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A decomposition of the adjoint representation of $U_q(sl_2)$

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Abstract. A decomposition of the adjoint representation into indecomposable modules for rank one quantum algebras is given. A problem related to the uniqueness of the decomposition is presented.

$U_q(sl_2)$

Definition of $U_q(sl_2)$. It is confusing, but there are three objects all of which are called $U_q(sl_2)$. Thus, first of all, we give the definition of these, and we call them $U_q^{(l)}$, $U_q^{(m)}$, $U_q^{(s)}$, respectively.

DEFINITION. *Let $K = \mathbb{Q}(q)$ be the field of rational functions. $U_q^{(l)}$is the associative algebra over K defined by the following generators and relations:*
Generators are e,f, $k^{\frac{1}{2}}$ and $k^{-\frac{1}{2}}$, relations are

$$k^{\frac{1}{2}}ek^{-\frac{1}{2}} = qe, \quad k^{\frac{1}{2}}fk^{-\frac{1}{2}} = q^{-1}f$$

$$k^{\frac{1}{2}}k^{-\frac{1}{2}} = k^{-\frac{1}{2}}k^{\frac{1}{2}} = 1, \quad ef - fe = \frac{k^2 - k^{-2}}{q^2 - q^{-2}}$$

DEFINITION. *$U_q^{(m)}$is the subalgebra of $U_q^{(l)}$generated by e,f,k and k^{-1}.*

DEFINITION. *$U_q^{(s)}$is the subalgebra of $U_q^{(m)}$generated by $ek,k^{-1}f,k^2$ and k^{-2}.*

To be more precise, $U_q^{(m)}$and $U_q^{(s)}$should also be defined by generators and relations, but it is convenient for us to define these as above.

An element $C = fe + \frac{q^2k^2+q^{-2}k^{-2}}{(q^2-q^{-2})^2}$ is called the Casimir element.

Adjoint action. These three rank one quantum algebras have natural adjoint action arising from their Hopf algebra structure. Suppose that (U, Δ, S, ϵ) is a Hopf algebra. $a \in U$ acts on U as the endmorphism which sends x to $\sum a_i^{(1)} x S(a_i^{(2)})$
where, $\Delta(a) = \sum a_i^{(1)} \otimes a_i^{(2)}$.

This action is called the adjoint action of U.

Returning to our case, $U_q^{(l)}$ has a Hopf algebra structure as follows: Its comultiplication is the algebra homomorphism which is uniquely determined by

$$\begin{aligned}\Delta\ : e &\mapsto e \otimes k^{-1} + k \otimes e\\ f &\mapsto f \otimes k^{-1} + k \otimes f\\ k^{\frac{1}{2}} &\mapsto k^{\frac{1}{2}} \otimes k^{\frac{1}{2}}\end{aligned}$$

Its antipode (which is an antihomomorphism) and its counit are,

$$S\ :\ e \mapsto -q^{-2}e, \quad f \mapsto -q^2 f, \quad k^{\frac{1}{2}} \mapsto k^{-\frac{1}{2}}$$

$$\epsilon\ :\ e \mapsto 0, \quad f \mapsto 0, \quad k^{\frac{1}{2}} \mapsto 1$$

This Hopf algebra structure naturally induces those for $U_q^{(m)}, U_q^{(s)}$. Summarizing the above, we have reached the following more concrete definition of the adjoint representation of $U_q(\mathfrak{sl}_2)$.

DEFINITION. *$U_q^{(l)}$ becomes a $U_q^{(l)}$-module by*

$$\begin{aligned}Ad(e)x &= exk - q^{-2}kxe\\ Ad(f)x &= fxk - q^2kxf \qquad (x \in U_q^{(l)})\\ Ad(k^{\frac{1}{2}})x &= k^{\frac{1}{2}}xk^{-\frac{1}{2}}\end{aligned}$$

We denote it by (Ad, U_q^{ad}), and we call it the adjoint representation.

BASIC LEMMAS

Simultaneous eigenvectors for $Ad(k^{\frac{1}{2}})$ and $Ad(C)$. We start with determining concrete form of simultaneous eigenvectors for $Ad(k^{\frac{1}{2}})$ and $Ad(C)$.

LEMMA1. *Let $\overline{K}$ be the algebraic closure of K. Then any simultaneous eigenvector for $Ad(k^{\frac{1}{2}})$ and $Ad(C)$ in $U_q^{(l)} \otimes \overline{K}$ is either of the following forms(up to non zero scalar):*

(1) $k^{-2n-m}e^m(1 + \sum_{j=1}^{n-1} a_j^{(m)}(C)k^{2j})(n = 1, 2, ...)$

(*where, $a_j^{(m)}(X)$ is a polynomial of degree equal or less than j.*)

(2) $k^{-m}e^m$

(3) $k^{-2n-m}f^m(1 + \sum_{j=1}^{n-1} a_j^{(m)}(C)k^{2j})(n = 1, 2, ...)$

(4) $k^{-m}f^m$

$(m = 0, 1, 2, ...)$

Using this lemma, one can easily prove the following Lemma2.

Lemma 2.

(1) *There is no highest weight vector whose highest weight is q to the negative power.*
(2) *There is no lowest weight vector whose lowest weight is q to the positive power.*
(3) *Let V be a submodule of U_q^{ad}, then $V \neq 0$ if and only if $V^0 := V \cap K[k^{\frac{1}{2}}, k^{-\frac{1}{2}}, C] \neq 0$*
(4) *Let $\{V_\alpha\}$be a set of submodules of U_q^{ad}, then $\sum V_\alpha = \oplus V_\alpha$ is equivalent to $\sum V_\alpha^0 = \oplus V_\alpha^0$*

Concrete submodules. Now we give definition of certain submodules of U_q^{ad}.

Definition.

(1) $V_{half} = \sum_{n,m \in \mathbb{Z}} K[C]k^{n+\frac{1}{2}}e^m + K[C]k^{n+\frac{1}{2}}f^m$
(2) $V_{odd} = \sum_{n+m=\text{odd};n,m\in\mathbb{Z}} K[C]k^n e^m + K[C]k^n f^m$
(3) $V_{even} = \sum_{n+m=\text{even};n,m\in\mathbb{Z}} K[C]k^n e^m + K[C]k^n f^m$

Definition.

(1) $V_{n+\frac{1}{2}} = Ad(U_q^{(l)})k^{n+\frac{1}{2}} \quad (n \in \mathbb{Z})$
(2) $V_{2n+1} = Ad(U_q^{(l)})k^{2n+1} \quad (n \in \mathbb{Z})$
(3) $V_{2n} = Ad(U_q^{(l)})C^{-n}k^2 \quad (n \in \mathbb{Z}_{\leq 0})$
(4) $V_{2n} = Ad(U_q^{(l)})k^{-n+2}e^n + Ad(U_q^{(l)})k^{-n+2}f^n \quad (n \in \mathbb{Z}_{>0})$

Then,it is easy to see the following.

Proposition 3.

(1) $U_q^{(l)} = V_{half} \oplus V_{odd} \oplus V_{even}$
(2) $U_q^{(m)} = V_{odd} \oplus V_{even}$
(3) $U_q^{(s)} = V_{even}$

lemma for indecomposability. The next lemma is for proving that V_n's are indecomposable as $U_q^{(*)}$ -module ($* = l, m, s$). But we have to remark that for V_{2n} ($n \in \mathbb{Z}_{>0}$), we need one more fact that $Ad(f)^n(k^{-n+2}e^n)$ coincides with $Ad(e)^n(k^{-n+2}f^n)$ up to scalar. We can prove this fact by induction on n.

Lemma 4. *Let V be a submodule of U_q^{ad}. If V^0 is generated by one element as $K[Ad(C)]$-module, and it has no $Ad(C)$-eigenvector, then V is indecomposable.*

Proposition 5. *V_n's are indecomposable.*

Main Result

Theorem. To give decomposition of the adjoint represetation, it is enough to prove that

THEOREM.

$$V_{half} = \oplus_{n\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} V_n$$
$$V_{odd} = \oplus_{\mathrm{n=odd}} V_n$$
$$V_{even} = (\oplus_{\mathrm{n=even}} V_n) \oplus V_{soc}$$
$$(\text{where, } V_{soc} = \oplus K[C]Ad(U_q^{(l)})k^{-n}e^n)$$

These $Ad(U_q^{(l)})k^{-n}e^n$ (n=0,1,..) are irreducible modules. One may call V_{soc} the socle part of U_q^{ad} since any irreducible submodule is contained in it.

We can give module structure of these direct summands. Let $X(n) = U_q^{(l)}/U_q^{(l)}(k^{\frac{1}{2}} - q^n)$. It is naturally a left module. Then, V_{2n}(n=1,2,...) is an amalgamated sum of $X(n)$ and $X(-n)$ respectively, and all other summands are isomorphic to $X(0)$. Furthermore, V_{2n} (n=0,1,2,...) are mutually nonisomorphic.

V_{half} **and** V_{odd}. Direct calculation of $Ad(e)(k^n e^m)$ and $Ad(f)(k^n f^m)$ shows that $V_{half} = Ad(U_q^{(l)})(V_{half})^0$ and $V_{odd} = Ad(U_q^{(l)})(V_{odd})^0$. Thus it is enough to give decomposition of $(V_{half})^0$ and $(V_{odd})^0$ into indecomposable $K[Ad(C)]$-modules.

We can show $C^p k^{n+\frac{1}{2}} \in (V_{half})^0$ and $C^p k^{2n+1} \in (V_{odd})^0$ by induction on p.

To prove that $(V_{odd})^0 = \sum_{n=\mathrm{odd}} V_n^0$, we introduce a filtration $\{F_n = \sum_{j\geq i+n} KC^i k^{2j+1}\}$ of $K[Ad(C)]$-modules. Then it is easy to see that $V_{2n+1}^0 \cong F_n/F_{n+1}$. It completes the proof of the theorem for V_{odd}. The similar argument is valid for V_{half}.

V_{even}. The proof of the theorem for V_{even} splits into two parts. First part is to prove $(V_{even})^0 = (V_{soc})^0 \oplus (\oplus V_{2n}^0)$. It is the consequence of the following lemma.

LEMMA 6. *Let* $V^+ = \sum_{j\leq 0} K[C]k^{2j}$, $V^- = \sum_{j>0} K[C]k^{2j}$, *then*

(1) $V^+ = \oplus_{p,n\geq 0} KC^p Ad(f)^n(k^{-n}e^n)$
(2) $V^- = (\oplus_{n\leq 0} K[Ad(C)]C^{-n}k^2) \oplus (\oplus_{n>0} K[Ad(C)]k^{2n+2})$
(3) $Ad(f)^n(k^{-n+2}e^n) \equiv k^{2n+2}$ *up to nonzero scalar modulo* $V^+ \oplus (\oplus_{j\leq 0} C^{-j}k^2) \oplus (\oplus_{0<j<n} K[Ad(C)]k^{2j+2})$

Second part is to prove that $V_{soc} + \sum V_{2n}$ coincides with the whole space V_{even}. Since $k^n e^m$ is in the image of $Ad(e)$ and $k^n f^m$ is in the image of $Ad(f)$ if $n + m \neq 2$, it is enough to see the following.

LEMMA 7.

(1) $Ad(f)^j(k^{-n-j+2}e^{n+j}) \equiv f_{j,n}(C)k^{-n+2}e^n$ *modulo* $ImAd(e)$ *where* $f_{j,n}(X)$ *is a polynomial of degree* j.

(2) $Ad(e)^j(k^{-n-j+2}f^{n+j}) \equiv f_{j,n}(C)k^{-n+2}f^n$ modulo $ImAd(f)$.

REMARK ON THE UNIQUENESS OF THE DECOMPOSITION

In the previous section, we gave a decomposition of the adjoint representation of $U_q(sl_2)$ into indecomposable modules. Then it is natural to consider the uniqueness problem of the decomposition up to isomorphism. From this, it arises an interesting problem, which is as follows.

Let r, s, n be non negative integers such that $r + s = n$. Let $\{p_i\}$ $(r+1 \le i \le n-1)$ be a set of mutually distinct prime elements of $K[X]$. We put

$$I = \{A = (a_{ij}) \in M(n, n, K[X]) \mid a_{ij} \equiv 0 \ mod\ p_i..p_{j-1}(i > r)\}$$

Let $I^\times$ be the group consisting of invertible elements of I. Then, what should be natural representatives of $I^\times \backslash I / I^\times$?.

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