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<th>Finite size approximation for representations of $U_q(\widehat{\mathfrak{sl}}(n))$ (Combinatorial Aspects in Representation Theory and Geometry)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1991), 765: 62-65</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82283">http://hdl.handle.net/2433/82283</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Finite size approximation for representations of $U_q(\widehat{sl}(n))$

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1. The present note is an elucidation of an observation made in [1] concerning the crystal base of integrable representations of $U_q(\widehat{sl}(n))$.

Let $U_q = U_q(\widehat{sl}(2))$ denote the quantized affine algebra of type $A_1^{(1)}$. Just as in the classical case $q = 1$, it admits the following two classes of representations of particular interest:

(1) Highest weight representations. These are irreducible modules $L(\Lambda)$ with dominant integral highest weight $\Lambda$. For simplicity we consider here the level 1 representations $L(\Lambda_i)$ ($i = 0, 1$) where the $\Lambda_i$ denote the fundamental weights.

(2) Finite dimensional representations. These are level 0, non-highest weight representations (cf.[C]). For example, the natural representation $V = \mathbb{C}^2$ of $U_q(\widehat{sl}(2))$ can be made a $U_q(\widehat{sl}(2))$-module by letting the Chevalley generators act on $V$ as follows:

$$e_0 = f_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_1 = f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_0 = t_1^{-1} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix},$$

where $t_i = q^{h_i}$. (Here we follow the notations of [2]).

Given two modules $L$, $L'$ over $U_q$ one can form their tensor product $L \otimes L'$ via the comultiplication

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i.$$

Let us consider the $N$-fold tensor product $V^{\otimes N}$ of $V = \mathbb{C}^2$. Our objective here is to show the following fact

$$\lim_{N \to \infty} V^{\otimes N} \sim L(\Lambda_0) \cup L(\Lambda_1), \quad (*)$$

whose meaning will be made clear below.
2. The algebra $U_q$ loses meaning at $q = 0$. However, Kashiwara's theory of crystal base [2] tells that on each integrable module $L$ one can define the action of 'the Chevalley generators at $q = 0'$ $\tilde{e}_i, \tilde{f}_i$. Moreover there exists a unique canonical base $B = B(L)$ of $L$ 'at $q = 0'$, such that

If $u, v \in B$, then $\tilde{f}_i u = v \iff u = \tilde{e}_i v$

holds. For precise statements see [2]. The above situation is represented as

\[ u \xrightarrow{i} v. \]

This equips $B$ with a structure of colored (by the index $i = 0, 1$), oriented graph, called the crystal graph of $L$. It is known also that the crystal base $B$ has a unique canonical extension to nonzero $q$ [3].

There are some subtle points for finite-dimensional representations, since they are not integrable in the sense of [2]; but one can still consider crystal graphs for them. For instance $V = \mathbb{C}^2$ has the crystal graph

\[ U_0 \xleftarrow{1} \quad \quad \xrightarrow{0} U_1 \]

with $u_i$ denoting the natural base of $V$.

According to [2] the crystal graph behaves remarkably nicely under tensor products. The vertices of $B(L_1 \otimes L_2)$ are simply $B(L_1) \times B(L_2)$ as a set. The edges of the graph are described by a simple rule [2], color–by–color. It is an amusing exercise to work out the crystal graphs for $B(V^{\otimes N})$ using this rule. Their vertices consist of sequences $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ with $\xi_i \in \{0, 1\}$, representing the vectors $u_{\xi_1} \otimes \cdots \otimes u_{\xi_N}$. We show how they look like at the end of this note.

3. Let $B_i^N$ ($i = 0, 1$) be the full subgraph of the crystal graph $B(V^{\otimes N})$, whose vertices consist of sequences $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ with $\xi_N = i$. From the figure for $N = 2, 3, 4$ the following is already apparent:

**Theorem.** There is an imbedding of graphs $B_i^N \hookrightarrow B_{i+1}^{N+1}$ given by $v \mapsto v \otimes u_{i+1}$, where the suffix $i$ is to be read modulo 2. As $N$ even $\to \infty$, $B_i^N$ converges to the crystal graph $B(L(\Lambda_i))$ of the highest weight representation $L(\Lambda_i)$ (with the arrows reversed, because of conventions).

Thus the equality (*) makes sense in the language of crystal base. The proof of the theorem can be done by straightforward induction using Kashiwara's rule. As a consequence, $L(\Lambda_i)$ has a basis labeled by infinite sequences (called paths)
\( \xi = (\xi_1, \xi_2, \cdots ) \), whose 'tail' is \( \cdots 010101 \cdots \) (i.e. \( \xi_j \equiv j + i - 1 \mod 2 \) for \( j \gg 0 \)). Though we have omitted here, there is also a formula for the weight of these base vectors given in terms of the paths [1]. This type of result has an important application in solvable lattice models of statistical mechanics [4]; in fact the whole story was motivated by the latter.

4. In [1] a similar result is established for \( U_q(\widehat{\mathfrak{sl}}(n)) \). Integrable representations of arbitrary level \( l \) can be 'approximated' by taking \( V = S^l(C^n) \), the \( l \)-th symmetric power of the standard representation \( C^n \).

Remark. At the stage of writing this note, Kashiwara found a simple explanation to this phenomenon.

References

[1] M. Jimbo, K. C. Misra, M. Okado and T. Miwa, Combinatorics of representations of \( U_q(\widehat{\mathfrak{sl}}(n)) \) at \( q = 0 \), preprint RIMS 709 (1990)


$N=2$

$N=3$

$N=4$

\[ \iota k \]

$= \sum \! \! \! \! ^{t}k^\otimes u_I \otimes u_t \otimes u_0$

\[ \neq \]

\[ \text{Convergence of crystal graphs for } V^\otimes N \]

\[ \begin{align*}
\text{---} & = \frac{1}{2} \\
\text{------} & = \frac{1}{2} \\
0110 & = u_0 \wedge u_1 \wedge u_2 \wedge u_3 \text{ etc.}
\end{align*} \]