}$. The equality (2.3) allows us to obtain the asymptotics of the function $\Theta^s(x, \lambda)$ for large λ .

2.4. On Localization of Multiple Fourier Integrals. We start with the following very known assertion.

Lemma 3. Let $A(\xi)$ be an arbitrary ellitic polynomial of order m .Then

$$\| (1+\hat{A}(\mathcal{D}))^{\frac{a}{m}} f \|_{L_p(R^N)} \leq c \| f \|_{L_p^a(R^N)} \, , \qquad (2.14)$$

where $a>0, \quad 1 \leq p \leq \infty$.

In this lemma $\hat{A}(\mathcal{D})$ is the closure (selfadjoint operator) of the positive elliptic operator $A(\mathcal{D}) = \sum_{|\alpha|=m} a_\alpha \mathcal{D}^\alpha$ with the domain of definition $C_0^\infty(\mathbb{R}^N)$ .For this reason we can define the power $(1+\hat{A}(\mathcal{D}))^{\frac{a}{m}}$ by the spectral theorem.

We note that for $p=2$ the lemma follows from the obvious inequality $(1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha)^2 \leq c\,(1+|\xi|^2)^m$.

Now we return to study the Riesz means $\mathcal{E}_\lambda^s f$ .Note that the operator $\hat{A}$ (and its powers) is commutable with its resolution of the identity $\mathcal{E}_\lambda$ .Therefore we can write $\mathcal{E}_\lambda^s f = (1+\hat{A})^{-\frac{a}{m}} \mathcal{E}_\lambda^s (1+\hat{A})^{\frac{a}{m}}$ .Here the operator $(1+\hat{A})^{-\frac{a}{m}}$ is defined by the spectral theorem and it is not hard to see that $(1+\hat{A})^{-\frac{a}{m}} \mathcal{E}_\lambda^s$ is an integral operator with the kernal

$$e_a^s(x,\lambda) = \int_0^\lambda (1+t)^{-\frac{a}{m}} \, d\, e^s(x,t) \, .$$

Hence using the estimates (2.6) and (2.10) one can establish L_∞ and L_2 estimates for the kernal $e_a^s(x,\lambda)$,which corresponds to an elliptic polynomial $A(\xi) \in A_r$ .Having done this we get for a function $f \in L_1^a(R^N)$,which is equal to zero in $U \subset R^N$,

$$|\mathcal{G}_\lambda^s f(x)| \leq c\,\|(\hat{A}+1)^{\frac{a}{m}} f\|_{L_1(R^N)} \leq c\,\|f\|_{L_1^a(R^N)}$$

where $a + Re\,s \geq N-1-\frac{r}{2}$ and if $f \in L_2^a(\mathbb{R}^N)$ then for $a + Re\,s \geq \frac{N-1}{2}$ we have

$$|\mathcal{G}_\lambda^s f(x)| \leq c\,\|f\|_{L_2^a(R^N)},$$

where $x \in K$, K is an arbitrary compact of U.

Applying to the last two estimates the interpolation method we get

$$|\mathcal{G}_\lambda^s f(x)| \leq c\,\|f\|_{L_p^a(R^N)}, \qquad x \in K$$

where $1 \leq p \leq 2$ and $a+s \geq \delta_r(p)$. Theorem 7 follows from this estimate, since $C_0^\infty(\mathbb{R}^N)$ is dense in $L_p^a(\mathbb{R}^N)$.

2.5. On Convergence Almost Everywhere. We denote by E_* so-called the maximal operator

$$E_*^s f(x) = \sup_{\lambda > 1} |E_\lambda^s f(x)|, \quad f \in L_p(T^N)$$

(the maximal operator $\mathcal{G}_*^s f$ defines in the same way). The investigation of convergence almost everywhere of the Riesz means $E_\lambda^s f$ is based on estimates of majorants $E_*^s f$ in L_p for p closed to 1 and L_2 and on a subsequent application of Steins interpolation theorem (see[1]).

Estimates of $E_*^s f$ in L_p are founded on asymptotic estimates of the function $e^s(x,\lambda)$ established by Randol [13] (note to use these results we need convexsity of the set Ω_A) and the following generalization of the Hardy-Littlewood inequality.

Let $\omega = \frac{x}{|x|}$, $G(\omega)$ be a positive function

$$\|G\|_{L_1(S^{N-1})} = \int_{S^{N-1}} G(\omega)\, d\sigma_\omega < \infty .$$

We define an operator M acting in $L_p(\mathbb{R}^N)$ as

$$Mg(x) = \sup_{\varepsilon>0} \varepsilon^{-N} \int_{|y|<\varepsilon} |g(x-y)|\, G\left(\frac{y}{|y|}\right) dy ,$$

where $g \in L_p(\mathbb{R}^N)$, $p \geq 1$. If $G \equiv 1$ then Mg is the usual maximal function of Hardy-Littlewood.

<u>Lemma 4</u>. There exists constant $c = c(p) > 0$ depending on $p > 1$ so that

$$\|Mg\|_{L_p(\mathbb{R}^N)} \leq c\, \|G\|_{L_1(S^{N-1})} \|g\|_{L_p(\mathbb{R}^N)} . \quad (2.15)$$

<u>Proof</u>. We have

$$Mg(x) = \sup_{\varepsilon>0} \varepsilon^{-N} \int_{S^{N-1}} G(\omega) \int_0^{\varepsilon} |g(x-r\omega)|\, r^{N-1}\, dr\, d\omega$$

$$\leq \int_{S^{N-1}} G(\omega) \sup_{\varepsilon>0} \varepsilon^{-1} \int_{-\varepsilon}^{\varepsilon} |g(x-r\omega)|\, dr\, d\omega .$$

According to the Minkowski's inequality we get

$$\|Mg\|_{L_p(\mathbb{R}^N)} \leq \int_{S^{N-1}} G(\omega) \left\{ \int_{R^N} \left[\sup_{\varepsilon>0} \varepsilon^{-1} \int_{-\varepsilon}^{\varepsilon} |g(x-r\omega)|\, dr \right]^p dx \right\}^{1/p} d\omega .$$

For fixed ω using the one-dimensional Hardy-Littlewood inequality we obtain (2.15).

In conclusion we consider the following elliptic polynomial of order 4

$$L(\xi,\zeta) = \frac{1}{4\varepsilon^4}\left(|\xi|^4 + |\zeta|^4\right) + \left(4 - \frac{1}{2\varepsilon^4}\right)|\xi|^2|\zeta|^2,$$

where $\xi \in R^k$, $\zeta \in R^\ell$, $0<\varepsilon< 2^{-\frac{1}{2}}$. We note that the set $\{(\xi,\zeta)\in R^{k+\ell};\ L(\xi,\zeta)<1\}$ is not convex.Furthermore,the set of stationary points corresponding to the direction e=(0,...,0,1) of the integral (2.7),connected with the polynomial $L(\xi,\zeta)$, is $k-1$ -dimensional sphere. Nevertheless for the Riesz means $E_\lambda^s f$,corresponding to $L(\xi,\zeta)$ the Theorem 9 holds.

Acknowledgment. The author is deeply gratefull to Professor Hidenori Fujiwara for discussion of these results.

REFERENCES

1. Elias M.Stein.Localization and summability of multiple Fourier series //Acta. Math. 1958.v.100, № 1-2,p.93-147.

2. Sh.A.Alimov, V.A.Ilin and E.M.Nikishin. Convergence problems of multiple trigonometric series and spectral decompositions // Russian Math.Surveys. 1976.v.31,p.28-83.

3. Sh.A.Alimov, R.R.Ashurov and A.K.Pulatov. The multiple Fouirier series and integrals // Itogi nauki i techniki.

Sovremennie problemi matematiki.Fundamentalnie Napravlenija. 1989. v.42,p.5-105; English transl. in Math.Encyclopaedia. Springer-Verlag. 1990.v.42.

4. Lars Hormander.Some Recent Advances in the Basic Sciences v.2 Belfer Graduate School Sci. Yeshiva Univ.New York 1969.p.155-202.

5. V.A.Ilin and Sh.A.Alimov. Convergence conditions for spectral decompositions corresponding to self-adjoint extensions of elliptic operator // Differential Equetions. 1971.v.7.p.670-710.

6. T.Peetre. Remark on eigenfunction expansions for elliptic operators with constant coefficients // Math.Scand. 1964.v.15. p.83-92.

7. S.M.Nikolskii. Aproximtion of Functions of Several Variables and Imbedding Theorems. Springer-Verlag.New York.1975.

8. A.N.Varchenko //Functional analysis.1983.v.17.p.1-6.

9. M.V.Fedoryuk. The Saddle-Point Method. [in Russian],Nauka. Moscow.1977.

10. G.Bergendal.Convergence and summability of eigenfunction expansions connected with elliptic differential operators //Medd.Lunds Unniver. Mat. Sem. 1959.v.14.p.1-63.

11.Elias M.Stein.Interpolation of linear operators // Trans.Amer. Math.Soc. 1956. v.83. p.482-492.

12.A.N.Varchenko and S.A.Molchanov // Dokl.Akad.Nauk.SSSR. 1977. v.233. № 1 . p.11-14.

13. Burton Randol. On the asymptotic behaviour of the Fourier transform of the indicator function of a convex set // Trans.Amer.Math.Soc. 1969. v.139. p.279-285.