A decomposition of the adjoint representation of $U_q(\mathfrak{sl}_2)$ (Combinatorial Aspects in Representation Theory and Geometry)

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A decomposition of the adjoint representation of $U_q(\mathfrak{sl}_2)$

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Abstract. A decomposition of the adjoint representation into indecomposable modules for rank one quantum algebras is given. A problem related to the uniqueness of the decomposition is presented.

$U_q(\mathfrak{sl}_2)$

Definition of $U_q(\mathfrak{sl}_2)$. It is confusing, but there are three objects all of which are called $U_q(\mathfrak{sl}_2)$. Thus, first of all, we give the definition of these, and we call them $U_q^{(l)}$, $U_q^{(m)}$, $U_q^{(s)}$, respectively.

DEFINITION. Let $K = \mathbb{Q}(q)$ be the field of rational functions. $U_q^{(l)}$ is the associative algebra over $K$ defined by the following generators and relations:

Generators are $e,f,k^\frac{1}{2}$ and $k^{-\frac{1}{2}}$, relations are

\[ k^\frac{1}{2}ek^{-\frac{1}{2}} = qe, \quad k^\frac{1}{2}fk^{-\frac{1}{2}} = q^{-1}f \]
\[ k^\frac{1}{2}k^{-\frac{1}{2}} = k^{-\frac{1}{2}}k^\frac{1}{2} = 1, \quad ef - fe = \frac{k^2 - k^{-2}}{q^2 - q^{-2}} \]

DEFINITION. $U_q^{(m)}$ is the subalgebra of $U_q^{(l)}$ generated by $e,f,k$ and $k^{-1}$.

DEFINITION. $U_q^{(s)}$ is the subalgebra of $U_q^{(m)}$ generated by $ek,k^{-1}f,k^2$ and $k^{-2}$.

To be more precise, $U_q^{(m)}$ and $U_q^{(s)}$ should also be defined by generators and relations, but it is convenient for us to define these as above.

An element $C = fe + \frac{q^2h^2 + q^{-2}h^{-2}}{q^2 - q^{-2}}$ is called the Casimir element.

Adjoint action. These three rank one quantum algebras have natural adjoint action arising from their Hopf algebra structure. Suppose that $(U,\Delta,S,\epsilon)$ is a Hopf algebra. $a \in U$ acts on $U$ as the endmorphism which sends $z$ to $\sum a_i^{(1)} S(a_i^{(2)})$

where, $\Delta(a) = \sum a_i^{(1)} \otimes a_i^{(2)}$.

This action is called the adjoint action of $U$. 
Returning to our case, $U_q^{(l)}$ has a Hopf algebra structure as follows: Its comultiplication is the algebra homomorphism which is uniquely determined by

$$\Delta : e \mapsto e \otimes k^{-1} + k \otimes e$$
$$f \mapsto f \otimes k^{-1} + k \otimes f$$
$$k^{\frac{1}{2}} \mapsto k^{\frac{1}{2}} \otimes k^{\frac{1}{2}}$$

Its antipode (which is an antihomomorphism) and its counit are,

$$S : e \mapsto -q^{-2}e, \quad f \mapsto -q^2f, \quad k^{\frac{1}{2}} \mapsto k^{-\frac{1}{2}}$$
$$\epsilon : e \mapsto 0, \quad f \mapsto 0, \quad k^{\frac{1}{2}} \mapsto 1$$

This Hopf algebra structure naturally induces those for $U_q^{(m)}, U_q^{(\cdot)}$. Summarizing the above, we have reached the following more concrete definition of the adjoint representation of $U_q(sl_2)$.

**DEFINITION.** $U_q^{(l)}$ becomes a $U_q^{(l)}$-module by

$$Ad(e)x = exk - q^{-2}kxe$$
$$Ad(f)x = fxk - q^2kxf \quad (x \in U_q^{(l)})$$
$$Ad(k^{\frac{1}{2}})x = k^{\frac{1}{2}}xk^{-\frac{1}{2}}$$

We denote it by $(Ad, U_q^{ad})$, and we call it the adjoint representation.

**BASIC LEMMAS**

**Simultaneous eigenvectors for $Ad(k^{\frac{1}{2}})$ and $Ad(C)$**. We start with determining concrete form of simultaneous eigenvectors for $Ad(k^{\frac{1}{2}})$ and $Ad(C)$.

**LEMMA 1.** Let $\overline{K}$ be the algebraic closure of $K$. Then any simultaneous eigenvector for $Ad(k^{\frac{1}{2}})$ and $Ad(C)$ in $U_q^{(l)} \otimes \overline{K}$ is either of the following forms (up to non zero scalar):

1. $k^{-2n-m}e^m(1 + \sum_{j=1}^{n-1} a_j^{(m)}(C)k^{2j})(n = 1, 2, ...)$
   (where, $a_j^{(m)}(X)$ is a polynomial of degree equal or less than $j$.)
2. $k^{-n-m}e^m$
3. $k^{-2n-m}f^m(1 + \sum_{j=1}^{n-1} a_j^{(m)}(C)k^{2j})(n = 1, 2, ...)$
4. $k^{-m}f^m$
   ($m = 0, 1, 2, ...$)

Using this lemma, one can easily prove the following Lemma 2.
There is no highest weight vector whose highest weight is $q$ to the negative power.

There is no lowest weight vector whose lowest weight is $q$ to the positive power.

Let $V$ be a submodule of $U_q^{ad}$, then $V \neq 0$ if and only if $V^0 := V \cap K \left[ k^{\frac{1}{2}}, k^{-\frac{1}{2}}, C \right] \neq 0$

Let $\{ V_\alpha \}$ be a set of submodules of $U_q^{ad}$, then $\sum V_\alpha = \oplus V_\alpha$ is equivalent to $\sum V_\alpha^0 = \oplus V_\alpha^0$

Concrete submodules. Now we give definition of certain submodules of $U_q^{ad}$.

**DEFINITION.**

1. $V_{\text{half}} = \sum_{n,m \in \mathbb{Z}} K[C] k^{n+\frac{1}{2}} e^m + K[C] k^{n+\frac{1}{2}} f^m$
2. $V_{\text{odd}} = \sum_{n+m = \text{odd}, n,m \in \mathbb{Z}} K[C] k^n e^m + K[C] k^n f^m$
3. $V_{\text{even}} = \sum_{n+m = \text{even}, n,m \in \mathbb{Z}} K[C] k^n e^m + K[C] k^n f^m$

**DEFINITION.**

1. $V_{n+\frac{1}{2}} = Ad(U_q^{(l)}) k^{n+\frac{1}{2}}$ (n $\in \mathbb{Z}$)
2. $V_{2n+1} = Ad(U_q^{(l)}) k^{2n+1}$ (n $\in \mathbb{Z}$)
3. $V_{2n} = Ad(U_q^{(l)}) C^{-n} k^2$ (n $\in \mathbb{Z}_{\leq 0}$)
4. $V_{n} = Ad(U_q^{(l)}) k^{-n+2} e^n + Ad(U_q^{(l)}) k^{-n+2} f^n$ (n $\in \mathbb{Z}_{> 0}$)

Then, it is easy to see the following.

**PROPOSITION3.**

1. $U_q^{(l)} = V_{\text{half}} \oplus V_{\text{odd}} \oplus V_{\text{even}}$
2. $U_q^{(m)} = V_{\text{odd}} \oplus V_{\text{even}}$
3. $U_q^{(s)} = V_{\text{even}}$

Lemma for indecomposability. The next lemma is for proving that $V_n$'s are indecomposable as $U_q^{(*)}$-module ($*$ = $l$, $m$, $s$). But we have to remark that for $V_{2n}$ (n $\in \mathbb{Z}_{> 0}$), we need one more fact that $Ad(f)^n(k^{-n+2}e^n)$ coincides with $Ad(e)^n(k^{-n+2}f^n)$ up to scalar.

We can prove this fact by induction on n.

**LEMMA4.** Let V be a submodule of $U_q^{ad}$. If $V^0$ is generated by one element as $K[Ad(C)]$-module, and it has no $Ad(C)$-eigenvector, then $V$ is indecomposable.

**PROPOSITION5.** $V_n$'s are indecomposable.
**Main Result**

**Theorem.** To give decomposition of the adjoint representation, it is enough to prove that

**THEOREM.**

\[
V_{\text{half}} = \oplus_{n \in \mathbb{Z} \backslash 2} V_n \\
V_{\text{odd}} = \oplus_{n = \text{odd}} V_n \\
V_{\text{even}} = (\oplus_{n = \text{even}} V_n) \oplus V_{\text{soc}}
\]

(where, \( V_{\text{soc}} = \oplus K[C]Ad(U_q^{(1)})k^{-n}e^n \))

These \( Ad(U_q^{(1)})k^{-n}e^n \) (\( n = 0,1,\ldots \)) are irreducible modules. One may call \( V_{\text{soc}} \) the socle part of \( U_q^{ad} \) since any irreducible submodule is contained in it.

We can give module structure of these direct summands. Let \( X(n) = U_q^{(1)}/U_q^{(1)}(k^{\frac{1}{2}} - q^n) \). It is naturally a left module. Then, \( V_{2n}(n = 1,2,\ldots) \) is an amalgamated sum of \( X(n) \) and \( X(-n) \) respectively, and all other summands are isomorphic to \( X(0) \). Furthermore, \( V_{2n} \) (\( n = 0,1,2,\ldots \)) are mutually nonisomorphic.

**V_{\text{half}} and V_{\text{odd}}.** Direct calculation of \( Ad(e)(k^n e^m) \) and \( Ad(f)(k^n f^m) \) shows that \( V_{\text{half}} = Ad(U_q^{(1)})(V_{\text{half}})^0 \) and \( V_{\text{odd}} = Ad(U_q^{(1)})(V_{\text{odd}})^0 \). Thus it is enough to give decomposition of \( (V_{\text{half}})^0 \) and \( (V_{\text{odd}})^0 \) into indecomposable \( K[Ad(C)] \)-modules.

We can show \( C^n k^{n+\frac{1}{2}} \in (V_{\text{half}})^0 \) and \( C^n k^{2n+1} \in (V_{\text{odd}})^0 \) by induction on \( p \).

To prove that \( (V_{\text{odd}})^0 = \sum_{n = \text{odd}} V_n^0 \), we introduce a filtration \( \{ F_n = \sum_{i \geq n} K[C]k^{2i+1} \} \) of \( K[Ad(C)] \)-modules. Then it is easy to see that \( V_{2n+1}^0 \cong F_n/F_{n+1} \). It completes the proof of the theorem for \( V_{\text{odd}} \). The similar argument is valid for \( V_{\text{half}} \).

**V_{\text{even}}.** The proof of the theorem for \( V_{\text{even}} \) splits into two parts. First part is to prove \( (V_{\text{even}})^0 = (V_{\text{soc}})^0 \oplus (\oplus V_{2n}) \). It is the consequence of the following lemma.

**Lemma 6.** Let \( V^+ = \sum_{j \leq 0} K[C]k^{3j}, V^- = \sum_{j > 0} K[C]k^{3j} \), then

1. \( V^+ = \oplus_{p,n \geq 0} K^p Ad(f)^n (k^{-n}e^n) \)
2. \( V^- = (\oplus_{n \leq 0} K[Ad(C)]k^{-n}^2) \oplus (\oplus_{n > 0} K[Ad(C)]k^{2n+2}) \)
3. \( Ad(f)^n (k^{-n+2}e^n) \equiv k^{2n+2} \) up to nonzero scalar modulo \( V^+ \oplus (\oplus_{j \leq 0} C^{-j}k^2) \oplus (\oplus_{0 < j < n} K[Ad(C)]k^{2j+2}) \)

Second part is to prove that \( V_{\text{soc}} + \sum V_{2n} \) coincides with the whole space \( V_{\text{even}} \). Since \( k^n e^m \) is in the image of \( Ad(e) \) and \( k^n f^m \) is in the image of \( Ad(f) \) if \( n + m \neq 2 \), it is enough to see the following.
**Lemma 7.**

1. $Ad(f^j(k^{-n-j+2}e^{n+j}) \equiv f_{j,n}(C)k^{-n+2}e^n$ modulo $ImAd(e)$ where $f_{j,n}(X)$ is a polynomial of degree $j$.
2. $Ad(e^j(k^{-n-j+2}f^{n+j}) \equiv f_{j,n}(C)k^{-n+2}f^n$ modulo $ImAd(f)$.

**Remark on the uniqueness of the decomposition**

In the previous section, we gave a decomposition of the adjoint representation of $U_q(sl_2)$ into indecomposable modules. Then it is natural to consider the uniqueness problem of the decomposition up to isomorphism. From this, it arises an interesting problem, which is as follows.

Let $r, s, n$ be non-negative integers such that $r + s = n$. Let $\{p_i\}$ ($r + 1 \leq i \leq n - 1$) be a set of mutually distinct prime elements of $K[X]$. We put

\[ I = \{ A = (a_{ij}) \in M(n, n, K[X]) \mid a_{ij} \equiv 0 \mod p_1 \ldots p_{j-1} (i > r) \} \]

Let $I^\times$ be the group consisting of invertible elements of $I$. Then, what should be natural representatives of $I^\times \backslash I/I^\times$?

**References**


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