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Developable of a Curve and
Determinacy Relative to Osculation-Type

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Introduction

The ruled surface by tangent lines to a space curve is called the developable surface of the curve. More generally, the developable of a curve in \((n+1)\)-dimensional projective space is defined as the hypersurface "ruled" by osculating \((n-1)\)-subspaces to the curve.

Consider a \(C^\infty\) curve \(\gamma : M \rightarrow \mathbb{R}P^{n+1}\), where \(M\) is a 1-dimensional manifold. We call the germ \(\gamma_p\) at a point \(p \in M\) of finite osculation-type (or simply, of finite type) \(a = (a_1,a_2,\ldots,a_{n+1})\) if there exist a \(C^\infty\) coordinate \(t\) of \((M,p)\) and an affine coordinate \((z_1,\ldots,z_{n+1})\) of \(\mathbb{R}P^{n+1}\) centered at \(\gamma(p)\) such that \(\gamma\) is represented by

\[
z_1 = t^{a_1} + o(t^{a_1}), \quad \ldots, \quad z_{n+1} = t^{a_{n+1}} + o(t^{a_{n+1}}),
\]

where each \(a_i\) is a natural number and \(1 \leq a_1 < \cdots < a_{n+1}\).

A point \(p \in M\) is called an ordinary point if \(\gamma_p\) is of type \((1,2,\ldots,n,n+1)\), and, otherwise, it is called a special point.

For each \(p \in M\) where \(\gamma_p\) is of finite type and for each \(i\), \((0 \leq i \leq n+1)\), there exists the most osculating linear subspace to \(\gamma\) at \(p\) in \(T_{\gamma(p)}\mathbb{R}P^{n+1}\) of dimension \(i\). We call it the osculating \(i\)-subspace and denote by \(O_i(\gamma,p)\). The corresponding projective subspace of \(\mathbb{R}P^{n+1}\) through \(p\) of dimension \(i\) is also denoted by \(O_i(\gamma,p)\). The type of a curve therefore describes the order of tangency to each osculating subspace, and it is the simplest local projective invariant of the curve.

We can define the osculating \(i\)-bundle \(O_i(\gamma)\) in the pullback \(\gamma^{-1}T\mathbb{R}P^{n+1}\). The natural parametrization

\[
de\gamma : O_{n-1}(\gamma) \rightarrow \mathbb{R}P^{n+1}
\]
defined by \((p,q) \mapsto q\), where \(q \in O_{n-1}(\gamma,p) (\subset \mathbb{R}P^{n+1})\), is called also a developable of \(\gamma\).

There are several results on the classification of developables of curves under the \(C^{\infty}\) right-left equivalence.

For a space curve \(\gamma\), at each ordinary point \(p\), the developable has cuspidal singularities along \(\gamma\) and \(\text{dev}(\gamma)_p\) is equivalent to \((z,t) \mapsto (z,t^2,t^3)\).

Cleave [C], Gaffney-du Plessis [GP] and Shcherbak [S1] prove that, at a point \(p\) of type \((1,2,4)\), \(\text{dev}(\gamma)_p\) is equivalent to \((z,t) \mapsto (z,t^2,zt^3)\).

Mond [M1][M2] gives \(C^{\infty}\) normal forms of developable of curves of type \((1,2,2+r)\), \(r \leq 5\), and of type \((1,3,4)\).

In the case of arbitrary dimension, Shcherbak, in [S1], shows the the developable of a curve of type \((2,3,\ldots,n+1,n+2)\) is equivalent to the (parametrization of) \(n\)-dimensional swallowtail, generalizing the observation of Arnol'd [A] for a curve of type \((2,3,4)\) based on the Legendre singularity theory.

In the connection with the study of projections of wave front sets, Shcherbak, further in [S2], gives the \(C^{\infty}\) normal form of the union of the developable of a curve-germ \(\gamma_p\) of type \((1,2,\ldots,n,n+2)\) and the osculating hyperplane \(O_n(\gamma,p)\). See also [K].

We can notice that the type of a curve determines the local \(C^{\infty}\) class of the developable of the curve in the above mentioned cases.

Inspired with these previous results, we are led to the natural problem that whether a type of a curve-germ \(\gamma_p\) determines the \(C^{\infty}\) class of map-germ \(\text{dev}(\gamma)_p\) or not.

If such determinacy for a type \(a\) is established once, then to have the normal form of developables of curves of type \(a\) is reduced to just a calculation of an example. The purpose of this paper is to announce the complete solution of this determinacy problem.

**Theorem 1.** A type \(a\) of a curve-germ in \(\mathbb{R}P^{n+1}\) determines \(C^{\infty}\) class of developable if and only if \(a\) is one of following types:

- \((I)_{n,r} a = (1,2,\ldots,n,n+r)\), \(r = 1,2,\ldots\),
- \((II)_{n,i} a = (1,2,\ldots,i,i+2,\ldots,n+1,n+2)\), \(0 \leq i \leq n-1\),
- \((III)_{n} a = (3,4,\ldots,n+2,n+3)\),
- \((IV) a = (3,5)\), \((V) a = (1,3,5)\).
Further, in this case, for any $\gamma_p$ of type a, the map-germ $\text{dev}(\gamma)_p$ is $C^\infty$ right left equivalent to $(z', U(z', t), U_r(z', t)) : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^{n+1}, 0$, where $(z', t) = (z_1, \ldots, z_{n-1}, t)$ is a coordinate of $(\mathbb{R}^n, 0)$,

$$U(z', t) = \frac{t^{a_n}}{a_n!} + z_1 \frac{t^{a_n-a_1}}{(a_n - a_1)!} + \cdots + z_{n-1} \frac{t^{a_n-a_{n-1}}}{(a_n - a_{n-1})!},$$

$r = a_{n+1} - a_n$ and

$$U_r(z', t) = \int_0^t \frac{t'}{r!} \frac{\partial U}{\partial t} dt.$$

Notice that the developable appears as a component of the envelope of one-parameter family of osculating hyperplanes to a curve-germ $\gamma_p$. In the case $a_{n+1} - a_n > 1$, the envelope also has a component $O_n(\gamma, p)$ itself. In this case therefore it is natural to classify developables by diffeomorphisms preserving $O_n(\gamma, p)$. Then we have

**Theorem 2.** A type a of a curve-germ in $\mathbb{R}P^{n+1}$ determines $C^\infty$ class of envelope of osculating hyperplanes if and only if a is one of types $(I)_{n,r}, r = 1, 2, \ldots, (II)_{n,i}$ and $(III)_{n}, n \geq 2$, in Theorem 1.

**Theorem 3.** A type a of a curve-germ $\gamma_p$ in $\mathbb{R}P^{n+1}$ determines $C^\infty$ class of the union of developable and $O_n(\gamma, p)$ if and only if a is one of types $(I)_{n,r}$ and $(II)_{n,i}$ in Theorem 1.

These results unifies and generalizes the results of [C], [G-P] on $(I)_{2,2}$, the results of [A], [S1], [S2], on $(I)_{n,2}$ and $(II)_{n,0}$, and the results of [M1] [M2] on $(I)_{2,r}, (r \leq 5)$, and $(II)_{2,1}$.

The proofs of Theorems 1, 2 and 3 will be given in a forthcoming paper.
Mond's theorem

Based on Theorem 1, we reprove the following result due to Mond [M1], [M2, Corollary 0.2]:

**Corollary.** Let \( \gamma : \mathbb{R}, 0 \rightarrow \mathbb{R}P^3 \) be a curve-germ of type \((1, 2, 2 + r)\). Then \( \text{dev}(\gamma) : \mathbb{R}^2, 0 \rightarrow \mathbb{R}P^3 \) is a topological embedding if \( r \) is odd, and \( \text{dev}(\gamma) \) has a single curve of selfintersection if \( r \) is even.

**Proof:** By Theorem 1, \( \text{dev}(\gamma) \) is \( C^\infty \) equivalent to the germ at 0 of

\[
f(z, t) = (z, \frac{t^2}{2} + zt, \int_0^t \frac{s^r}{r!}(s + z)ds) : \mathbb{R}^2 \rightarrow \mathbb{R}^3.
\]

Now, assume \( f(x_1, t_1) = f(x_2, t_2), (x_i, t_i) \in \mathbb{R}^2, i = 1, 2 \). Then we see \( x_1 = x_2, x_1 = -(1/2)(t_1 + t_2) \) and \( \int_{t_1}^{t_2} s^r(s + x_1)ds = 0 \). Thus, setting \( \sigma = s + x_1 \), we have

\[
\int_{-a}^{a} (\sigma - x_1)^r \sigma d\sigma = 0 \quad \cdots (*)
\]

where \( a = (1/2)(t_2 - t_1) \).

If \( r \) is odd, then the left hand side of (*) is equal to an integral from \(-a\) to \( a\) with almost everywhere positive integrand. Hence we have \( a = 0 \). This means that \( (x_1, t_1) = (x_2, t_2) \) and that \( f \) is injective.

By a similar argument, if \( r \) is even, then we have \( x_1 = 0 \) or \( (x_1, t_1) = (x_2, t_2) \).

Since \( f \) is a finite mapping and \( f|\{z = 0\} = (0, t^2/2, (r + 1\{r^r + 2/(r + 2)\}) \), we see \( f \) is an embedding in the complement of a double point curve \( \{z = 0\} \).

**References**


