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<td>資料の種類</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>作成者</td>
<td>NAKASHIMA, Toshiki</td>
</tr>
<tr>
<td>資料号</td>
<td>数理解析研究所講究録 1991年8月号 1991年8月号 1991年8月号</td>
</tr>
<tr>
<td>号数</td>
<td>765</td>
</tr>
<tr>
<td>ページ</td>
<td>79-90</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82280">http://hdl.handle.net/2433/82280</a></td>
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<td>作成年</td>
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Kyoto University
Crystal Graph and Littlewood Richardson rule

Toshiki NAKASHIMA (RIMS)

§0. Introduction

Recently, Professor Kashiwara constructed "Crystal base". We shall introduce the crystal base and the crystal graph associated with it, and their applications.

§1. Crystal base

1.1 Let $g$ be a finite dimensional simple Lie algebra with the Cartan subalgebra $t$ the set of simple roots $\{\alpha_i \in t^*\}$ and the set of simple coroots $\{h_i \in t\}$. We take an inner product $(\ , \ )$ on $t^*$ such that $(\alpha_i, \alpha_j) \in \mathbb{Z}_{>0}$ and $<h_i, \lambda>=\frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $\lambda \in t^*$. Then the $q$-analogue $U_q(g)$ is the algebra over $\mathbb{Q}(q)$ generated by $e_i, f_i$ and the invertible elements $t_i$ satisfying the following relations;

\begin{align}
(1.1.1) \quad & t_i t_j = t_j t_i \\
(1.1.2) \quad & t_i e_j t_i^{-1} = q^{2(\alpha_j, \alpha_i)} e_j \\
(1.1.3) \quad & t_i f_j t_i^{-1} = q^{-2(\alpha_j, \alpha_i)} f_j \\
(1.1.4) \quad & [e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \quad \text{where} \quad q_i = q^{(\alpha_i, \alpha_i)}
\end{align}

For $i \neq j$, we have, setting $b = 1 - <h_i, \alpha_j >$

\begin{align}
(1.1.4) \quad & \sum_{\mu=0}^{b} e_i^{(\mu)} e_j e_i^{(b-\mu)} = 0 \\
(1.1.4) \quad & \sum_{\mu=0}^{b} f_i^{(\mu)} f_j f_i^{(b-\mu)} = 0
\end{align}

Here $e_i^{(k)} = \frac{e_i}{[k]_i!}$, $f_i^{(k)} = \frac{e_i}{[k]_i!}$, $[n]_i = \frac{q^i - q^{-i}}{q_i - q_i^{-1}}$ and $[k]_i! = \prod_{n=1}^{k} [n]_i$. 

For a finite dimensional $U_q(g)$-module $M$, we set for $\lambda \in P = \{\lambda \in \mathfrak{t}^*; <h_i, \lambda> \in \mathbb{Z}\}$ $M_{\lambda} = \{u \in M; t_i u = q^{2(\alpha_i; \lambda)} u\}$. We call $M$ integrable if $M = \bigoplus M_{\lambda}$. Then we have

\[(1.1.5)\quad M_{\lambda} = \bigoplus_{0 \leq k \leq <h_i, \lambda>} f^{(k)}(M_{\lambda} \cap \text{Ker} \ e_i)\]

We define the operators $\tilde{e}_i, \tilde{f}_i$ acting on $M$ by

\[(1.1.6)\quad \tilde{e}_i f^{(k)}(u) = f^{(k-1)}(u) \quad \text{and} \quad \tilde{f}_i f^{(k)}(u) = f^{(k+1)}(u)\]

for $u \in M_{\lambda} \cap \text{Ker} \ e_i$ and $(\lambda, k)$ as above.

**Definition 1.1.1.** A pair $(L, M)$ is called a crystal base of a finite-dimensional integrable representation $M$ if the following condition are satisfied:

\[(1.1.7)\quad L \text{ is a free sub-}A\text{-module of } M \text{ such that } Q(q) \otimes_A L \cong M.\]

Here $A$ is the ring of rational functions regular at $q = 0$

\[(1.1.8)\quad B \text{ is a base of the } Q\text{-vector space } L/\mathbb{Q}L\]

\[(1.1.9)\quad L = \bigoplus L_{\lambda}, B = \bigsqcup B_{\lambda}\]

where $L_{\lambda} = L \cap M_{\lambda}$ and $B_{\lambda} = B \cap (L_{\lambda}/\mathbb{Q}L_{\lambda})$

\[(1.1.10)\quad \tilde{f}_i L \subset L, \quad \tilde{e}_i L \subset L\]

\[(1.1.11)\quad \tilde{f}_i B \subset B \cup \{0\} \quad \text{and} \quad \tilde{e}_i B \subset B \cup \{0\}\]

\[(1.1.12)\quad \text{For } u, v \in B \text{ and } i \in I, u = \tilde{e}_i v \text{ if and only if } v = \tilde{f}_i u.\]

Then the following results are proved in [k] when $g = A_n, B_n, C_n$ and $D_n$ and announce in [k] in general case. Let $\lambda \in P_+ = \{\lambda \in \mathfrak{t}^*; <h_i, \lambda> \in \mathbb{Z}_{\geq 0}\}$ and $V(\lambda)$ be the irreducible integrable $U_q(g)$-module generated by the highest weight vector $u_\lambda$ of weight $\lambda$. Let $L(\lambda)$ be the sub $A$-module generated by the vectors of the form $\tilde{f}_i \cdots \tilde{f}_i u_\lambda$ and let $B(\lambda)$ be the subset of the $L(\lambda)/qL(\lambda)$ consisting of the non-zero vector of the form $\tilde{f}_i \cdots \tilde{f}_i u_\lambda \text{ mod } qL(\lambda)$.
**Theorem 1.1.2.** $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.

**Theorem 1.1.3.** Let $M \in \mathcal{O}_{\text{int}}$ and $(L, B)$ is a crystal base of $M$. Then there is an isomorphism

$$M \cong \oplus_{j} V(\lambda_j) \text{ by which } (L, B) \cong \oplus_{j} (L(\lambda_j), B(\lambda_j)).$$

**Theorem 1.1.4.** Let $(L_j, B_j)$ be a crystal base of an integrable $U_q(\mathfrak{g})$-module $M_j$ ($j = 1, 2$). Set $L = L_1 \otimes_A L_2 \subset M_1 \otimes M_2$ and $B = \{b_1 \otimes b_2; b_j \in B_j (j = 1, 2)\} \subset L/qL$. Then we have

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if there exists } n \geq 1 \text{ such that } f_i^n b_1 \neq 0 \text{ and } e_i^n b_2 = 0. \\ b_1 \otimes \tilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if there exists } n \geq 1 \text{ such that } e_i^n b_1 \neq 0 \text{ and } f_i^n b_2 = 0. \\ \tilde{e}_i b_1 \otimes b_2 & \text{otherwise.} \end{cases}$$

**Definition 1.1.5.** A crystal graph of a crystal base $(L, B)$ is the colored oriented graph $B$, by the rule:

$$u \xrightarrow{i} v \iff v = \tilde{f}_i u.$$

**Example 1.1.6**

Let $V_i$ is the $i + 1$-dimensional irreducible representation of $U_q(s\mathfrak{l}_2)$.

(i) For $g = s\mathfrak{l}_2$, the crystal graph of $V_2$ is given by as follows;

```
  *    *    *    *
```

(ii) For $g = s\mathfrak{l}_2$, the crystal graph of $V_3$ is given by as follows;

```
  *    *    *    *
  *    *    *    *
```

(iii) By Theorem 1.1.4, the crystal graph of $V_3 \otimes V_2$ is described by as follows;

```
V_3  *    *    *    *
V_2  *    *    *    *
  ↓    ↓    ↓    ↓
  *    *    *    *
  ↓    ↓    ↓    ↓
  *    *    *    *
```
Here, we get that the crystal graph of $V_3 \otimes V_2$ are decomposed into three connected components. They correspond to $V_3$, $V_3$ and $V_1$ respectively.

§2. Remarks on the crystal graphs

2.1 Let us investigate first the crystal graph of the tensor product of the vector representation of $U_q(sl_2)$. The crystal graph of the vector representation is $u_+ \rightarrow u_-$. The crystal graph of the trivial representation is $u_0$. We shall calculate $\tilde{e}(u_{i_1} \otimes \cdots \otimes u_{i_N})$ and $\tilde{f}(u_{i_1} \otimes \cdots \otimes u_{i_N})$, where $e$ and $f$ are generators of $U_q(sl_2)$, and $i_1, \cdots, i_N = +, -, 0$.

Proposition 2.1.1. For $u = u_{i_1} \otimes \cdots \otimes u_{i_N}$ ($i_j = +, -, 0$), the actions of $\tilde{e}$ and $\tilde{f}$ are given by the following three steps:

(I) We neglect $u_0$

(II) If there is $u_+ \otimes u_-$ in $u$, then we neglect such a pair.

(III) Then $\tilde{e}$ changes the $u_-$ in the most right to $u_+$ and $\tilde{f}$ changes the $u_+$ in the most left to $u_-$. If there is no $u_-$ (resp. $u_+$), then $\tilde{e}u = 0$ (resp. $\tilde{f}u = 0$).

Example

For $u = u_- \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_0 \otimes u_+ \\ \otimes u_+$

$\tilde{e}u = u_+ \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_+ \otimes u_-

\otimes u_-$

$\tilde{f}u = u_- \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_- \otimes u_- \otimes u_-$

2.2 Now let $U_q(g)$ be the $q$-analogue as in §1. and let $\lambda_1, \cdots, \lambda_N \in P_+$ and $\lambda = \sum \lambda_i$. Then there is a unique embedding $V(\lambda) \hookrightarrow V(\lambda_1) \otimes \cdots \otimes V(\lambda_N)$ sending $u_\lambda$ to $u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_N}$.

Hence $B(\lambda)$ is embedded into $\otimes_{j=1}^{N} B(\lambda_j)$.

Proposition 2.2.1. Assume the following condition for any $k$ ($1 \leq k < N$).

(2.2.1) $\quad$ If $u \in B(\lambda_{k+1})$ satisfies

(i) $u_{\lambda_k} \otimes u \in B(\lambda_k + \lambda_{k+1})$

(ii) $\tilde{e}u = 0$ for any $i$ such that $< h_i, \lambda_\nu > = 0$ for $\nu \leq k$,

then $u = u_{\lambda_{k+1}}$.
Then we have

\[(2.2.2) \quad V(\lambda) \cong \bigcap_{k=1}^{N-1} V(\lambda_1) \otimes \cdots \otimes V(\lambda_{k-1}) \otimes V(\lambda_k + \lambda_{k+1}) \otimes V(\lambda_{k+2}) \otimes \cdots B(\lambda_N), \]

\[(2.2.3) \quad B(\lambda) \cong \bigcap_{k=1}^{N-1} B(\lambda_1) \otimes \cdots \otimes B(\lambda_{k-1}) \otimes B(\lambda_k + \lambda_{k+1}) \otimes B(\lambda_{k+2}) \otimes \cdots B(\lambda_N). \]

§3. Crystal Graphs for $U_q(C_n)$-modules

3.1 Notation

In the rest of this paper we shall treat the $C_n$-case. Let $(\epsilon_1, \cdots, \epsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of $C_n$ such that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ ($1 \leq i < n$) and $\alpha_n = 2\epsilon_n$ form the set of simple roots. Hence, $\alpha_n$ is the long roots and $\alpha_1, \cdots, \alpha_{n-1}$ are short roots. Let $\{\Lambda_i\}_{1 \leq i \leq n}$ be the dual base of $\{h_i\}_{1 \leq i \leq n}$. Hence $\Lambda_i = \epsilon_1 + \cdots + \epsilon_i$ ($1 \leq i \leq n$).

3.2 The crystal graph of the vector representation.

First let us consider the vector representation $V(\Lambda_1) = V_\square$. Letting $\[i, i \]$ (1 \leq i \leq n) be the base of $Q(q)^{\oplus 2}$, the vector representation of $U_q(C_n)$ is explicitly constructed as follows;

\[t_i \[j\] = q^{2(h_i, \epsilon_j)}[j], \quad t_i \[j\] = q^{2(h_i, -\epsilon_j)}[j].\]

(3.2.1) $e_i \[j\] = \delta_{i+1,j}[\overline{\epsilon}], \quad e_i \[\overline{\epsilon}\] = \delta_{i,j}[\epsilon]. \quad (1 \leq i < n, 1 \leq j \leq n)$

$\[j\] = \delta_{i,j}[\epsilon], \quad \[\epsilon\] = \delta_{i+1,j}[\overline{\epsilon}]$

and

\[e_n \[j\] = 0, \quad e_n \[\overline{\epsilon}\] = \delta_{j,n}[\overline{\epsilon}]\]

(3.2.2) $f_n \[j\] = \delta_{j,n}[\epsilon], \quad f_n \[\epsilon\] = 0. \quad (1 \leq j \leq n)$

Here, we understand $\[i\] = \[i\] = 0$ unless $1 \leq j \leq n$. Then the crystal base $(L(V_\square), B(V_\square))$ is given by

\[(3.2.3) \quad L(V_\square) = \bigoplus_{i=1}^{n} (A \[i\] \oplus A \[\overline{i}\]) \quad B(V_\square) = \{[i], [\overline{i}] ; 1 \leq i \leq n\},\]
and the crystal graph of $V_\square$ is given by;

\[(3.2.4)\] \[
\begin{array}{c}
1 \\
\rightarrow \\
2 \\
\rightarrow \\
\vdots \\
\rightarrow \\
n \\
\rightarrow \\
n-1 \\
\rightarrow \\
n \\
\rightarrow \\
n-1 \\
\rightarrow \\
n \\
\rightarrow \\
2 \\
\rightarrow \\
1
\end{array}
\]

Remark that we have

\[(3.2.5)\] \[
\tilde{e}_i^2 = \tilde{f}_i^2 = 0 \quad \text{on } B(V_\square)
\]

Hence, the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(V_\square)^{\otimes m}$ is given by Proposition 2.1.1.

3.3 The crystal graph of the fundamental representations

The representation $V(A_N)$ with highest weight $A_N$ ($1 \leq N \leq n$) is embedded into $V_\square^{\otimes N}$. Similarly to the $A_n$-case, the connected component of the crystal graph of $B(V_\square)^{\otimes N}$ containing $\bigotimes_{i=1}^{N}$ is that of $B(A_N)$.

We write $i_1 \prec i_2 \prec \cdots \prec i_N$ for $\bigotimes_{i=1}^{N}$.

We denote by $u_{A_N}$ the highest weight vector $\bigotimes_{i=1}^{N}$. We give the linear order on $\{i, \overline{i} ; 1 \leq i \leq n \}$ by

\[(3.3.1)\] \[
i_1 \prec i_2 \prec \cdots \prec n < \overline{n} < \cdots < \overline{2} < \overline{1}.
\]

This ordering is derived by the crystal graph (3.2.4) of $V_\square$. We set

\[(3.3.2)\] \[
I_N^{(G)} = \left\{ \begin{array}{c}
i_1 \\
\vdots \\
i_N \\
\in B(V_\square)^{\otimes N} ; (1) \quad 1 \leq i_1 \prec \cdots i_N \preceq \overline{1}, \\
(2) \quad \text{if } i_k = p \text{ and } i_l = \overline{p}, \quad \text{then } k + (N - l + 1) \leq p
\end{array} \right\}.
\]
Proposition 3.3.2. $B(\Lambda_N)$ coincides with $I_N^{(C)}$

Remark 3.3.3.

(i) $\tilde{e}_i^3 = \tilde{f}_i^3 = 0$ for $1 \leq i \leq n$ and $\tilde{e}_n^2 = \tilde{f}_n^2 = 0$ on $B(\Lambda_N)$.

(ii) If $u \in B(\Lambda_N)$ satisfies $\tilde{f}_i^2 u \neq 0$, then $u$ contains $i$ and $i+1$ but neither $i+1$ nor $i$. If $u \in B(\Lambda_N)$ satisfies $\tilde{e}_i^2 u \neq 0$, then $u$ contains $i+1$ and $\overline{i}$ but neither $i$ nor $i+1$.

(iii) If $u \in B(\Lambda_N)$ satisfies $\tilde{f}_i u \neq 0$ and $\tilde{e}_i u \neq 0$, then $u$ contains $i+1$, $\overline{i+1}$ but neither $i$ nor $\overline{i}$.

3.4 The crystal graph of $V(\Lambda_M + \Lambda_N)$

Now, we shall investigate the crystal graph of $V(\Lambda_M + \Lambda_N)$ with $1 \leq M \leq N \leq n$. By embedding $V(\Lambda_M + \Lambda_N)$ into $V(\Lambda_M) \otimes V(\Lambda_N)$, $B(\Lambda_M + \Lambda_N)$ is the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containing $u_{\Lambda_M} \otimes u_{\Lambda_N}$.

For $u = \begin{array}{c} j_1 \\ \vdots \\ j_M \end{array} \in B(\Lambda_M)$ and $v = \begin{array}{c} i_1 \\ \vdots \\ i_N \end{array} \in B(\Lambda_N)$,

the vector $u \otimes v \in B(\Lambda_M) \otimes B(\Lambda_N)$ will be denoted by $\begin{array}{c} i_1 \\ \vdots \\ i_M \\ j_1 \\ \vdots \\ j_N \end{array}$

Definition 3.3.1. For $1 \leq i \leq j \leq n$, we say that $w \in B(\Lambda_M) \otimes B(\Lambda_N)$ is in $(i, j)$-configuration if $w$ holds the following; (3.4.1) There exist $1 \leq p \leq q \leq r \leq s$ such that $i_p = i$, $j_q = j$, $j_r = \overline{j}$ or $i_p = i$, $i_q = j$, $i_r = \overline{j}$, $j_s = \overline{i}$. Remark that when $i = j$, we understand that $p = q$ and $r = s$. Now, we define $p(i, j; w) = (q - p) + (s - r)$, remark that if there exist another $1 \leq p' \leq q' < r' \leq s'$ such that gives $(i, j)$-configuration on $w$, we take the largest one as $p(i, j; w)$. Let us set
\[ I_{(M,N)}^{(C)} = \{ w \in B(\Lambda_M) \otimes B(\Lambda_N) \mid \begin{array}{lllll} w & \text{satisfies the conditions} & \\ & (M.N.1) & \text{and} & (M.N.2) \end{array} \} \]

(M.N.1) \[ i_k \leq j_k \quad \text{for} \quad 1 \leq k \leq M \]

(M.N.2) \[ \text{if } w \text{ is in the } (i,j)-\text{configuration, then } p(i,j;w) < j - i. \]

Remark that any vector of \( I_{(M,N)}^{(C)} \) is not in the \((i,i)\)-configuration, because \( p(i,j;w) \geq 0 \).

**Proposition 3.4.3.** \( B(\Lambda_M + \Lambda_N) \) coincides with \( I_{(M,N)}^{(C)} \).

### 3.5 The crystal graph of \( V(\lambda) \)

Let \( \lambda = \sum_{i=1}^{p} \Lambda_{l_i} \quad (1 \leq l_1 \leq l_2 \leq \cdots \leq n) \) be a dominant integral weight. Let us consider the crystal graph of \( B(\lambda) \). By Lemma 3.4.4, we can apply Proposition 2.2.1 and hence

\[
B(\lambda) = \left\{ u_1 \otimes \cdots \otimes u_p \in B(V_\lambda) \otimes \cdots \otimes B(\Lambda_j) \mid u_i \otimes u_{i+1} \in B(\Lambda_{l_i} + \Lambda_{l_{i+1}}) \quad \text{for} \quad 1 \leq i \leq p \right\}
\]

For the Young diagram \( Y \) with the columns of \( l_i \quad (1 \leq i \leq p) \), we define a C-semi-standard tableau with shape \( Y \) with elements \( \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\} \) in each boxes of \( Y \) satisfying the following conditions;

(3.5.1) Letting \( t_{i,j} \) be the element of \( \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\} \) at the \( i \)-th column and \( j \)-th row, we have

\[ t_{i,j} \leq t_{i+1,j} \quad \text{and} \quad t_{i,j} < t_{i,j+1} \]

(3.5.2) For \( 1 \leq p \leq q \leq n \), if \( t_{i,j} = p \), \( t_{i+1,j'} = \overline{p} \) and if \( t_{i,k} = q \), \( t_{i,k'} = \overline{q} \) (resp. \( t_{i+1,k} = q \), \( t_{i+1,k'} = \overline{q} \)) then \( (k - j) + (j' - k') < q - p \).

**Theorem 3.5.1.** \( B(\lambda) \) coincides with the set of the C-semi-standard tableaux with shape \( Y \). The actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) are described by Proposition 2.1.2 by identifying \( i \) and \( i + 1 \) with \( u_+ \), \( i + 1 \) and \( i \) with \( u_- \), and others with \( u_0 \).
Littlewood Richardson rule for $C_n.$

In this section, we give the rule to decompose $V_Y \otimes V_{Y'}$ ($Y$ and $Y'$ are Young diagrams with depth $n$) in terms of crystal graph.

4.1 The following lemma plays a significant role in this rule.

**Lemma 4.1.1.** For $u \in B(V_Y)$ and $v \in B(V_{Y'})$,

$$\tilde{e}_i(u \otimes v) = 0 \text{ (for any } i) \iff \bullet \tilde{e}_i u = 0$$

$$\bullet \tilde{e}_i^{<h_i,\lambda>+1} v = 0 \text{ (for any } i)$$

where $\lambda$ is the weight of $Y$.

4.2. Decomposition of $V_Y \otimes V_{\square}$

**Lemma 4.2.1.** For a Young diagram $Y = (l_1, l_2, \cdots, l_n)$, when we identify $Y$ and the highest element $u_Y$ of $B(V_Y)$, where $u_Y$ is the following;

```
111............111
222............222
........
i...........ii
ii...........ii
...........
11
22
33
44
55
66
77
88
```

\#\{i \in u_Y\} = l_i

a) For $u \in B(V_{\square})$ ($j = 1, 2, \cdots, n$), $Y \otimes u$ is the highest element of $B(V_Y \otimes V_{\square})$ if and only if $l_{j-1} - l_j > 0$. b) For $u \in B(V_{\square})$ ($j = 1, 2, \cdots, n$), $Y \otimes u$ is the highest element of $B(V_Y \otimes V_{\square})$ if and only if $l_j - l_{j+1} > 0$

Remark that we assume $l_0 = \infty$ and $l_{n+1} = 0$.

**Proof** We can easily obtain the result by Lemma 4.1.1 and the following facts;

For any $i$,

$$\tilde{e}_i^{<h_i,\lambda>+1} [j] = 0 \iff <h_{j-1}, \lambda > = l_{j-1} - l_j > 0$$

$$\tilde{e}_i^{<h_i,\lambda>+1} \uparrow_j = 0 \iff <h_j, \lambda > = l_j - l_{j+1} > 0$$

$q.e.d.$

Now, we get the following proposition.
Proposition 4.2.2. Let \( Y = (l_1, l_2, \cdots, l_n) \) be a Young diagram and \( V_Y \) be a finite-dimensional irreducible \( C_\mathfrak{n} \)-module characterized by \( Y \),

\[
V_Y \otimes V_\square \cong \bigoplus_{j=1}^{n} V(Y \leftarrow j) \oplus \bigoplus_{j=1}^{n} V(Y \leftarrow \overline{j})
\]

where \( (Y \leftarrow j) = (l_1, \cdots, l_j + 1, \cdots, l_n) \) and \( (Y \leftarrow \overline{j}) = (l_1, \cdots, l_j - 1, \cdots, l_n) \).

Remark 4.2.3. If \( Y \) is not a Young diagram, then \( V_Y \) means a 0-dimensional vector space.

Proof. By Lemma 4.1.1 and Lemma 4.2.1 we can identify \( u \otimes \begin{array}{c} \text{3} \end{array} \) (resp. \( u \otimes \begin{array}{c} \text{3} \end{array} \)) with highest condition with a Young diagram \( (Y \leftarrow j) \) (resp. \( (Y \leftarrow \overline{j}) \)). Hence, \( Y \otimes \begin{array}{c} \text{3} \end{array} \) (resp. \( Y \otimes \begin{array}{c} \text{3} \end{array} \)) is the highest element of \( B(V_Y \otimes V_\square) \) if and only if \( (Y \leftarrow j) \) (resp. \( (Y \leftarrow \overline{j}) \)) is a Young diagram. Since both \( Y \otimes \begin{array}{c} 1 \end{array} \) (resp. \( Y \otimes \begin{array}{c} 3 \end{array} \)) and \( (Y \leftarrow j) \) (resp. \( (Y \leftarrow \overline{j}) \)) have the same weight, \( u \otimes \begin{array}{c} 1 \end{array} \) (resp. \( u \otimes \begin{array}{c} 3 \end{array} \)) (1 \( \leq j \leq n \)) with the highest condition can be identified with a Young diagram \( (Y \leftarrow j) \) (resp. \( (Y \leftarrow \overline{j}) \)).

\( Q.E.D. \)

Example 4.2.4. For \( g = C_3 \) and \( Y = (2, 2, 1) \)

\[
B(V_\square) = \{ \begin{array}{c} 1 \end{array}, \begin{array}{c} 2 \end{array}, \begin{array}{c} 3 \end{array}, \begin{array}{c} 3 \end{array}, \begin{array}{c} 3 \end{array}, \begin{array}{c} 1 \end{array} \}
\]

\[
Y \otimes \begin{array}{c} 1 \end{array} = (Y \leftarrow 1) = (3, 2, 1) = \begin{array}{c} \text{3} \end{array}, \quad Y \otimes \begin{array}{c} 3 \end{array} = (Y \leftarrow \overline{3}) = (2, 2, 0) = \begin{array}{c} \text{3} \end{array}, \quad Y \otimes \begin{array}{c} 3 \end{array} = (Y \leftarrow \overline{3}) = (2, 2, 0) = \begin{array}{c} \text{3} \end{array}
\]

\[
Y \otimes \begin{array}{c} 2 \end{array} = (Y \leftarrow 2) = (2, 3, 1) \times \quad Y \otimes \begin{array}{c} 2 \end{array} = (Y \leftarrow \overline{2}) = (2, 1, 1) = \begin{array}{c} \text{2} \end{array}, \quad Y \otimes \begin{array}{c} 3 \end{array} = (Y \leftarrow 3) = (2, 2, 2) = \begin{array}{c} \text{3} \end{array}, \quad Y \otimes \begin{array}{c} 3 \end{array} = (Y \leftarrow \overline{1}) = (1, 2, 1) \times
\]

Then, we get

\[
\begin{array}{c} \text{3} \end{array} \otimes \square + \begin{array}{c} \text{3} \end{array} + \begin{array}{c} \text{3} \end{array} + \begin{array}{c} 1 \end{array}
\]

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4.3. Decomposition of $V_Y \otimes V_{Y'}$.

We shall treat a general case. Let $Y$ and $Y'$ be Young diagrams. We give a combinatorial description for irreducible decomposition of $V_Y \otimes V_{Y'}$. By the following lemma and the way of the construction of the crystal graph, we know that the previous elementary case plays a significant role in a general case.

**Lemma 4.3.1.** Let $J = \{1, 2, \cdots, p\}$ be a finite index set and $V_j$ ($j \in J$) be a finite dimensional irreducible representation of $U_q(C_n)$. For $u_1 \otimes u_2 \otimes \cdots \otimes u_p \in B(\bigotimes_{j \in J} V_j)$, following two assertions are equivalent;

(A) $u_1 \otimes u_2 \otimes \cdots \otimes u_p$ is the highest element of $B(\bigotimes_{j \in J} V_j)$

(B) For any $j \in J$, $u_1 \otimes u_2 \otimes \cdots \otimes u_p$ is the highest element of $B(V_1 \otimes \cdots V_j)$

**Proof** First assuming (B), we get (A) easily. Next we assume (A). For any $j \in J$ we can consider $u_1 \otimes u_2 \otimes \cdots \otimes u_p = (u_1 \otimes \cdots u_j \otimes u_{j+1} \otimes \cdots \otimes u_p) \in B(V_1 \otimes \cdots V_j) \otimes B(V_{j+1} \otimes \cdots V_p)$. By Lemma 4.1.1, if $(u_1 \otimes \cdots u_j \otimes u_{j+1} \otimes \cdots \otimes u_p)$ satisfies the highest condition, $u_1 \otimes \cdots \otimes u_j$ also satisfies the highest condition. Hence, we get (B).

$q.e.d.$

Here, by Proposition 4.2.2 and Lemma 4.3.1, we obtain the following theorem.

**Theorem 4.3.2.** Let $Y$ and $Y'$ be Young diagrams. Let $m$ be \#$Y'$. Then we obtain the following;

$$V_Y \otimes V_{Y'} \cong \bigoplus_{\emptyset \cdots \emptyset \fbox{ }} \in B(V_{Y'}) V_{((((Y \leftarrow j_1) \leftarrow j_2) \cdots) \cdots \leftarrow j_m})$$

where $V_{((((Y \leftarrow j_1) \leftarrow j_2) \cdots) \cdots \leftarrow j_m)}$ is a 0-dimensional vector space if there exists $k \in \{1, \cdots, m\}$ such that $((((Y \leftarrow j_1) \leftarrow j_2) \cdots) \cdots \leftarrow j_k)$ is not a Young diagram.
Example 4.3.3. For $g = C_2$, $Y = (2,2)$ and $Y' = (1,1)$.

$$B(V_{Y'}) = \{ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 2 \end{array} \}$$

$Y \otimes \frac{1}{2} = (( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - 1) \leftarrow 2) = ( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - 2) = \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{array}$

$Y \otimes \frac{1}{2} = (( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - 1) \leftarrow \frac{1}{2}) = ( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - \frac{1}{2}) \times \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}$

$Y \otimes \frac{1}{2} = (( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - 2) \leftarrow \frac{1}{2}) = ( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - \frac{1}{2}) \times \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}$

$Y \otimes \frac{1}{2} = (( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - \frac{1}{2}) \leftarrow 1) = ( \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{array} - 1) \times \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}$

Then we get (we omit "V")

$\begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \odot \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} = \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}$

Remark We have already obtained the similar conclusions for $A_n$, $B_n$ and $D_n$.

References

