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Developable of a Curve and Determinacy Relative to Osculation-Type

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Introduction

The ruled surface by tangent lines to a space curve is called the developable surface of the curve. More generally, the developable of a curve in $(n+1)$-dimensional projective space is defined as the hypersurface "ruled" by osculating $(n-1)$-subspaces to the curve.

Consider a C^∞ curve $\gamma : M \longrightarrow \mathbf{R}P^{n+1}$, where M is a 1-dimensional manifold. We call the germ γ_p at a point $p \in M$ of finite osculation-type (or simply, of finite type) $\mathbf{a} = (a_1, a_2, \ldots, a_{n+1})$ if there exist a C^∞ coordinate t of (M, p) and an affine coordinate $(x_1, \ldots, x_{n+1})$ of $\mathbf{R}P^{n+1}$ centered at $\gamma(p)$ such that γ is represented by

$$x_1 = t^{a_1} + o(t^{a_1}), \quad \ldots, \quad x_{n+1} = t^{a_{n+1}} + o(t^{a_{n+1}}),$$

where each a_i is a natural number and $1 \leq a_1 < \cdots < a_{n+1}$.

A point $p \in M$ is called an ordinary point if γ_p is of type $(1, 2, \ldots, n, n+1)$, and, otherwise, it is called a special point.

For each $p \in M$ where γ_p is of finite type and for each i, $(0 \leq i \leq n+1)$, there exists the most osculating linear subspace to γ at p in $T_{\gamma(p)}\mathbf{R}P^{n+1}$ of dimension i. We call it the osculating i-subspace and denote by $O_i(\gamma, p)$. The corresponding projective subspace of $\mathbf{R}P^{n+1}$ through p of dimension i is also denoted by $O_i(\gamma, p)$. The type of a curve therefore describles the order of tangency to each osculating subspace, and it is the simplest local projective invariant of the curve.

We can define the osculating i-bundle $O_i(\gamma)$ in the pullback $\gamma^{-1}T\mathbf{R}P^{n+1}$. The natural parametrization

$$\mathrm{dev}(\gamma) : O_{n-1}(\gamma) \longrightarrow \mathbf{R}P^{n+1}$$

defined by $(p,q) \mapsto q$, where $q \in O_{n-1}(\gamma,p)(\subset \mathbf{R}P^{n+1})$, is called also a developable of γ.

There are several results on the classification of developables of curves under the C^∞ right-left equivalence.

For a space curve γ, at each ordinary point p, the developable has cuspidal singularities along γ and $\mathrm{dev}(\gamma)_p$ is equivalent to $(x,t) \mapsto (x,t^2,t^3)$.

Cleave [C], Gaffney-du Plessis [GP] and Shcherbak [S1] prove that, at a point p of type $(1,2,4)$, $\mathrm{dev}(\gamma)_p$ is equivalent to $(x,t) \mapsto (x,t^2,xt^3)$.

Mond [M1][M2] gives C^∞ normal forms of developable of curves of type $(1,2,2+r)$, $r \leq 5$, and of type $(1,3,4)$.

In the case of arbitrary dimension, Shcherbak, in [S1], shows the the developable of a curve of type $(2,3,\ldots,n+1,n+2)$ is equivalent to the (parametrization of) n-dimensional swallowtail, generarizing the observation of Arnol'd [A] for a curve of type $(2,3,4)$ based on the Legendre singularity theory.

In the connection with the study of projections of wave front sets, Shcherbak, further in [S2], gives the C^∞ normal form of the union of the developable of a curve-germ γ_p of type $(1,2,\ldots,n,n+2)$ and the osculating hyperplane $O_n(\gamma,p)$. See also [K].

We can notice that the type of a curve determines the local C^∞ class of the developable of the curve in the above mentioned cases.

Inspired with these previous results, we are led to the natural problem that whether a type of a curve-germ γ_p determines the C^∞ class of map-germ $\mathrm{dev}(\gamma)_p$ or not.

If such determinacy for a type $\mathbf{a}$ is established once, then to have the normal form of developables of curves of type $\mathbf{a}$ is reduced to just a calculation of an example. The purpose of this paper is to announce the complete solution of this determinacy problem.

THEOREM 1. *A type* $\mathbf{a}$ *of a curve-germ in* $\mathbf{R}P^{n+1}$ *determines* C^∞ *class of developable if and only if* $\mathbf{a}$ *is one of following types:*

$(I)_{n,r}$ $\mathbf{a} = (1,2,\ldots,n,n+r)$, $\quad r = 1,2,\ldots,$

$(II)_{n,i}$ $\mathbf{a} = (1,2,\ldots,i,i+2,\ldots,n+1,n+2)$, $\quad 0 \leq i \leq n-1,$

$(III)_n$ $\mathbf{a} = (3,4,\ldots,n+2,n+3)$,

(IV) $\mathbf{a} = (3,5)$, $\quad (V)$ $\mathbf{a} = (1,3,5)$.

Further, in this case, for any γ_p of type **a**, *the map-germ $dev(\gamma)_p$ is C^∞ right left equivalent to $(x', U(x',t), U_r(x',t)) : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^{n+1}, 0$, where $(x',t) = (x_1, \ldots, x_{n-1}, t)$ is a coordinate of $(\mathbf{R}^n, 0)$,*

$$U(x',t) = \frac{t^{a_n}}{a_n!} + x_1 \frac{t^{a_n - a_1}}{(a_n - a_1)!} + \cdots + x_{n-1} \frac{t^{a_n - a_{n-1}}}{(a_n - a_{n-1})!},$$

$r = a_{n+1} - a_n$ *and*

$$U_r(x',t) = \int_0^t \frac{t^r}{r!} \frac{\partial U}{\partial t} dt.$$

Notice that the developable apears as a component of the envelope of one-parameter family of osculating hyperplanes to a curve-germ γ_p. In the case $a_{n+1} - a_n > 1$, the envelope also has a component $O_n(\gamma, p)$ itself. In this case therefore it is natural to classify developables by diffeomorphisms preserving $O_n(\gamma, p)$. Then we have

THEOREM 2. *A type* **a** *of a curve-germ in $\mathbf{R}P^{n+1}$ determines C^∞ class of envelope of osculating hyperplanes if and only if* **a** *is one of types $(I)_{n,r}, r = 1, 2, \ldots, (II)_{n,i}$ and $(III)_n, n \geq 2$, in Theorem 1.*

THEOREM 3. *A type* **a** *of a curve-germ γ_p in $\mathbf{R}P^{n+1}$ determines C^∞ class of the union of developable and $O_n(\gamma, p)$ if and only if* **a** *is one of types $(I)_{n,r}$ and $(II)_{n,i}$ in Theorem 1.*

These results unifies and generalizes the results of [C], [G-P] on $(\mathrm{I})_{2,2}$, the results of [A], [S1], [S2], on $(\mathrm{I})_{n,2}$ and $(\mathrm{II})_{n,0}$, and the results of [M1] [M2] on $(\mathrm{I})_{2,r}, (r \leq 5)$, and $(\mathrm{II})_{2,1}$.

The proofs of Theorems 1,2 and 3 will be given in a forthcoming paper.

Mond's theorem

Based on Theorem 1, we reprove the following result due to Mond [M1], [M2, Corollary 0.2]:

COROLLARY. *Let $\gamma : \mathbf{R}, 0 \longrightarrow \mathbf{R}P^3$ be a curve-germ of type $(1, 2, 2+r)$. Then dev(γ) : $\mathbf{R}^2, 0 \longrightarrow \mathbf{R}P^3$ is a topological embedding if r is odd, and dev(γ) has a single curve of selfintersection if r is even.*

PROOF: By Theorem 1, dev(γ) is C^∞ equivalent to the germ at 0 of

$$f(x,t) = (x, \frac{t^2}{2} + xt, \int_0^t \frac{s^r}{r!}(s+x)ds) : \mathbf{R}^2 \longrightarrow \mathbf{R}^3.$$

Now, assume $f(x_1,t_1) = f(x_2,t_2), (x_i,t_i) \in \mathbf{R}^2, i = 1,2$. Then we see $x_1 = x_2, x_1 = -(1/2)(t_1+t_2)$ and $\int_{t_1}^{t_2} s^r(s+x_1)ds = 0$. Thus, setting $\sigma = s + x_1$, we have

$$\int_{-a}^{a} (\sigma - x_1)^r \sigma d\sigma = 0 \qquad \cdots (*),$$

where $a = (1/2)(t_2 - t_1)$.

If r is odd, then the left hand side of (*) is equal to an integral from $-a$ to a with almost everywhere positive integrand. Hence we have $a = 0$. This means that $(x_1,t_1) = (x_2,t_2)$ and that f is injective.

By a similar argument, if r is even, then we have $x_1 = 0$ or $(x_1,t_1) = (x_2,t_2)$.

Since f is a finite mapping and $f|\{x=0\} = (0, t^2/2, (r+1)\{t^{r+2}/(r+2)!\})$, we see f is an embedding in the complement of a double point curve $\{x = 0\}$.

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