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Dynkin graphs and combinations of singularities on some algebraic varieties

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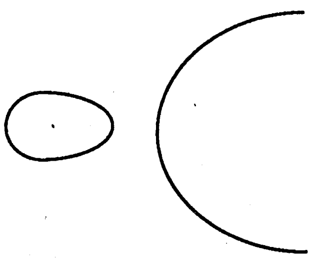
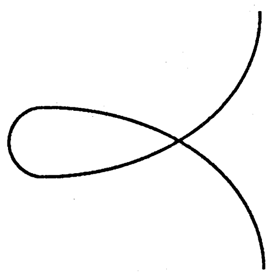
In this article I would like to explain my recent results on Dynkin graphs and global theory of singularities on algebraic varieties. We assume that every variety is defined over the complex field $\boldsymbol{C}$.

First, let us consider plane cubic curves, that is, curves of degree 3 in the 2-dimensional projective space. It is easy to give the following classification.

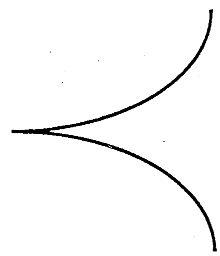


smooth
(normal defining polynomial)
$y^2=x^3+ax+b$
$(4a^3+27b^2\neq 0)$



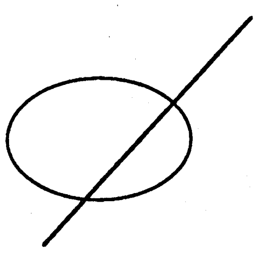
nodal
$y^2=x^3+x^2$

A_1

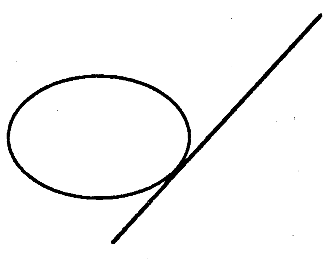


cuspidal
$y^2=x^3$

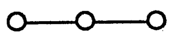
A_2

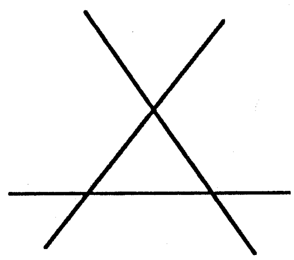


$2A_1$

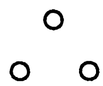


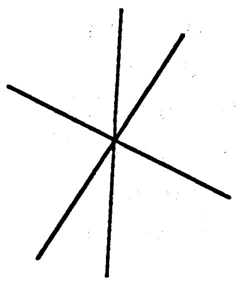
A_3

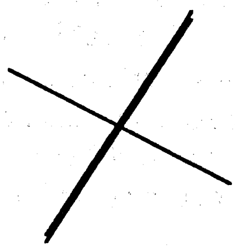


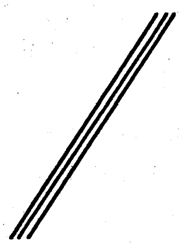


$3A_1$

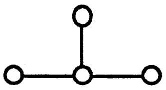








D_4



The first one is the smooth one. In spite that we consider curves over the complex field, above we have drawn the pictures of real points. Because a smooth cubic curve over the complex field is a 2-dimensional manifold in a 4-dimensional manifold $\mathbf{P}^2(\mathbf{C})$, it is impossible to draw correct pictures. Anyway, we have 9 types of cubic curves. The last two have multiple components, and they are not worth calling cubic curves, if we treat them as figures.

Here I explain an important concept in singularity theory. We have a series of singularities with the same names as Dynkin graphs without multiple edges.

$$A_k: x^{k+1} + y^2(+z^2) = 0 \qquad D_l: x^{l-1} + xy^2(+z^2) = 0$$
$$E_6: x^3 + y^4(+z^2) = 0 \qquad E_7: x^3y + y^3(+z^2) = 0 \qquad E_8: x^3 + y^5(+z^2) = 0$$

When we consider singularities on curves, we abbreviate the above terms $+z^2$. When we consider surface singularities, we add the above terms $+z^2$. If a singularity is defined by one of these equations under a suitable local coordinate, we call the singularity one of type A, D or E.

For example, if a curve singularity is defined by a power series of 2 variables x, y with cubic terms in the beginning part, and if the sum of cubic terms defines a homogeneous polynomial without a multiple root, then we can make this power series into $x^3 + xy^2$ by a suitable coordinate change. Therefore, the singularity is of type D_4.

The above 7-th cubic curve has a unique singularity and it is of type D_4. We draw a Dynkin graph of type D_4 beneath the 7-th curve. By the same method we can associate a Dynkin graph (possibly with multiple components) to each cubic curve without a multiple component. We have the empty graph, A_1, A_2, $2A_1$, A_3, $3A_1$ and D_4.

Now, perhaps you can notice that classification of cubic curves corresponds to subgraphs of D_4. 7 types of cubic curves have one-to-one correspondences with 7 kinds of subgraphs of D_4. This is not an accidental coincide. We can give theoretical explanation. Moreover, we can observe similar facts for curves and surfaces of low degree in a projective space. For such objects there exists a common law dominating possible appearance of singularities.

By my study so far the basic frame of the common theory can be explained as follows. For a given class of objects, a basic Dynkin graph can be determined, and a certain operation by which we can make a new Dynkin graph from a given Dynkin graph is defined. The set of all Dynkin graphs made by the operation from the basic graph coincides with the set of possible combinations of singularities.

A class of objects

basic Dynkin graph $\xrightarrow{\text{operation}}$ {Dynkin graphs made from the basic one}

||

possible combinations of singularities

Needless to say, in the case of cubic curves, the basic graph = D_4, the operation = taking a subgraph, and the set of graphs made from the basic one consists of 7 graphs.

By the study so far, I know that the following geometrical objects are dominated by theory described by the above frame.

- plane curves cubic, quartic, -, sextic
- space surfaces cubic, quartic
- deformation fibers of a singularity
  - rational double points A_k, D_l, E_6, E_7, E_8
  - simple elliptic singularities P_8, X_9, J_{10}
  - 9 of 14 triangle singularities
  - 6 quadrilateral singularities (singularities of rectangles)

Here the concept of a deformation fiber is to be explained. Let $f(x,y,z)=0$ be a power series with 3 variables defining a singularity at the origin. Let $g(x,y,z)$ be an arbitrary power series. We consider the zero-locus defined in a ball with a sufficiently small radius $\varepsilon > 0$ with the center at the origin by the following equality.

$$f(x,y,z) + tg(x,y,z) = 0$$

Here t is a complex number with $0 < |t| < \delta$, where δ is a sufficiently small positive number compared with ε. This locus is called a deformation fiber of the singularity defined by f. We can show that the combination of singularities on it is independent from the choice of t.

Now, we have a classification list of hypersurface singularities due to Arnold. (Arnold [1].) It should be remarked that the singularities in the above item "♠ deformation fibers ..." appear in the beginning part of his list.

So far we considered singularities. Next we would like to consider elliptic surfaces, because they are also related to Dynkin graphs of type A, D or E. A compact complex surface X with a morphism Φ to a curve C, $\Phi: X \to C$ is called an elliptic surface, if the inverse image $\Phi^{-1}(c)$ is a connected smooth curve of genus 1 for a general point $c \in C$.

In this article because of a technical reason, we assume moreover that there is a section of Φ, that is, a morphism $s: C \to X$ such that the composition Φs is the identity on C.

An inverse image of a point on C is called a fiber in this case. Possible singular fibers in an elliptic surface are classified by Kodaira. (Kodaira [2].)

(1) An irreducible fiber is one of the following.

(1.1) an elliptic curve = a curve of genus 1

(1.2) a rational curve with an A_1 singular point (It is isomorphic to a plane nodal cubic curve.)

(1.3) a rational curve with an A_2 singular point (It is isomorphic to a plane cuspidal cubic curve.)

(2) A reducible fiber is a union of smooth rational curves. The graph of intersection of these smooth rational curves (In this graph the set of vertices has one-to-one correspondence with the set of smooth rational curves. If two rational curves are disjoint, then the corresponding two vertices are not connected. If they have intersection-number 1, then the corresponding two vertices are connected by a single edge. If they have intersection-number 2, then the corresponding vertices are connected by a bold edge. An intersection-number ≥ 3 never appears.) coincides with an *extended* Dynkin graph of type A_k, D_l, E_6, E_7 or E_8.

Therefore we can associate a Dynkin graph to each singular fiber in a natural manner. (We associate the empty graph to an irreducible fiber. We associate a Dynkin graph, though the graph of intersection is the corresponding *extended* Dynkin graph.) Under this correspondence we can apply the basic frame explained above also to describe possible combinations of singular fibers on elliptic surfaces. So far we have obtained results to this direction for rational elliptic surfaces and K3 elliptic surfaces with a singular fiber of type D_4.

- elliptic surfaces
  - rational elliptic surfaces
  - K3 elliptic surfaces with a D_4-fiber.

We divide the above mentioned objects into 3 grades I, II, and III.

In the first grade the operation is the simplest one — taking a subgraph. Cubic curves and rational double points fall into this grade. The basic graph for a rational double point coincides with the graph of the name of the singularity.

I. operation = taking a subgraph

In the second grade a rather complicated operation called an elementary transformation is introduced. In an elementary transformation, first we replace every component to the associated extended Dynkin graph. In the second step we choose a proper subgraph.

An elementary transformation

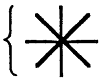
1. a Dynkin graph $\rightarrow$ an extended Dynkin graph.
2. Choose a proper subgraph of the extended graph.

In the second grade as the operation we use elementary transformations repeated twice:

II. operation = elementary transformations repeated twice

Quartic curves, cubic surfaces, simple elliptic singularities and rational elliptic surfaces fall into this grade. The basic graph is one of type E.

Consider quartic curves as an example. Now, 4 lines intersecting at one point is a reducible quartic curve. However, the singularity on it is not of type A, D or E. We have to exclude this case. Excluding only this case, the set of possible combinations of singularities coincides with the set of Dynkin graphs which can be made from a basic graph E_7 by elementary transformations repeated twice.

{plane quartic curves without a multiple component} — {  }

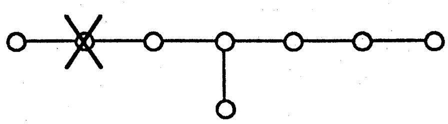
$\Downarrow$

E_7 ⟶ {possible combinations of singularities}

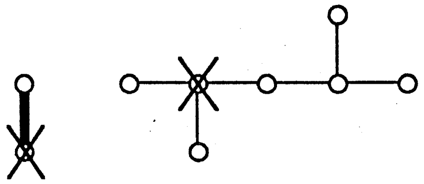
elementary transformation

2 times

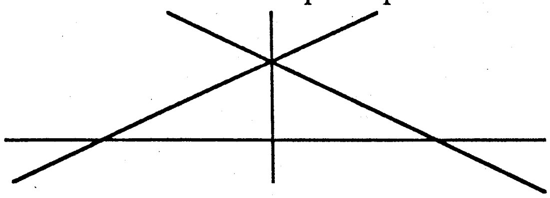
In an elementary transformation starting from E_7 we can erase a vertex as in the figure below.



Under this choice we get the graph D_6+A_1. We can repeat an elementary transformation once more. In the second transformation vertices can be erased as in the figure below.



(Note that the edge in the extended Dynkin graph of type A_1 is bold.) We get the graph D_4+3A_1. This corresponds to the reducible quartic curve below — three lines intersection at one point plus another general line.



The remaining classes fall into the third grade. In the third grade the third operation called a tie transformation has to be introduced, and the number of the basic graphs is not necessarily 1.

III. operation = elementary transformation & tie transformation
repeated 2 times
(Four kinds of combinations, i.e., "elementary" twice, "tie" twice, "elememtary" after "tie", and "tie" after "elementary" are all permitted.)
basic graphs — possibly 2 or more
special vertices (black vertices) have to be erased until the final stage.

For example for quartic surfaces we have 9 basic graphs. For sextic curves we have 4 basic graphs.

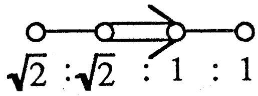
Here I explain the case of one of 6 quadrilateral singularities called $J_{3,0}$. $J_{3,0}$ is defined by the following equality.

$$J_{3,0}: x^3+ax^2y^3+y^9+bxy^7+z^2=0 \quad (4a^3+27\neq 0)$$

We consider only deformation fibers with ADE singularities only. In this case we have a unique exception $3A_3+2A_2$. The basic graph is E_8+F_4.

The set of all possible combinations of singularities minus $3A_3+2A_2$ coincides with the set of Dynkin graphs with only components of type A, D or E made from E_8+F_4 by 2 kinds of transformations repeated 2 times.

Here we need to give some explanation on the Dynkin graph F_4, because it has a double edge and an arrow. Now, every vertex in a Dynkin graph corresponds to a vector called a *root* in an Euclidean space. If the graph has an arrow with a multiple edge, it indicates that the length of roots at both ends are different. Therefore for F_4 two vertices correspond to shorter roots. In fact the ratio of length is as in the following.



Because length is different, we can replace these vertices of shorter roots by black vertices. In our theory we use the expression including black vertices for the graph F_4.

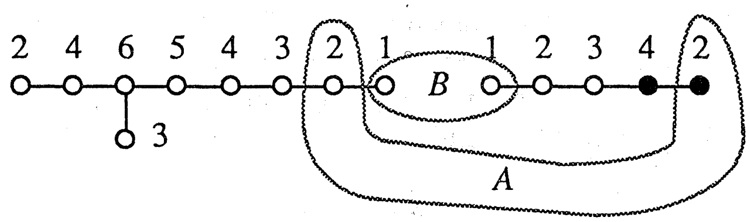
F_4 : 

Here I explain the concept of tie transformations using this case.

At the first step of a tie transformation we make every component to the corresponding extended Dynkin graph, and moreover we attach the corresponding coefficient of the maximal root to each vertex of the extended graph.

At the second step we choose two subsets A and B of vertices satisfying the following three conditions <a>, <b> and <c>.

Thus the basic graph E_8+F_4 becomes the graph like the following.



The conditions:

<a> $A \cap B = \varnothing$

<b> Let $\tilde{G}_0$ be an arbitrary component of the extended graph. Let N be the sum of integers attached to $B \cap \tilde{G}_0$. Let $n_1, n_2, \cdots, n_l$ be all the attached integers to $A \cap \tilde{G}_0$. Then, $G.C.D.(N, n_1, n_2, \cdots, n_l) = 1$.

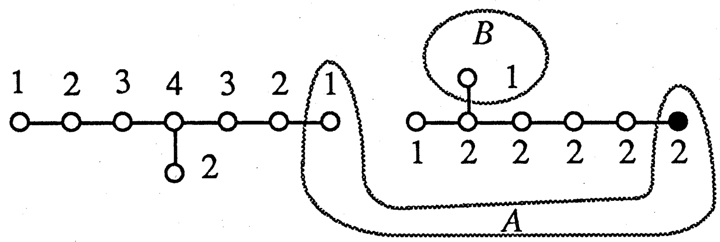
<c> The resulting graph after the third step is a Dynkin graph.

At the third step we erase all attached integers and all vertices in A

together with edges issuing from them. Next we draw a new white vertex called θ and connect each vertex in B to θ by an edge.

Under the choice of A and B in the above figure, we get the graph $E_7 + B_6$. This is an example of a tie transformation.

Now, we can apply a transformation once more. At the second step of the second tie transformation we get the following graph under a choice of A and B.



Under the above choice, we get $2E_7$ as the result of the second tie transformation. Note that $2E_7$ contains no black vertex.

Here I explain the role played by black vertices. If we start from a graph with a black vertex, we can make a graph with a black vertex after the second transformation. However, obviously such a graph does not correspond to a combination of singularities on a surface. Thus in the third grade we consider the set of Dynkin graphs with only white vertex made from a basic graph by two transformation. This set coincides with the set of possible combination of singularities on surfaces in the given class.

Therefore we can conclude that there is a deformation fiber of $J_{3,0}$ with $2E_7$ singularities.

{deformation fibers of $J_{3,0}$ with only ADE singularities}

$\Downarrow$

{possible combinations of singularities} – {$3A_3 + 2A_2$ }

$||$

$E_8 + F_4 \longrightarrow$ {Dynkin graphs made from $E_8 + F_4$ } – {Dynkin graphs with black vertices}

elementary transf.
tie transf. } 2 times

Here I conclude the explanation of my theory. I do not explain why these facts can hold.

For the classes in the grade I, they follow from Grothendieck-Brieskorn theory related to simple algebraic groups. For the grade II and the grade III we apply the theory of periods of algebraic surfaces. By periods we reduce our problem to the theory of integral bilinear forms. With the aid of the theory of reflection groups we can show the above facts.

It is slightly strange that for the grade II and III no Lie groups appear in the theory at present, though the grade I is closely related to Lie groups. It is quite natural that we expect a relation between Lie groups and the grade II and III.

I hope that someone in the world — perhaps a reader of this article — can find out the relation to Lie groups.

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