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# Congruences for $M$ and $(M-1)$-curves with odd branches on a hyperboloid

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## 1. Introduction.

In [2], Gudkov completed the isotopic classification of nonsingular irreducible real algebraic curves of order 8 on a hyperboloid and conjectured two congruences concerning $M$ and $(M-1)$-curves of order $4n$ with odd branches, where $n$ is a positive integer. In this paper we partially prove his conjecture (see Corollary 1 and Remark 2) as a corollary of our main theorem. Although the main theorem needs some conditions in its statement, it treats curves of general even order. As an appendix, we give the complete isotopic classification of curves of bidegree (4,4) on a hyperboloid (see Remark 4 and the table in §5). Since curves of bidegree (4,4) are necessarily of order 8, restrictions for curves of order 8 are also applicable to curves of bidegree (4,4). However, the existence problem should be studied separately. Therefore the author thinks that the table is worthy of notice.

## 2. Formulation of the main theorem.

Let $H$ be a nonsingular quadric surface defined by a real polynomial in the complex projective 3-space $\mathbf{P}^3$. It is well-known that the real part $\mathbf{R}H(=H\cap\mathbf{R}P^3)$ of $H$ is homeomorphic to a 2-sphere $S^2$ or a 2-torus $T^2$. In this paper we restrict ourselves to the latter case. Such a quadric surface is called a *hyperboloid*. By an appropriate real linear automorphism of $\mathbf{P}^3$, $H$ is transformed into the quadric surface $\{X_0X_3-X_1X_2=0\}$. As is well known, there is a biholomorphic map between this surface and $\mathbf{P}^1\times\mathbf{P}^1$, which is given by some real polynomials. Hence, in what follows, we often identify $H$ with $\mathbf{P}^1\times\mathbf{P}^1$.

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Let $A(\subset H)$ be a nonsingular irreducible algebraic curve defined by real polynomials. Then the divisor class $[A]$ is written as

$$d[\infty \times \mathbf{P}^1] + r[\mathbf{P}^1 \times \infty]$$

in $\mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^1)$ for some non-negative integers $d$ and $r$. We call $(d, r)$ the *bidegree* of $A$. Then there exists a real bihomogeneous polynomial $F(X_0, X_1; Y_0, Y_1)$ of bidegree $(d, r)$ such that $A$ is the zero locus of $F$ in $\mathbf{P}^1 \times \mathbf{P}^1$. We may think of $\mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^1)$ as a subgroup of $H^2(\mathbf{P}^1 \times \mathbf{P}^1; \mathbf{Z})$. The *order* of $A$ is the intersection number

$$[A] \cdot ([\infty \times \mathbf{P}^1] + [\mathbf{P}^1 \times \infty]) = d + r.$$

Now we set $\mathbf{R}A = A \cap \mathbf{R}H$. We say $\mathbf{R}A$ is an *$(M - i)$-curve of bidegree* $(d, r)$ if the number of *branches*, i.e., connected components of that is

$$(d - 1)(r - 1) + 1 - i.$$

We note that Gudkov (see [2]) defines *$(M - i)$-curves of a fixed even order* $2m$ to be curves with

$$(m - 1)^2 + 1 - i$$

branches. For a branch $C$ of $\mathbf{R}A$, the homology class [C] is written as

$$s[\infty \times \mathbf{R}P^1] + t[\mathbf{R}P^1 \times \infty]$$

in $H_1(\mathbf{R}P^1 \times \mathbf{R}P^1; \mathbf{Z})$, where $s$ and $t$ are some integers. Following [2], we call $(s, t)$ the *torsion* of the branch $C$. We say the branch $C$ is *odd* (or *even*) if the intersection number

$$[C] \cdot ([\infty \times \mathbf{R}P^1] + [\mathbf{R}P^1 \times \infty]) = t - s$$

is odd (or even). We say $C$ is an *oval* (or *non-oval*) if $(s, t) = (0, 0)$ (or otherwise). We note that the torsions are equal to the same fixed value for all non-ovals of $\mathbf{R}A$ if we give them appropriate orientations.

Now suppose that both $d$ and $r$ are even. We note that if $\mathbf{R}A$ is of even order and has odd branches, then the number of odd branches is even, and hence, both $d$ and $r$ are even (cf. Theorem below). Then we can define $B^+$ (resp. $B^-$) to be the set $\{F \geq 0\}$(resp. $\{F \leq 0\}$) in $\mathbf{R}P^1 \times \mathbf{R}P^1$. Moreover, we can take a double covering $Y \to \mathbf{P}^1 \times \mathbf{P}^1$ branched along $A$. Let $T^+$ and $T^-$ be the two lifts of the complex conjugation of $\mathbf{P}^1 \times \mathbf{P}^1$, and let $\mathbf{R}Y^+$ and $\mathbf{R}Y^-$ be their fixed point sets. Then the restrictions of the covering map make $\mathbf{R}Y^+$ and $\mathbf{R}Y^-$ be double coverings of $B^+$ and $B^-$ branched along $\mathbf{R}A$ respectively.

REMARK 1 (see [5,Remark 3.2]). For $\mathbf{R}A$ with non-ovals of torsion $(s,t)$, $\mathbf{R}Y^\pm$ can be regarded as the doubles of $B^\pm$ through the covering map if and only if $\frac{d}{2}t + \frac{r}{2}s$ is even.

There are several articles (for instance, [2], [7], [5], [6], and [1]) on real algebraic curves on a hyperboloid.

Our new results are as follows.

THEOREM. *Let $A$ be a nonsingular irreducible real algebraic curve of bidegree $(d,r)$ and of even order on a hyperboloid. Suppose that $\mathbf{R}A$ has odd branches of torsion $(s,t)$ with $s$ odd and $t$ even, and $r \equiv 0 \pmod 4$.*

(i) *If $\mathbf{R}A$ is an $M$-curve of bidegree $(d,r)$, then we have* $\chi(B^\pm) \equiv \frac{dr}{2} \pmod 8$.

(ii) *If $\mathbf{R}A$ is an $(M-1)$-curve of bidegree $(d,r)$, then we have* $\chi(B^\pm) \equiv \frac{dr}{2} \pm 1 \pmod 8$.

COROLLARY 1 (related to Gudkov's conjecture (see [2])). *Let $A$ be a nonsingular irreducible real algebraic curve of order $8n$ on a hyperboloid. Suppose that $\mathbf{R}A$ has odd branches.*

(i) *If $\mathbf{R}A$ is an $M$-curve of order $8n$, then we have* $\chi(B^\pm) \equiv 0 \pmod 8$.

(ii) *If $\mathbf{R}A$ is an $(M-1)$-curve of order $8n$, then we have* $\chi(B^\pm) \equiv \pm 1 \pmod 8$.

REMARK 2. In general, $M$ and $(M-1)$-curves of order $4n$ are also that of bidegree $(2n,2n)$. By (4) in the proof of Lemma 1 below, for an $M$-curve (resp. $(M-1)$-curve) of bidegree $(2n,2n)$ with non-ovals, we automatically have $\dim H_*(B^+;\mathbf{Z}_2) = (2n-1)^2 + 1$ (resp. $(2n-1)^2$). Hence, we do not need to assume this equality (cf. [2]). Gudkov's conjecture is just unproved in the case $n$ is odd.

We will apply Corollary 1 to curves of bidegree (4,4) in §5.

**3. Some basic results.**

In this section we only assume that both $d$ and $r$ are even and $\mathbf{R}A$ has non-ovals that are not necessarily odd and show some properties of the double covering $\pi : Y \to \mathbf{P}^1 \times \mathbf{P}^1$ and the involutions $T^{\pm}$. In what follows, we treat only $T^+$. For $T^-$, we have only to replace '+' by '−' for the reason that the hyperboloid is divided into some annuli by non-ovals.

We first note that $Y$ is a simply connected compact nonsingular complex algebraic surface. Hence, in particular, we see that $H_1(Y;\mathbf{Z}) = 0$ and $H^2(Y;\mathbf{Z})$ is free. Moreover, we can regard the divisor class group Pic($Y$) as a subgroup of $H^2(Y;\mathbf{Z})$. We set

$$E_1 = \{x \in H^2(Y;\mathbf{Z}) | (T^+)^*(x) = x\}$$

and

$$E_{-1} = \{x \in H^2(Y;\mathbf{Z}) | (T^+)^*(x) = -x\}.$$

We note that $E_1$ and $E_{-1}$ are orthogonal each other with respect to the intersection form. Let $Q_{\pm 1}$ denote the restrictions of the form to $E_{\pm 1}$, and $\sigma_{\pm 1}$ their signature.

We consider the second Wu class $v_2$ ($\in H^2(Y;\mathbf{Z}_2)$) of $Y$ (for instance, see [8]). This class has the property that $x \cdot v_2 = x^2$ for every $x$ in $H^2(Y;\mathbf{Z}_2)$. Since $w_1 = 0$, we have $v_2 = w_2$ by Wu's formula, where $w_i$ is the $i$−th Stiefel-Whitney class of $Y$. Hence we have

$$v_2 = (c_1)_{\text{mod } 2} = (-[K_Y])_{\text{mod } 2}, \tag{1}$$

where $c_1$ is the first Chern class of $Y$ and $K_Y$ is the canonical divisor of $Y$. We note that

$$[K_Y] = (\frac{d}{2} - 2)h_1 + (\frac{r}{2} - 2)h_2, \tag{2}$$

where we set $h_1 = \pi^*[\infty \times \mathbf{P}^1]$ and $h_2 = \pi^*[\mathbf{P}^1 \times \infty]$. We note that $h_1$ and $h_2$ are contained in $E_{-1}$. Since $c_1$ is contained in $E_{-1}$, we see that $E_1$ is always an even lattice. We set

$$E = \{x \in H_2(Y;\mathbf{Z}_2) | T_*^+(x) = x\},$$

and let

$$\alpha_2 : H_2(Y/T^+, \mathbf{R}Y^+; \mathbf{Z}_2) \oplus H_2(\mathbf{R}Y^+; \mathbf{Z}_2) \to H_2(Y; \mathbf{Z}_2)$$

be the homomorphism in the Smith exact sequence for the involution $T^+ : Y \to Y$. Since $H_1(Y; \mathbf{Z}) = 0$, as in the proof of [4, Lemma 3.7], we have

$$\text{(3)} \qquad \mathrm{Im}\alpha_2 = E.$$

LEMMA 1 (cf. [5, Remark3.1]). *A curve* $\mathbf{R}A$ *with non-ovals is an* $(M - i)$*-curve if and only if the pair* $(Y, T^+)$ *is an* $(M - (i + 2))$*-manifold in the sense of* [8], *i.e.,*

$$\dim H_*(\mathbf{R}Y^+; \mathbf{Z}_2) = \dim H_*(Y; \mathbf{Z}_2) - 2(i + 2).$$

PROOF: For a curve with non-ovals, it is easy (cf. [5, §3]) to verify that

$$\text{(4)} \qquad \dim H_*(B^+; \mathbf{Z}_2) = \sharp\{\text{branches of } \mathbf{R}A\}$$

and

$$\dim H_*(\mathbf{R}Y^+; \mathbf{Z}_2) = 2 \cdot \sharp\{\text{branches of } \mathbf{R}A\}.$$

We note that we must add 2 to the right-hand side of (4) if the curve has only ovals and $B^+$ contains the exterior of all the ovals. Here, however, we assume that the curve has some non-ovals. On the other hand, we have (see also [5, §3])

$$\dim H_*(Y; \mathbf{Z}_2) = \chi(Y) = 6 + 2(d - 1)(r - 1).$$

Thus we have the required result. ∎

By Lemma 1 and [4, Lemma 3.7], we have

$$\text{(5)} \qquad |\det Q_1| = |\det Q_{-1}| = 2^{i+2}$$

for an $(M - i)$-curve with non-ovals.

According to (2.4) of [8], we have

$$\sigma_1 - \sigma_{-1} = -\chi(\mathbf{R}Y^+).$$

Since the signature $\sigma(Y)$ of $Y$ is equal to $\sigma_1 + \sigma_{-1}$, we have

$$\sigma(Y) + \chi(\mathbf{R}Y^+) = 2\sigma_{-1}.$$

On the other hand, we have $\sigma(Y) = -dr$ (see [5, §3]) and $\chi(\mathbf{R}Y^+) = 2\chi(B^+)$. Thus we have

$$\chi(B^+) - \frac{dr}{2} = \sigma_{-1}. \tag{6}$$

LEMMA 2. *Let $A$ be a nonsingular irreducible real algebraic curve of bidegree $(d, r)$ on a hyperboloid. Suppose that $\mathbf{R}A$ has odd branches of torsion $(s, t)$ with $s$ odd and $t$ even, and $r \equiv 0 \pmod 4$. Then $h_1 \cdot x$ is even for every $x$ in $E_{-1}$.*

PROOF: We note that $t$ is the intersection number $[C] \cdot [\infty \times \mathbf{R}P^1]$ for each odd branch $C$ of $\mathbf{R}A$. We may think that $\infty \times \mathbf{P}^1$ intersects the curve $A$ transversely in $\mathbf{P}^1 \times \mathbf{P}^1$. Since $\infty \times \mathbf{P}^1$ and $A$ are real curves, i.e., invariant under the complex conjugation, $\infty \times \mathbf{R}P^1$ intersects $\mathbf{R}A$ transversely in $\mathbf{R}P^1 \times \mathbf{R}P^1$. The inverse image $\pi^{-1}(\infty \times \mathbf{P}^1)$ is a nonsingular real curve in $Y$ and represents the cohomology class $h_1$. We set $K = \mathbf{R}Y^+ \cap \pi^{-1}(\infty \times \mathbf{P}^1)$. Then we have

$$K = \pi^{-1}(B^+) \cap \pi^{-1}(\infty \times \mathbf{P}^1) = \pi^{-1}(B^+ \cap \infty \times \mathbf{R}P^1).$$

By the assumption, we see that $\frac{d}{2}t + \frac{r}{2}s$ is even. Hence, $\mathbf{R}Y^+$ can be regarded as the double of $B^+$ through the covering map $\pi$ (recall Remark 1), and $K$ is also regarded as the double of $B^+ \cap \infty \times \mathbf{R}P^1$.

We will show that the cycle $K$ is a $\mathbf{Z}_2$-boundary in $\mathbf{R}Y^+$. To prove this, it suffices to prove that, for every connected component $B_k^+$ of $B^+$, $\pi^{-1}(B_k^+ \cap \infty \times \mathbf{R}P^1)$ is a $\mathbf{Z}_2$-boundary in $\mathbf{R}Y^+$.

We first consider the case when the boundary $\partial B_k^+$ of $B_k^+$ consists of two odd branches, denoted by $C_1$ and $C_2$, and some ovals. $C_1$ and $C_2$ divide the hyperboloid into two annuli.

$B_k^+$ is obtained from one of them (say $R$) by removing the interiors of some ovals. Since $[C_1]\cdot[\infty\times\mathbf{R}P^1]$ is even, $\infty\times\mathbf{R}P^1$ meets $R$ in an even number of intervals joining $C_1$ to $C_2$ together with some arcs each joining $C_1$ (or $C_2$) to itself. Such a union of intervals and arcs always bounds, dividing $R$ into regions $R_+$ and $R_-$. Then $\pi^{-1}(B_k^+\cap\infty\times\mathbf{R}P^1)$ bounds $\pi^{-1}(B_k^+\cap R_+)$.

In the case when the boundary $\partial B_k^+$ consists of only some ovals, $B_k^+$ is indeed the interior of an oval by removing the interiors of some nested ovals since we assume that $\mathbf{R}A$ has some non-ovals (odd branches). Hence we obtain the same fact as above by a similar (but simpler) argument.

Thus we see that

$$[K]=0 \text{ in } H_1(\mathbf{R}Y^+;\mathbf{Z}_2).$$

Since $K$ is a disjoint union of $S^1$, the total Stiefel-Whitney class $w(K)$ is equal to 1. Hence, $\mathbf{R}Y^+$ and $K$ satisfy the conditions a) and b) of [3, Remark 2.2]. By this remark, [3, Lemma 2.3] is applicable to the involution $T^+:Y\to Y$ and $\pi^{-1}(\infty\times\mathbf{P}^1)$. By this lemma, $(h_1)_{\mathrm{mod}\ 2}$ ($\in H_2(Y;\mathbf{Z}_2)$) is orthogonal to $\mathrm{Im}\alpha_2$. By (3), $(h_1)_{\mathrm{mod}\ 2}$ is orthogonal to $E$. Hence, $h_1\cdot x$ is even for every $x$ in $E_{-1}$. ∎

LEMMA 3 (cf. [3, Lemma3.1]). *Let $G$ be a free abelian group of finite rank, and $Q: G\times G\to\mathbf{Z}$ be an even symmetric bilinear form. Suppose that there exists a primitive element $u$ in $G$ such that $Q(u,u)\equiv 0 \pmod 8$ and $Q(u,x)$ is even for every $x$ in $G$.*

(i) *If* $|\det Q|=4$, *then* $\mathrm{Sign}Q\equiv 0 \pmod 8$.

(ii) *If* $|\det Q|=8$, *then* $\mathrm{Sign}Q\equiv\pm1 \pmod 8$.

PROOF: (i) has already appeared in [3, Lemma 3.1]. As in the proof of this lemma, we define an even integral symmetric bilinear form $\tilde{Q}$ by using $Q$. If $|\det Q|=8$, then $|\det\tilde{Q}|=2$. Hence, $\mathrm{Sign}Q=\mathrm{Sign}\tilde{Q}\equiv\pm1 \pmod 8$. ∎

## 4. Proof of theorem.

Now we prove the theorem. By Lemma 2, $h_1 \cdot x$ is even for every $x$ in $E_{-1}$. $\frac{r}{2}$ is assumed to be even. If $\frac{d}{2}$ is also even, then, by (1) and (2), $H^2(Y;\mathbf{Z})$ is an even lattice, and hence, so is $E_{-1}$. If $\frac{d}{2}$ is odd, then, by (1) and (2), $v_2 = (h_1)_{\mathrm{mod}\ 2}$. Therefore, $x^2 \equiv h_1 \cdot x \equiv 0 \pmod 2$ for every $x$ in $E_{-1}$, that is, $E_{-1}$ is an even lattice. It is easy to check that $h_1$ is primitive in $H^2(Y;\mathbf{Z})$, hence in $E_{-1}$, and $h_1^2 = 0$. If $\mathbf{R}A$ is an $M$-curve (resp. $(M-1)$-curve), then, by (5), we have $|\mathrm{det} Q_{-1}| = 4$ (resp. 8). Thus, by Lemma 3 and the formula (6), we have the required results. ∎

## 5. Isotopic classification of curves of bidegree (4,4).

We say two real algebraic curves $\mathbf{R}A$ and $\mathbf{R}A'$ on a hyperboloid $\mathbf{R}H$ are *isotopic* if there exists a continuous map

$$\Phi : \mathbf{R}H \times [0,1] \to \mathbf{R}H$$

such that $\varphi_t := \Phi(\ ,t)$ are homeomorphisms, $\varphi_0$ is the identity map, and $\varphi_1(\mathbf{R}A) = \mathbf{R}A'$.

For a curve of bidegree (4,4), the number of non-ovals is 0, 2, or 4. If it is 4, then we have $|s| \leq 1$ and $|t| \leq 1$, where $(s,t)$ is the torsion of the non-ovals, and the curve has no more branches (Notation: $4(s,t)$). If we have $(|s|,|t|) = (1,2)$ or $(2,1)$, then the curve has no more branches (Notation: $2(s,t)$). If the number of non-ovals is 2 and $|s| \leq 1$ and $|t| \leq 1$, then the non-ovals divide $\mathbf{R}P^1 \times \mathbf{R}P^1$ into two annuli and the interior of each oval does not contain any other ovals (Notation: $2(s,t;m,n)$, where $m$ and $n$ $(m \geq n)$ denote the numbers of ovals contained the two annuli respectively.)

For curves of bidegree (4,4) with non-ovals, the notations defined above describe the isotopy classes.

Convention: We regard two isotopy classes $2(s,t;m,n)$ and $2(s',t';m,n)$ as equivalent if

$$(|s|,|t|) = (|s'|,|t'|) \text{ or } (|t'|,|s'|).$$

As for $4(s,t)$ and $2(s,t)$, we define the equivalence relation in the same way.

Let $\frac{\ m\ }{n}$ denote the equivalence class which contains $2(1,0;m,n)$ and $/m/n$ the class which contains $2(1,1;m,n)$.

From Corollary 1, we obtain the following.

COROLLARY 2 ([2, THEOREM D1]). *Let* $\mathbf{R}A$ *be a nonsingular real algebraic curve of bidegree* $(4,4)$ *in* $\mathbf{R}P^1 \times \mathbf{R}P^1$. *Suppose that* $\mathbf{R}A$ *has non-ovals of torsion* $(\pm 1, 0)$ *or* $(0, \pm 1)$.

(i) *If* $\mathbf{R}A$ *is an* $M$*-curve, then its isotopy type is* $\overline{\frac{8}{0}}$ *or* $\overline{\frac{4}{4}}$.

(ii) *If* $\mathbf{R}A$ *is an* $(M-1)$*-curve, then its isotopy type is* $\overline{\frac{7}{0}}$ *or* $\overline{\frac{4}{3}}$.

REMARK 3. The author confesses that the existence of curves of type $\overline{\frac{6}{2}}$ asserted in [5] is an error.

REMARK 4. Corollary 2 is the last restrictions for isotopy types of curves of bidegree (4,4). In fact, we can realize all the isotopy types listed in [5, Table 1.1] that satisfy this corollary. The existence of some isotopy types is announced in [5] and [6]. We can show the existence of the others by checking that the corresponding curves of order 8 constructed in [2] and [7] are just of bidegree (4,4). The following table gives all the isotopy types, which actually exist, of nonsingular real algebraic curves of bidegree (4,4) in $\mathbf{R}P^1 \times \mathbf{R}P^1$, where we use the well-known notations for the curves which have only ovals (see [2] and [5]).

(Table is inserted here.)

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Curves without non-ovals (46 types)

$\frac{9}{1}$ $\frac{5}{1}4$ $\frac{1}{1}8$

$\frac{8}{1}$ $\frac{5}{1}3$ $\frac{4}{1}4$ $\frac{1}{1}7$ $9$

$\frac{7}{1}$ $\frac{6}{1}1$ $\frac{5}{1}2$ $\frac{4}{1}3$ $\frac{3}{1}4$ $\frac{2}{1}5$ $\frac{1}{1}6$ $8$

$\frac{6}{1}$ $\frac{5}{1}1$ $\frac{4}{1}2$ $\frac{3}{1}3$ $\frac{2}{1}4$ $\frac{1}{1}5$ $7$

$\frac{5}{1}$ $\frac{4}{1}1$ $\frac{3}{1}2$ $\frac{2}{1}3$ $\frac{1}{1}4$ $6$

$\frac{4}{1}$ $\frac{3}{1}1$ $\frac{2}{1}2$ $\frac{1}{1}3$ $5$

$\frac{3}{1}$ $\frac{2}{1}1$ $\frac{1}{1}2$ $4$ $\frac{1}{1}\frac{1}{1}$

$\frac{2}{1}$ $\frac{1}{1}1$ $3$

$\frac{1}{1}$ $2$

$1$

$\phi$

Curves with two non-ovals of torsion (1,0) (20 types)

$$\overline{\frac{8}{0}} \quad \overline{\frac{4}{4}}$$

$$\overline{\frac{7}{0}} \quad \overline{\frac{4}{3}}$$

$$\overline{\frac{m}{n}} \qquad (m \geq n \geq 0 \text{ and } m+n \leq 6)$$

Curves with two non-ovals of torsion (1,1) (25 types)

$$/m/n \qquad (m \geq n \geq 0 \text{ and } m+n \leq 8)$$

Otherwise (3 types)

$$4(1,0) \quad 4(1,1) \quad 2(2,1)$$

TABLE. All the isotopy types of curves of bidegree (4,4) (94 types)