Torelli theorem for certain rational surfaces and root system of type A

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For an integer $n \geq 2$, let $\Sigma_n$ be the $n$--th Hirzebruch surface defined by

\begin{equation}
\{(\zeta_0 : \zeta_1 : \zeta_2)(s : t) \in \mathbb{P}^2 \times \mathbb{P}^1 | s^n \zeta_0 = t^n \zeta_1\},
\end{equation}

where $\mathbb{P}^k$ is $n$-dimensional complex projective space. Let $X_n$ be a surface obtained by blowing up $n + 1$ points of $\Sigma_n$ and $D$ be an anti-canonical divisor on $X_n$ such that $D$ consists of four nonsingular rational curves and its intersection diagram is a circle (thus $D$ forms a square).

We study the isomorphism classes of the pairs $(X_n, D)$. The isomorphism classes can be characterized in terms of the root system of type $A$. E. Looijenga investigated the isomorphism classes of rational surfaces with anti-canonical divisors [L]. We deal with another class of rational surfaces. The method and formulation are very similar to those of Looijenga's.

1. Hirzebruch surfaces

We assume $n \geq 3$. $\Sigma_n$ is a subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ (cf (0.1)). Let $\pi: \Sigma_n \rightarrow \mathbb{P}^1$ be the second projection. $\Sigma_n$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let $F$ be a fiber of the projection $\pi: \Sigma_n \rightarrow \mathbb{P}^1$ and $S$ be the section defined by $\zeta_0 = \zeta_1 = 0$.

**Definition.** We say that $n + 1$ points $P_1, \ldots, P_{n+1}$ of $\Sigma_n$ are in 'general position' if they satisfy the following conditions: (1) $P_i \neq P_j$ for $i \neq j$ and (2) there exists a nonsingular curve in the complete linear system $|nF + S|$ passing through $P_1, \ldots, P_{n+1}$.

**Remark.** If $P_1, \ldots, P_{n+1}$ are in general position, then $P_i \not\in S$ and no two of $P_i$ are on a fiber.

Let $p: X_n \rightarrow \Sigma_n$ be the morphism obtained by blowing up $n + 1$ points $P_1, \ldots, P_{n+1}$ in general position.

**Lemma 1.1.** If $D$ is an anti-canonical divisor on $X_n$ and satisfies the following conditions:

1. $D$ is the strict transform of an anti-canonical divisor $D'$ on $\Sigma_n$,
(2) $D'$ consists of four irreducible components and its intersection diagram is a circle,

(3) $P_1, \ldots, P_{n+1}$ are on only one component of $D'$ and not on other components,

then

$$D = F_1 + F_2 + S + C,$$

where $F_i$ is a strict transform of a fiber of the projection $\pi: \Sigma_n \rightarrow \mathbb{P}^1$, $S$ is the strict transform of the $(-n)$-section of $\Sigma_n$ and $C$ is the strict transform of the unique nonsingular curve of $|nF + S|$ passing through $P_1, \ldots, P_{n+1}$.

**NOTATION.** We say that an anti-canonical divisor $D$ on $X_n$ is of '♯-type' if it satisfies the condition of lemma 1.1. We denote by $F_0$ and $F_{\infty}$ the components of $D$ which are the strict transforms of the fibers of $\pi$.

### 2. Homology and Root System

Let $X_n$ and $D = F_0 + F_{\infty} + S + C$ be as in §1. Consider the homology exact sequence:

$$
\begin{array}{cccccc}
\ldots & \rightarrow & H_3(X_n; \mathbb{Z}) & \rightarrow & H_3(X_n, X_n - D; \mathbb{Z}) & \rightarrow \\
& & \Downarrow & & \Downarrow & \\
& 2\rightarrow & H_2(X_n - D; \mathbb{Z}) & \rightarrow & H_2(X_n; \mathbb{Z}) & \rightarrow \\
& & \rightarrow & & \rightarrow & \\
& & \rightarrow & & \rightarrow & \\
\end{array}
$$

We extend the intersection form in $H_2(X_n; \mathbb{Z})$ to $H_2(X_n; \mathbb{Z}) \otimes \mathbb{R}$. Let

$$Q = \text{ker} j_* \subset H_2(X_n; \mathbb{Z})$$

and

$$R = \{ \alpha \in Q | \alpha \cdot \alpha = -2 \}.$$

**Lemma 2.1.** $R$ is a root system of type $A_n$ in $Q \otimes \mathbb{R}$ and $Q$ is generated by $R$. The set $\{e_i - e_{i-1} | 1 \leq i \leq n\}$ is the basis of $R$, where $e_i$ is the class of the exceptional curve $E_i = p^{-1}(P_i)$.

We now have the short exact sequence:

$$0 \rightarrow H_3(X_n, X_n - D; \mathbb{Z}) \rightarrow H_2(X_n - D; \mathbb{Z}) \rightarrow Q \rightarrow 0.$$

**Lemma 2.2 (K.Irie).**

$$H_3(X_n, X_n - D; \mathbb{Z}) \simeq \mathbb{Z}.$$
Let $\epsilon$ be the generator of $H_3(X_n, X_n - D; \mathbb{Z})$. We next consider a meromorphic 2-form on $X_n$ which has poles only along $D$.

**Lemma 2.3.** There exists a unique meromorphic 2-form $\omega$ on $X_n$ such that

1. $\omega$ has poles only along $D$,
2. $\omega(\partial_*(\epsilon)) = 1$.

Furthermore, we can choose an affine coordinate $z$ on $C(\subset D)$ such that $F_0 \cap C = 0$, $F_\infty \cap C = \infty$ and

$$\text{Res}_C \omega = \frac{1}{(2\pi i)^2} \frac{dz}{z}.$$

It follows from this lemma, we can define a character $\chi : Q \rightarrow \mathbb{C}^*$ by

$$\chi(i_*[\Gamma]) = \exp 2\pi i \int_{\Gamma} \omega,$$

where $\Gamma \in H_2(X_n - D; \mathbb{Z})$.

\[ \begin{array}{ccc}
H_2(X_n - D; \mathbb{Z}) & \xrightarrow{\exp 2\pi i \int_{\Gamma} \omega} & \mathbb{C}^* \\
\downarrow i_* & & \uparrow \chi \\
\mathbb{C} & & \\
\end{array} \]

3. **Torelli theorem for the pair $(X_n, D)$**

We first consider the value of $\chi$ at the class $e_i - e_j \in Q$, where $e_i$ and $e_j$ are the homology classes of the exceptional curves $E_i = p^{-1}(P_i)$ and $E_j = p^{-1}(P_j)$ respectively. Let $B_i = E_i \cap C$ and let $T$ be a closed tubular neighborhood of $C$ in $X_n$ such that $T \cap E_i$ and $T \cap E_j$ are fibers. Let $\gamma$ be an injective path in $C$ from $B_i$ to $B_j$ and let

$$\Gamma_{i,j} = (E_i \setminus (E_i \cap T)) \cup \partial T|_{\gamma} \cup (E_j \setminus (E_j \cap T)).$$

We can take the orientation such that $\Gamma_{i,j}$ is homologous to $E_i - E_j$ in $X_n$. Hence we have

$$i_*([\Gamma_{i,j}]) = e_i - e_j.$$
Since $E_i$ and $E_j$ are the inverse image of the points $P_i$ and $P_j$ respectively, we have

$$\int_{E_i \setminus (E_i \cap T)} \omega = \int_{E_j \setminus (E_j \cap T)} \omega = 0.$$ 

Therefore

$$\int_{\Gamma_{i,j}} \omega = \int_{\theta T|_{\gamma}} \omega.$$ 

By the residue formula, we have

$$\int_{\theta T|_{\gamma}} \omega = 2\pi i \int_{\gamma} \text{Res}_C \omega = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} = \frac{1}{2\pi i} \log \frac{t_j}{t_i} \quad (\text{mod } \mathbb{Z}),$$

where $t_i$ and $t_j$ are the affine coordinates of the points $B_i$ and $B_j$ respectively. Then we now have

$$\chi(e_i - e_j) = \exp 2\pi i \int_{\Gamma_{i,j}} \omega = \frac{t_j}{t_i}.$$

The important point is that this is the cross ratio of $C \cap F_0, C \cap F_\infty, B_j$ and $B_i$. Thus we have the theorem of Torelli type.

**Theorem.** Let $X_n$ and $X'_n$ be the surfaces defined in §1 and let $D$ and $D'$ be anticanonical divisors of #-type on $X_n$ and $X'_n$ respectively (cf. notation in §1). Let denote root lattices by $Q$ and $Q'$, root systems by $R$ and $R'$, and characters by $\chi$ and $\chi'$ defined as in §2 for $X_n$ and $X'_n$ respectively. If $\varphi : H_2(X_n; \mathbb{Z}) \to H_2(X'_n; \mathbb{Z})$ is an isometry such that

1. $\varphi([F_i]) = [F'_i]$,
2. $\varphi([C]) = [C']$,
3. $\varphi(R) = R'$,
4. $\varphi^*(\chi') = \chi$,

then there exists a unique isomorphism $\Phi : X_n \to X'_n$ which maps $F_i$ to $F'_i$ and $C$ to $C'$ and induces $\varphi$. 

4
REFERENCE
