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A simple approach to Thom polynomials for C^∞ maps : Vassil'ev complex for contact classes

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In recent works by V.A.Vassil'ev ([7]), he has constructed the "universal cochain complex" related to the hierarchy of degenerate singularities of smooth functions. This represents combinatrial aspects of complicated relations between distinct singularity types. The purpose of this note is to construct the "universal complex" for singularities of smooth mappings, analogous to Vassil'ev's one.

As usual we let $J^k(n,p)$ denote the space of k-jets of germs from $\mathbf{R}^n, 0$ to $\mathbf{R}^p, 0$, and K^k k-jets of the group of contact equivalence.

1. K^k-classification of $J^k(n,p)$ and Vassil'ev complex

DEFINITION(1.1). *Let γ be a stratification of $J^k(n,p)$ such that each stratum of γ is a semialgebraic set. γ is said to be a K^k-classification of $J^k(n,p)$ if γ satisfies the following properties.*
(1) Each stratum of γ is K^k invariant.
(2) If a stratum of γ has connected components L_1 and L_2, then there are two points $z_i \in L_i (i = 1,2)$ such that z_1 is K^k-equivalent to z_2.
(3) γ satisfies the Whitney regularity condition.

Remark(1.2) Let γ be a K^k-classification of $J^k(n,p)$. Then,
(1) γ is finite set. This is verifed from locally finiteness and the fact that elements of γ are invariant under the action of positive real numbers $\mathbf{R}^+$ on $J^k(n,p)$ which is defined by $t.j^k f := j^k(tf)$ where $t \in \mathbf{R}^+$.
(2) γ satisfies the frontier condition– for $X, Y \in \gamma$ with $\overline{X} \cap Y \neq \phi$, it holds that $Y \subset \overline{X}$. The reason is as follows. We let γ^c denote a stratification of $J^k(n,p)$ into all connected components of the strata of γ. Then, it follows from (3) in Definition(1.1) that γ^c satisfies the frontier condition (see Gibon et al.[1], p.61, (5.6) and (5.7)). Therefore, for $X, Y \in \gamma$ and some connected component L_1 of Y with $\overline{X} \cap L_1 \neq \phi$, we see $L_1 \subset \overline{X}$. For any connected component L_2 of Y, we can see that $\overline{X} \cap L_2 \neq \phi$ by using (1),(2) in Definition(1.1), and hence we have $L_2 \subset \overline{X}$ as well as L_1. Thus $Y \subset \overline{X}$.

PROPOSITION(1.3). *Let η be a locally finite partition of $J^k(n,p)$ into semialgebraic K^k invariant subsets. Then, there is a K^k-classification of $J^k(n,p)$, any stratum of which belongs to some element of η.*

In fact, we can construct a K^k-classification subordinate to η by using only the follwing operations: Boolean operations, taking the closure, partition into families of K^k-equivalent components on the linear connction, and removing the singular locus and bad point sets (e.g. Vassil'ev [7] or Gibson et al.[1]).

(1.4) Assume that we are given a K^k-classification γ of $J^k(n,p)$. For γ, we set

$C(\gamma) :=$ Z_2-module generated by all elements of γ,

$C^s(\gamma) :=$ Z_2-module generated by elements of γ with codimension $=s$ ($s \geq 0$).

Then, $C(\gamma) = \bigoplus_s C^s(\gamma)$.

Let us define the boundary operator $\delta_\gamma : C^s(\gamma) \to C^{s+1}(\gamma)$.

Let X be a strata of γ with codim s. From the frontier condition of γ (see (1.2-2)), $\overline{X}$ (closure of X) admits a stratification whose elements are all the strata $Y \in \gamma$ with $Y \subset \overline{X}$. This stratifcation of $\overline{X}$ induces the filtration $(V_i)_{i\geq 0}$ of $\overline{X}$ by codimension where V_i is the union of the strata with codimension $\geq s+i$ contained in $\overline{X}$. Set $m = \dim J^k(n,p)$, and let

$$\partial : H_{m-s}(V_0, V_1; Z_2) \to H_{m-s-1}(V_1, V_2; Z_2)$$

denote the connection homomorphism. Here we use closed supported homology groups. Let μ_X be the fundarmental class of $H_{m-s}(V_0, V_1; Z_2)$. For any $Y \in \gamma$ with $Y \subset V_1 - V_2$ and a point $y \in Y$, we define $[X;Y]_y \in Z_2$ by the value of $j_* \circ \partial(\mu_X)$ where $j_* : H_{m-s-1}(V_1, V_2; Z_2) \to H_{m-s-1}(V_1, V_1 - y; Z_2)$. From the conditions (1), (2) in Definition(1.1), we can see that $[X;Y]_y$ depends only on X and Y, and therefore we can define $[X;Y]$ by $[X;Y]_y$ for some $y \in Y$. For any $Y \in \gamma$ such that $Y \not\subset V_1 - V_2$, we set $[X;Y] := 0$. Now we can define $\delta_\gamma(X) \in C^{s+1}(\gamma)$ by $\sum_{Y\in\gamma} [X;Y]Y$.

LEMMA(1.5). $\delta_\gamma \circ \delta_\gamma = 0$.

Proof. For $X \in \gamma$ with codim $= s$, we show $\delta_\gamma \circ \delta_\gamma(X) = 0$. For $i \geq 0$, let C_X^{s+i}

denote the submodule of $C^{s+i}(\gamma)$ generated by the strata in $\overline{X}$ with codim $s+i$. If $V_i - V_{i+1} \neq \phi$, then $V_i - V_{i+1}$ is the union of some $Y_\alpha \in \gamma$. Let μ_α be the fundamental class of $H_{m-s-i}(\overline{Y_\alpha}, \overline{Y_\alpha} \cap V_{i+1})$. Then, we have a injective homomorphism $C_X^{s+i} \to H_{m-s-i}(V_i, V_{i+1}; Z_2)$ by sending Y_α to $j_{\alpha*}\mu_\alpha$ where $j_\alpha : Y_\alpha \hookrightarrow V_i$. Thus, we have the following commutative diagram, which implies that $\delta_\gamma \circ \delta_\gamma(X) = 0$.

$$\begin{array}{ccccc} C_X^s & \xrightarrow{\delta} & C_X^{s+1} & \xrightarrow{\delta} & C_X^{s+2} \\ \simeq\downarrow & & \downarrow & & \downarrow \\ H_{m-s}(V_0, V_1) & \xrightarrow{\partial} & H_{m-s-1}(V_1, V_2) & \xrightarrow{\partial} & H_{m-s-2}(V_2, V_3) \end{array}$$

Q.E.D.

(1.6) The set Γ of all K^k-classifications of $J^k(n,p)$ is partially ordered: For γ, γ' in Γ, $\gamma \leq \gamma'$ if any strutum of γ' is contained in some strata of γ.

For $\gamma, \gamma' \in \Gamma$, $\gamma \cap \gamma' = \{X \cap X' | X \in \gamma, X' \in \gamma'\}$ is a locally finite partition of $J^k(n,p)$ whose elements, $X \cap X'$, are K^k invariant semialgebraic sets, and from (1.2) it follows that there is a K^k-classification γ'' such that $\gamma \leq \gamma''$ and $\gamma' \leq \gamma''$. Thus Γ is a directed set.

If $\gamma \leq \gamma'$, then there is a natural homomorphism $(\rho^\gamma_{\gamma'}) : C(\gamma) \to C(\gamma')$: for $X \in \gamma$, $(\rho^\gamma_{\gamma'})(X) := \sum X'_i$, where $X'_i \in \gamma'$ with $X \subset X_i$. It is easy to see that $(\rho^\gamma_{\gamma'})$ commutes with δ_γ and $\delta_{\gamma'}$, and that $(\{C(\gamma)\}, \{(\rho^\gamma_{\gamma'})\})_{\gamma\in\Gamma}$ is an inductive system of cochain complices.

DEFINITION(1.7). $C(K^k_{n,p}) := \varinjlim C(\gamma)$.

$(\rho^\gamma_{\gamma'})$ induces a homomorphism $H^*(C(\gamma); Z_2) \to H^*(C(\gamma'); Z_2)$, and it is easy to see $H^*(C(K^k_{n,p}); Z_2) \simeq \varinjlim H^*(C(\gamma); Z_2)$.

2. Vassil'ev complex for K^∞-equivalence

$C(K^k_{n,p})$ defined in (1.7) depends only on positive intgers n, p and k. In this section, we construct a cochain complex depending only on an intger $l := p - n$, which can be considered an inductive limit of $C(K^k_{n,p})$'s in some sense.

In what follows in this section, we fix an intger l , and a positive intger n is always assumed $n+l>0$.

$J^k(n,n+l)$ is simply denoted by J^k_n. For $z=j^kf(0)\in J^k_n$, set

$$\mathrm{cork}(z):=\ \min(n,n+l)-\mathrm{rank}df(0),$$

and for a subset X of J^k_n, set

$$\mathrm{cork}(X):=\min\{\mathrm{cork}(z),z\in X\}.$$

For intgers m,n such that $m\geq n$, we define

$$i^n_m:J^k_n\longrightarrow J^k_m\quad ;\quad j^kf\mapsto j^k(f\times id_{\mathbf{R}^{m-n}}).$$

We have the following lemma.

LEMMA(2.1). *1) i^n_m is transverse to every K^k-orbit in J^k_m.*
2)For a semialgebraic smooth submanifold X of J^k_n invariant under the K^k-action, set

$$X(m):=K^k(i^n_m(X))=\{Hi^n_m(z)\in J^k_m|z\in X,H\in K^k(m,m+l)\}.$$

Then, $X(m)$ is a semialgebraic smooth submanifold of J^k_m and $\mathrm{codim}X(m)=\mathrm{codim}X$.
3)Let Y be a subset of J^k_m invariant under the K^k- action. Then, $(i^n_m)^{-1}Y=\phi$ if and only if $\mathrm{cork}(X)>\min(n,n+l)$.
4)Let Y be a smooth submanifold of J^k_m. If codim$Y<(n+1)(n+l+1)$, then $\mathrm{cork}(Y)\leq\min(n,n+l)$.

We omit the proof.

(2.2) Let γ^k_n be a K^k-classification of J^k_n and let $m\geq n$. We construct a K^k-classification of J^k_m induced from γ^k_n via i^n_m. Set $A:=\{z\in J^k_m|\mathrm{cork}z\leq\min(n,n+l)\}$ and $B:=\{z\in J^k_m|\mathrm{cork}z>\min(n,n+l)\}$. Then, $J^k_m=A\cup B$(disjoint). From (2),(3) in Lemma(2.1), it follows that A has a K^k-invariant semialgebraic stratification: $A=\bigcup_{X\in\gamma^k_n}X(m)$ where $X(m):=K^k(i^n_m(X))$ for $X\in\gamma^k_n$. Using Lemma(1.3), we obtain a K^k-classification of J^k_m (denoted by $(i^n_m)_*\gamma^k_n$) subordinate a K^k-invariant semialgebraic partition of J^k_m whose elements are B and all $X(m)$. Moreover, a cochain map $C(\gamma_n^k) \to C((i_m^n)_* \gamma_n^k)$ is defined by $X \mapsto X(m)$. When we take the inductive limit of cochain maps $C(\gamma) \to C((i_m^n)_* \gamma)$ over all K^k-classifications of J_n^k, we obtain a cochain map

$$(i_m^n)_\sharp : C(K_{n,n+l}^k) \to C(K_{m,m+l}^k).$$

(2.3) Let γ_n^k be a K^k-classification of J_n^k, and $\pi_k^r : J_n^r \to J_n^k (k \leq r)$ the natural projection. Then, J_n^r has a K^r-classification which consists of all $(\pi_k^r)^{-1} X$ where $X \in \gamma_n^k$. This K^r- classification of J_n^r is denoted by $(\pi_k^r)^* \gamma_n^k$. A cochain map $C(\gamma_n^k) \to C((\pi_k^r)^* \gamma_n^k)$ is defined by $X \mapsto (\pi_k^r)^{-1} X$, and we have

$$(\pi_k^r)^\sharp : C(K_{n,n+l}^k) \to C(K_{n,n+l}^r).$$

LEMMA(2.4). *The following diagram commutes.*

$$\begin{array}{ccc} C(K_{n,n+l}^k) & \xrightarrow{i} & C(K_{m,m+l}^k) \\ \pi \downarrow & & \pi \downarrow \\ C(K_{n,n+l}^r) & \xrightarrow{i} & C(K_{m,m+l}^r) \end{array}$$

This can be easily verified from the constructions of $(i_m^n)_\sharp$ and $(\pi_k^r)^\sharp$. .

DEFINITION(2.5). *For an intger l, $C(K(l)) := \varinjlim_{n,k} C(K_{n,n+l})$. We call the cochain complex $(C(K(l)), \delta)$ Vassil'ev complex for K^∞-equivalence.*

We have an approximation of $C^s(K(l))$ for small s .

PROPOSITION(2.6). *For an arbitrary intger $t > 0$, there are two intgers $k = k(t), n = n(t)$ such that the natural homomorphism $C^s(K_{n,n+l}^k) \to C^s(K(l))$ is an isomorphism for $0 \leq s \leq t$.*

It is enough to choose n satisfying $(n+1)(n+l+1) > t$ (by Lemma(2.1-4)) and sufficient large k (by the fact that $\varinjlim_k \mathrm{codim} W_k(n,p) = 0$, see Gibson [1]).

3. Calculation of $C(K(0))$

In this section, we assue $l = 0$ (this is the equidimensional case), and we give the intial part of $H^*(C(K(0)))$ without the detail of calculations.

From Mather [3,IV], we have the following proposition.

PROPOSITION(3.1). *Let k be sufficient large ($k \geq 9$) and $n \geq 2$. Then there exists K^k-invariant semialgebraic subset Δ_n^k of $J^k(n,n)$ which satisfies the following properties:*
1) codim$\Delta_n^k = 9$
2) $J^k(n,n) - \Delta_n^k$ contains finitely many K^k-orbits with the associated algebras $Q_k(I) \simeq \mathbf{R}[[x,y]]/I + \mathrm{m}^{k+1}$ listed below.

$$A_q : I = < x^{q+1}, y > \quad (0 \leq q \leq 8)$$
$$I_{a,b} : < x^a + y^b, xy > \quad (2 \leq a \leq b, a+b \leq 8)$$
$$II_{a,b} : < x^a + y^b, xy > \quad (2 \leq a \leq b, a+b \leq 8, \quad a,b: \text{ even })$$
$$IV_3 : < x^2 + y^2, x^3 >$$
$$G_7 : < x^2, y^3 >$$
$$G_8 : < x^2 + y^3, xy^2 > .$$

Thus we have a partition η of $J^k(2,2)$ where elements are Δ_2^k and K^k-orbits in $J^k(2,2)$ listed above. Let γ_2^k be the K^k-classification obtained from η by Lemma(1.3). From Lemma(2.6) we see

$$C^s(\gamma_2^k) \simeq C^s(K_{n,n}^k) \simeq C^s(K(0)) \quad \text{for} \quad s \leq 8, n \geq 2.$$

In the following theorem, We determine the value of the differential δ on generators of $C^s(K(0))$ ($s \leq 8$).

THEOREM(3.2). *The differential operator of $(C^s(K(0)), \delta)$ for $s \leq 8$ are described by the following formulae:*

$1)\delta A_s = 0 \quad (0 \leq s \leq 8)$, $\quad 2)\delta I_{2,2} = \delta II_{2,2} = I_{2,3}$,

$3)\delta I_{2,3} = 0$, $\quad 4)\delta I_{2,4} = \delta II_{2,4} = I_{2,5} + I_{3,4}$,

$5)\delta I_{3,3} = \delta IV_3 = G_7$, $\quad 6)\delta I_{2,5} = \delta I_{3,4} = 0$,

$7)\delta G_7 = 0$.

This result is obtained from direct calculus of $[X;Y]$ for all $X, Y \in \gamma_2^k$. Here we omit the detail (see Ohmoto [5] and Lander [2]).

COROLLARY(3.3). *Cohomology groups $H^s(C(K(0)); Z_2)(s \leq 8)$ are given in Table 1. Cycles of $C^s(K(0))$ whose classes are zero in $H^s(C(K(0)))$ are $I_{2,3}(s = 5)$, $I_{2,5}+I_{3,4}(s = 7)$ and $G_7(s = 7)$.*

$* =$	0	1	2	3	4	5	6	7
$H^*(C(K(0)))$	Z_2	Z_2	Z_2	Z_2	$(Z_2)^2$	Z_2	$(Z_2)^3$	$(Z_2)^2$
generator	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7
					$I_{22}+II_{22}$		$I_{24}+II_{24}$	I_{25} (or I_{34})
							$I_{33}+IV_3$	

Table 1

4. Thom polynomials

(4.1) We recall the definition of the Thom polynomials. Let Σ be a singularity type in a jet space $J^k(n,p)$ whese closure $\overline{\Sigma}$ carries a Z_2-fundamental class, and let $f : N^n \to P^p$ be a C^∞ map such that $j^k f$ is transverse to the subbundle $\overline{\Sigma(N,P)}$ of $J^k(N,P)$ associated to the fibre $\overline{\Sigma}$. Then there exists a polynomial $P_\Sigma(f)$ in the Whitney classes of the difference bundle $f^*TP - TN$ such that $P_\Sigma(f)$ is equal to the Poincaré dual of $[\overline{\Sigma(f)}]$ in $H^*(N; Z_2)$ where $\Sigma(f) = (j^k f)^{-1}(\Sigma(N,P))$. $P_\Sigma(f)$ is called the Thom polynomial of Σ for f. If N is noncompact, we assume that the homology groups of N are closed supported.

(4.2) $H^*(C(K(l)))$ is related to the Thom polynomials for smooth maps $f : N^n \to P^p$ with $l = p - n$. For any $c \in C^s(K^k_{n,p})$, there is some K^k-classification γ of $J^k(n,p)$ and $X_i \in \gamma$ such that $\sum_i X_i \in C^s(\gamma)$ represents c. Set $\Sigma_c := \bigcup_i X_i \subset J^k(n,p)$. It is easy to see that if c is a cycle, that is, $\delta_\gamma c = 0$, then Σ_c satisfies the condition on Σ in (4.1) (right-left invariant, and that $\overline{\Sigma}$ has a fundarmental class). Then, we can define the Thom polynomial $P_{\Sigma_c}(f_c)$ for generic maps $f_c : N \to P$.

In the case $n = p$, it follows immdiately from (3.3) that the Thom polynomials of type $I_{2,3}, I_{2,5} \cup I_{3,4}$ and G_7 are all zero (for $I_{2,3}$ and G_7, This has been known by Porteous [6]).

By more precise investigations of the adjacncy relations of contact orbits in $J^k(N,P)$ of type $A_q, I_{a,b}, \ldots$, we can have some relationships in Thom polynomial-:s, for instance, it hold that $P_{A_1}(f)P_{A_{2s}}(f) = P_{A_{2s+1}}(f)(s = 1,2,3)$ for generic smooth maps f (see [4]). Therefore it is expected that there would be something like combinatrial geometry in K^k-orbits of $J^k(N,P)$. To investigate the Vassil'ev complex for contact equivalence is the first step in this direction.

Refereneces

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