Kummer Surface with $D_4$-Symmetry

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For the simple root system $D_4$ there are exactly three linearly independent Weyl-group-invariant homogeneous polynomials of degree 4 on the Cartan subalgebra $V$. Since $V$ is 4-dimensional, the null locus $S$ of such an polynomial $\neq 0$ is a quartic surface in the associated projective space $\mathbb{P}(V) \cong \mathbb{P}_3(\mathbb{C})$. ($S$ has two parameters.) $S$ is smooth in general. In this note however we will only discuss a special case where $S$ is a Kummer quartic i.e. quartic surface with 16 nodes (ordinary double points). This case is introduced by imposing the following condition on $S$:

(A) Some (hence any by invariance) root-section of $S$ decomposes into two conics intersecting transversally.

For any root $r$ the section of $S$ by $r$ is the intersection of $S$ and the null plane $H_r := \{(x) \in \mathbb{P}(V) : r(x) = 0\}$. (This plane curve is in general irreducible.) From now on we assume that $S$ satisfies (A), so $S$ is now a Kummer surface.

$S$ has still one parameter. Explicitly $S$ is given by the equation

$$I_1(x) - (s^2 + 1)I_2(x) + 2s(s^2 + 3)I_3(x) = 0$$

where $s$ ($s^2 + 3 \neq 0, s = \pm 1$) is the parameter, $I_1(x) := \sum_{i=1}^{4} x_i^4$, $I_2(x) := \sum_{1 \leq i < j \leq 4} x_i^2 x_j^2$, $I_3(x) := x_1 x_2 x_3 x_4$ and the coordinates $(x_1, x_2, x_3, x_4)$ are so chosen that the roots are $\pm(x_i \pm x_j)$. The Weyl group is generated by the even sign changes and permutations of $x_1, x_2, x_3, x_4$. The 16 nodes are the orbit of $(s, 1, 1, 1)$. We see that the 16 nodes lie four by four on the 12 root-sections to be the inter-section points of the conics in (A). Each node is on exactly three root-sections.

For the definiteness of argument we fix a root $r$ and let $C_1, C_2$ be the conics such that $C_1 \cup C_2 = H_r \cap S$. Let $\{q_0, q_1, q_2, q_3\} = C_1 \cup C_2$. Recall now that the abelian surface
\( \mathcal{A} \) associated with \( S \) is the double cover of \( S \) branched over the 16 nodes; so the nodes are naturally imbedded into \( \mathcal{A} \); in particular \( \{q_0, q_1, q_2, q_3\} \subseteq \mathcal{A} \). We regard \( q_0 \) as the zero of \( \mathcal{A} \).

We remark that the inverse images \( E_1, E_2 \) of \( C_1, C_2 \) by \( \mathcal{A} \rightarrow S \) are elliptic curves. They are thus two subgroups of \( \mathcal{A} \) such that \( E_1 \cup E_2 = \{q_0, q_1, q_2, q_3\} \). We set \( G_0 := E_1 \cap E_2 \). This is a subgroup of the 2-torsion \( \mathcal{A}(2) \) of \( \mathcal{A} \). We also form the diagonal group \( \Delta_0 := \{(q_i, q_i)\}_{i=0,1,2,3} \) in the product group \( \mathcal{E} := E_1 \times E_2 \).

**Proposition 1.** The product mapping \( \mathcal{E} = E_1 \times E_2 \ni (x, y) \mapsto xy \in \mathcal{A} \) induces the isomorphism

\[
(1) \quad \mathcal{E}/\Delta_0 \cong \mathcal{A}.
\]

It follows also

\[
(2) \quad \mathcal{A}/G_0 \cong \mathcal{E}.
\]

**Remark.** So far we have only used the existence of a plane which cuts from a quartic two conics in a transversal position. This property is therefore a characterization of elliptic Kummer surfaces of degree 2.

We call such an isomorphism as (1) an *almost product structure* on \( \mathcal{A} \); (1) depends on the root \( r \) fixed above. Since there are 12 roots of \( D_4 \) up to sign, we have 12 almost product structures for \( \mathcal{A} \). But not all of them are different.

**Proposition 2.** The almost product structures associated with two roots are identical if and only if they are orthogonal (with respect to the Killing form \( \sum_{i=1}^{4} x_i^2 \)).

The existence of different almost product structures suggests that the original \( D_4 \)-symmetry should be explained by the symmetry of \( \mathcal{A} \) i.e. its non-trivial endomorphisms. This leads further to the natural question: what is the relation between the moduli of two elliptic curves \( E_1 \) and \( E_2 \) which should exist since we have only one parameter \( s \). The stabilizer of the Weyl symmetry at \( q_0 \) is isomorphic to \( S_3 \), so it contains an element of order 3. This fact proves
Proposition 3. There is an isogeny of degree 3 between $E_1$ and $E_2$.

By this result we can describe $E_1$ and $E_2$ by two lattices $L_1, L_2$ in $C$ in the following way:

(3) $3L_2 \subset L_1 \subset L_2, \quad [L_2 : L_1] = 3.$

(4) $E_1 = C/L_1, \quad E_2 = C/L_2.$

Then, by (1), we have also the isomorphism

(5) $(C \times C)/L \cong \mathcal{A}$

where $L$ is a lattice in $C \times C$ such that $2L \subset L_1 \times L_2 \subset L, \quad [L : L_1 \times L_2] = 4.$

Proposition 4. The lattice in (5) is given by

$L = \{(a, b) \in C \times C : 2a \in L_1, 2b \in L_2, a - b \in L_2\}.$

The stabilizer at $q_0$ is lifted to a subgroup of $\text{Aut}(\mathcal{A})$ generated by the elements which are induced by the matrices

$$M := \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad N := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Check that $ML = L, NL = L$ and that $M^3 = -1, N^2 = (MN)^2 = 1$. We close this note by remarking that the entire $D_4$-symmetry is generated by the stabilizer described above and the (translation) action of $\mathcal{A}(2)$ over $S = \mathcal{A}/\{\pm 1\}$.

The analytic counterpart of this story contains the parametric representation of $S$ by the Weierstrass $\sigma$-functions associated with $E_1$ and $E_2$; it also contains the explanation of the parameter $s$ and the isogeny between the elliptic curves by some modular models. This interesting topic will however be published elsewhere in a more general form.