数理解析研究所講究録
第 766 巻 1991 年 1-23

ON NON-CONVEX CURVES OF CONSTANT ANGLE

By

Shigetake MATSUURA
RIMS, Kyoto University

0. Introduction.

In the title, "non-convex" means, as usual, "not necessarily convex." And, to give a reasonably understandable exposition, it is preferable to start with the case of convex curves, because one has a good intuitive interpretation in that case.

The theory of curves of constant angle is a kind of generalization of that of curves of constant breadth. It is also an extensive generalization of the incomplete work of Blaschke [19] which generalized the classical example of ellipses, answering the problem posed by C. Neumann.

In view of the lack of space, we can scarcely give proofs of the results. We try, however, to make this exposition as readable as possible. We are obliged to omit many interesting and important results, especially solutions to various variational problems. But this brief sketch may be considered an introduction to the theory of curves of constant angle. Our subjects are a special kind of plane curves. However, from the beginning of definition of admissible curves, we need to make much use of functional analysis. Thus, our theory of curves of constant angle might be recognized as an example of *Functional-Analytic Geometry.*

A more complete version with full proofs shall be published elsewhere.

1. Notations.

To simplify the notations in this article, we introduce the following: let α be a given angle $0 < \alpha < \pi$, we put $\hat{\alpha} = \pi - \alpha$, and also use

$$c_1 = c_1(\alpha) = \sin\alpha, \qquad c_2 = c_2(\alpha) = \cos\alpha,$$

$$\tilde{c}_1 = \tilde{c}_1(\alpha) = \sin\frac{\alpha}{2}, \qquad \tilde{c}_2 = \tilde{c}_2(\alpha) = \cos\frac{\alpha}{2},$$

since these quantities will appear very frequently in formulas in this article.

2. Characteristic function χ_α and modified characteristic function $\tilde{\chi}_\alpha$.

Let $\Omega_\alpha = \min(\tilde{c}_1, \tilde{c}_2)$. The open intervals I_α, J_α are defined by

$$I_\alpha = (-\Omega_\alpha, \Omega_\alpha)$$
$$J_\alpha = \begin{cases} (\ 0\,, c_1) & \text{if} \quad 0 < \alpha \leq \pi/2 \\ (-c_2,\ 1) & \text{if} \quad \pi/2 \leq \alpha < \pi. \end{cases}$$

The *characteristic function* χ_α and the *modified characteristic function* $\tilde{\chi}_\alpha$ are defined by the formulas

$$\chi_\alpha(t) = c_1\sqrt{1-t^2} - c_2 t, \qquad t \in J_\alpha$$

$$\tilde{\chi}_\alpha(s) = \tilde{c}_1\sqrt{1-s^2} - \tilde{c}_2 s, \qquad s \in I_\alpha \quad \text{or} \quad s \in J_\alpha.$$

Proposition 2.1. *χ_α maps J_α onto J_α and is strictly monotone decreasing. χ_α has only the one fixed point $\tilde{c}_1$. And its inverse mapping χ_α^{-1} coincides χ_α.*

Schematically

$$\chi_\alpha : J_\alpha \rightleftarrows J_\alpha.$$

Proposition 2.2. *$\tilde{\chi}_\alpha$ maps J_α onto I_α and is strictly monotone decreasing. $\tilde{\chi}_\alpha$ maps $\tilde{c}_1$ to 0. And its inverse $\tilde{\chi}_\alpha^{-1}$ has the same expression as $\tilde{\chi}_\alpha$.*

Schematically

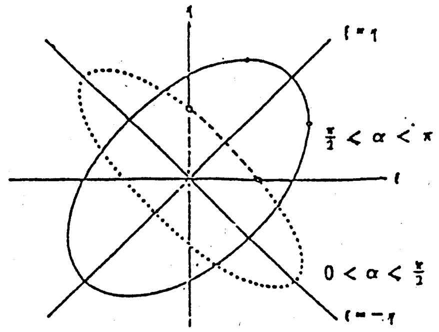
$$\tilde{\chi}_\alpha : J_\alpha \rightleftarrows I_\alpha.$$

In both schemes, $\rightleftarrows$ means that the mappings are one-to-one, onto and that their inverses have the same expressions. $\tilde{\chi}_\alpha$ has the *linearization effect* on χ_α, in fact, we have the following

Proposition 2.3. *If $w \in I_\alpha$, $p \in J_\alpha$, $w = \tilde{\chi}_\alpha(p)$, then we have*

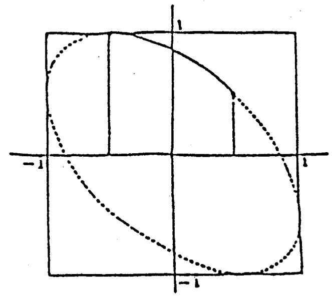
$$\tilde{\chi}_\alpha(\chi_\alpha(p)) = -w.$$

Fig. 2.1. Graph of χ_α, $\eta = \chi_\alpha(\xi)$



$$\xi^2 + \eta^2 + 2c_2\xi\eta = c_1^2$$

Fig. 2.2. Graph of $\tilde{\chi}_\alpha$, $p = \tilde{\chi}_\alpha(w)$



$$p^2 + \omega^2 + 2\tilde{c}_2 pw = \tilde{c}_1^2$$

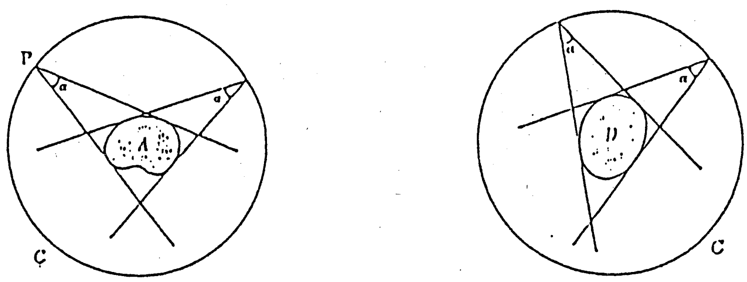
3. Curves of constant angle α (convex case).

Let C be the circle of radius R with center at the orgin $O = (0,0)$ of the plane R^2, and call it the *director circle*. [This terminology comes from the classical example of ellipses, $\alpha = \pi/2$.] Hereafter we assume $R = 1$, without loss of generality. Let A be a figure contained in C. A figure simply means here a subset of R^2. For a point $P \in C$ and we put

$$C(P;A) = \{\text{ray; starting at } P, \text{ passing a point of } A\}$$

(ray = closed half line). $C(P;A)$ is called the *sight-cone* at P for A. We assume that $C(P;A)$ is a closed convex cone with angle α at the vertex P.

Fig. 3.1



Suppose that the angle α is independent of P.

Then, there exists a closed convex set D with non-empty interior such that

$$\partial D \subseteq A \subseteq D$$

(In fact, $D = \bigcap_{P \in C} C(P; A)$ and the origin $O \in \overset{\circ}{D}$: the interior of D). ∂D designates the boundary of D.

$\Lambda = \partial D$ is by definition a closed convex curve.

It is clear that $C(P;\Lambda) = C(P;A) = C(P;D)$ for every $P \in C$. Thus, if we neglect the internal structure of A, it is enough to study D or $\Lambda = \partial D$. Λ is in fact a *strictly convex* curve, i.e. no part of it is a straight line segment. We call Λ a *convex curve of constant angle* α with the director circle C.

In general, to characterize a closed convex curve in $\mathbf{R}^2$, it is enough to obtain its *supporting function* $p(\theta)$

$$p(\theta) = \sup_{(x,y)\in\Lambda} (x\cos\theta + y\sin\theta).$$

Since Λ is strictly convex, p is a C^1-function with period 2π. Λ has the following *parametric representation*

$$\text{(3.1)} \qquad \begin{cases} x = x(\theta) = p(\theta)\cos\theta - \dot{p}(\theta)\sin\theta \\ y = y(\theta) = p(\theta)\sin\theta + \dot{p}(\theta)\cos\theta \end{cases} \qquad (0 \le \theta \le 2\pi)$$

where $\dot{p}(\theta)$ means the derivative $dp(\theta)/d\theta$. This is a continuous closed curve, i.e. $(x(\theta), y(\theta))$ depends continuously on θ but not C^1 in θ. In fact the second derivative $\ddot{p}(\theta)$ in the sense of distribution of L. Schwartz $\mathcal{D}'$ satisfies the inequality

$$p + \ddot{p} \ge 0$$

in the present convex case. This means that the left hand side is a Radon measure ≥ 0.

Theorem 3.1. *For a continuous function $p(\theta)$ of θ to be the supporting function of a convex curve of constant angle α, it is necessary and sufficient that it satisfies the following four conditions:*

(1°) *$p(\theta)$ is a function with period 2π*

(2°) *$p(\theta) \in J_\alpha$ for all θ*

(3°) *$p(\theta + \pi - \alpha) = \chi_\alpha(p(\theta))$ [functional equation]*

(4°) *$p + \ddot{p} \ge 0$ [differential inequality].*

Remark 1. These conditions imply $p \in C^1(\mathbf{R})$.

Remark 2. For every α, $0 < \alpha < \pi$, there exists a convex curve of constant angle α. In fact, if we employ the function $p(\theta) \equiv \tilde{c}_1$, we get a circle of radius $\tilde{c}_1$ concentric with C the director circle. We call this the *trivial* curve of constant angle α.

The totality of functions p satisfying the four conditions of the theorem above is denoted by $\wp_\alpha$ or more precisely $\wp_\alpha^{\text{convex}}$. The case that $\wp_\alpha$ is a singleton $\wp_\alpha = \{\tilde{c}_1\}$ is not interesting. The following theorem is the answer to the question: "Is $\wp_\alpha = \{\tilde{c}_1\}$? "

Theorem 3.2. *(I) If $\alpha/\pi \notin \mathbf{Q}$, then the answer is "yes."*
(II) Suppose $\alpha/\pi \in \mathbf{Q}$, and let $\alpha/\pi = \dfrac{m}{n}$ be its irreducible fraction representation.

(i) *If $mn =$ odd, then the answer is "yes."*

(ii) *If $mn =$ even, then the answer is "no."*

Remark. In the case (II),(ii) $\wp_\alpha = \wp_\alpha^{\text{convex}}$ is an infinite dimensional set. This means that, if $\wp_\alpha^{\text{convex}}$ is suitably topologized, it contains a non-empty open set of an infinite dimensional normed space.

4. The space $B_{2,\sigma}$ and its open convex cone $\wp$.

Let $C(\mathrm{T})$ be the space of all real-valued continuous functions on the torus $\mathrm{T} = \mathbf{R}/2\pi\mathbf{Z}$ equipped with the uniform norm. Then, its dual space $\mathcal{M}(\mathrm{T}) = C(\mathrm{T})'$ is the space of all Radon measures on T. Both $C(\mathrm{T})$ and $\mathcal{M}(\mathrm{T})$ are Banach spaces. Now, consider the space of functions

$$B_2 = \{u; u \in C(\mathrm{T}), \dot{u} \in C(T), \ddot{u} \in \mathcal{M}(\mathrm{T})\}.$$

This becomes naturally a Banach space. But, for later applications, the norm topology of $\mathcal{M}(\mathrm{T})$ is too strong. Thus we replace it by the $*$-weak topology $\sigma(\mathcal{M}(\mathrm{T}), C(\mathrm{T}))$, i.e. the simple convergence topology. Then we denote the space by $\mathcal{M}_\sigma(\mathrm{T})$. Correspondingly, we weaken the topology of B_2. The new topology is the pull-back by the one-to-one mapping

$$B_2 \ni u \to (u, \dot{u}, \ddot{u}) \in C(\mathrm{T}) \times C(\mathrm{T}) \times \mathcal{M}_\sigma(\mathrm{T})$$

when the target space is equipped with the product topology. Thus topologized B_2 is denoted by $B_{2,\sigma}$ which is a locally convex topological vector space but no more a Banach space.

Now consider

$$\wp = \{p; p \in B_{2,\sigma}, \min_{\theta\in\mathrm{T}} p(\theta) > 0\}.$$

This is clearly an open convex cone in $B_{2,\sigma}$. The space $\wp$ plays an important role when we introduce non-convex curves of constant angle.

For every $p \in \wp$, we consider a closed continuous plane curve Λ with the following parametric representation

$$\left\{\begin{array}{l} x = x(\theta) = p(\theta)\cos\theta - \dot{p}(\theta)\sin\theta \\ y = y(\theta) = p(\theta)\sin\theta + \dot{p}(\theta)\cos\theta \end{array}\right. \tag{4.1}$$

p is called the *generator* of Λ.

Remark. When Λ is a strictly convex curve, p is its supporting function. But for general non-convex curves, the meaning of $p(\theta)$ will be discussed later.

We put

$$T(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.2}$$

Then (4.1) takes the form

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = T(\theta)\begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix}. \tag{4.3}$$

Since $T(\theta)$ is an orthogonal matrix and $T(\theta)^{-1} = T(-\theta)$, we get

$$\begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix} = T(-\theta)\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} \tag{4.4}$$

and

$$x(\theta)^2 + y(\theta)^2 = p(\theta)^2 + \dot{p}(\theta)^2. \tag{4.5}$$

(4.3), (4.4) show that the generator p determines the curve Λ and conversely the curve Λ determines the generator p. Thus, we sometimes use the designations p_Λ and Λ_p.

When Λ is a strictly convex curve, its length $L[p]$ and the area $A[p]$ inscribed by Λ $(p = p_\Lambda)$ are calculated by the formulas

$$L[p] = \int (p + \ddot{p})[d\theta] = \int_0^{2\pi} p(\theta) d\theta \tag{4.6}$$

$$A[p] = \frac{1}{2} \int p(p + \ddot{p})[d\theta] = \frac{1}{2} \int_0^{2\pi} p^2 - \dot{p}^2 d\theta. \tag{4.7}$$

Even for non-convex case, we employ these formulas as representing the *oriented length* and the *oriented area* respectively.

To derive these formulas, we use the Leibniz formulas

$$(fg)^{\cdot} = \dot{f}g + f\dot{g}. \tag{4.8}$$

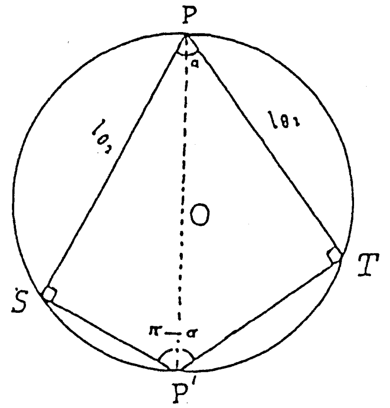
This holds in the sense of $\mathcal{D}'$, if $f \in C^1$ and g is a function of bounded variation, i.e. $\dot{g}$ is a Radon measure.

Both functional $L[p]$ and $A[p]$ are continuous even on $B_{2,\sigma}$. If $\alpha \neq \dfrac{n-1}{n}\pi$ $(n \geq 2)$, then $\wp_\alpha^{\text{convex}}$ is compact in $B_{2,\sigma}$. Therefore they attain their maxima and minima. To obtain the concrete solutions of these or other types of functionals, we are led to difficult *variational problems*. Some of them were solved but many remain unsolved.

5. Duality and dual curves.

We start with the case of convex curves with the generators $\in \wp_\alpha^{\text{convex}}$. Given a function $p \in \wp_\alpha^{\text{convex}}$, let Λ be the corresponding convex curve of constant angle α, and P be a point on the director circle C. Then, Λ spans a sector region of angle α at P (see the figure below).

Fig. 5.1



The two edge lines (rays) of the sector intersect with C at points S and T respectively. If we consider the point P' antipodal to P, i.e. the other end of the diameter of C passing P, then it is clear that the angle $SP'T = \hat{\alpha} = \pi - \alpha$. Now, let P move around the circle C, then P' also moves around C. In this situation, can or cannot the new sector of angle $\pi - \alpha$ with vertex P' with edge lines passing through S and T (respectively) be the sector corresponding to a curve $\hat{\Lambda}$ with a generator $\hat{p} \in \wp_{\hat{\alpha}}^{\text{convex}}$?

Remark. If $\alpha/\pi \notin Q$, then $\hat{\alpha}/\pi \notin Q$. Therefore $\wp_\alpha^{\text{convex}}, \wp_{\hat{\alpha}}^{\text{convex}}$ are the singletons $\{\tilde{c}_1(\alpha)\}$, $\{\tilde{c}_1(\hat{\alpha})\}$ [Theorem 3.2.]. In this case, we have trivially $\widehat{\tilde{c}_1(\alpha)} = \tilde{c}_1(\hat{\alpha})$.

Proposition 5.1. *If* $\alpha/\pi = \dfrac{m}{n}$ *(irreducible fraction),* $0 < \alpha \leq \pi/2$, n : *even* (m : *odd*), *then we have the duality mapping*

$$\hat{}: \wp_\alpha^{\text{convex}} \ni p \to \hat{p} \in \wp_{\hat{\alpha}}^{\text{convex}}$$

with the following properties:

(i) $\hat{}$ *is given by the formula*

$$(5.1) \qquad \hat{p}(\theta) = \sqrt{1 - p\left(\theta - \frac{\pi}{2}\right)^2}\,.$$

(ii) $\hat{}$ *is an injective mapping such that* $\hat{\hat{}}$ *is the identity on* $\wp_\alpha^{\mathtt{convex}}$, *i.e.* $\hat{\hat{p}} = p$.

(iii) *If* $\alpha = \dfrac{\pi}{2}$, *every element of* $\wp_\alpha^{\mathtt{convex}}$ *is self-dual, i.e.* $\hat{p} = p$. *Therefore,* $\hat{}$ *is onto mapping.*

(iv) *If* $0 < \alpha < \dfrac{\pi}{2}$ $\hat{}: \wp_\alpha^{\mathtt{convex}} \to \wp_{\hat{\alpha}}^{\mathtt{convex}}$ *is never onto.*

Remark. The last property (iv) show that, if $\dfrac{\pi}{2} < \alpha < \pi$, there exist always an element $p \in \wp_\alpha^{\mathtt{convex}}$ such that $\hat{p}$ defined by (5.1) is not the supporting function of a convex curve of constant angle $\hat{\alpha}$. It turns out later that this $\hat{p}$ can be the generator of a non-convex curve of constant angle $\hat{\alpha}$, otherwise $\hat{p} = (\hat{p}_1, \hat{p}_2)$ the generator of twin type curves (see below).

If $\alpha = \dfrac{m}{n}\pi$, n : odd, m : even, then $\hat{\alpha} = \dfrac{n-m}{n}\pi$. Thus both $n-m$ and n are odd. Therefore $\wp_{\hat{\alpha}}^{\mathtt{convex}}$ is the singleton $\{\tilde{c}_1(\hat{\alpha})\}$. In this situation, it is impossible, for a curve Λ of constant angle α different from the circle concentric with C, to have its dual curve $\hat{\Lambda}$ of single type. Natural observation of Fig 5.1 leads us to consider a pair functions $\hat{p} = (\hat{p}_1, \hat{p}_2)$ defined by the formulas

$$(5.2) \qquad \begin{cases} \hat{p}_1(\theta) = \sqrt{1 - p(\theta - \frac{\pi}{2})^2} \\ \hat{p}_2(\theta) = \sqrt{1 - p(\theta + \frac{\pi}{2})^2} \end{cases}$$

where p is the supporting function of Λ.

The pair of curves $\hat{\Lambda} = (\hat{\Lambda}_1, \hat{\Lambda}_2)$, $\hat{\Lambda}_j$ being the curve defined by the generator $\hat{p}_j$, would be the dual of Λ. This situation will be justified after we introduce the *admissible non-convex curves* of constant angle α of *twin type*. Then, it turns out that

$$(5.3) \qquad \hat{p} = (\hat{p}_1, \hat{p}_2) \in \wp_{\hat{\alpha}}^{\mathtt{twin}}.$$

By the formula (5.2), $\hat{p}_2(\theta) = \hat{p}_1(\theta + \pi)$, thus $\hat{\Lambda}_2$ is the symmetric image of $\hat{\Lambda}_1$ with respect to the center of symmetry the origin $O = (0, 0)$.

Conversely, given $p = (p_1, p_2) \in \wp_\alpha^{\text{twin}}$, thus satisfying

$$p_2(\theta) = p_1(\theta + \pi), \tag{5.4}$$

its dual $\hat{p} = (\hat{p}_1, \hat{p}_2)$ is defined by

$$\begin{cases} \hat{p}_1(\theta) = \sqrt{1 - p_2(\theta - \frac{\pi}{2})^2} \\ \hat{p}_2(\theta) = \sqrt{1 - p_1(\theta + \frac{\pi}{2})^2}\,. \end{cases} \tag{5.5}$$

By the formula (5.4) $\hat{p}_1(\theta) = \hat{p}_2(\theta)$. Thus we get a single function, therefore we identify the pair $\hat{p} = (\hat{p}_1, \hat{p}_2)$ with this single function, denote it again by the same letter $\hat{p}$ $(= \hat{p}_1 = \hat{p}_2)$. Then, even $p \in \wp_\alpha^{\text{convex}}$, $\alpha = \frac{m}{n}\pi$ (m : odd n : even), we get $\hat{\hat{p}} = p$. The similar argument shows that, starting with $p = (p_1, p_2) \in \wp_\alpha^{\text{twin}}$, $\alpha = \frac{m}{n}\pi(mn : \text{odd})$, we get $\hat{\hat{p}} = p$ by (5.4). (5.5) indicates the possibility of curves of constant angle α of double type and the space of generators $\wp_\alpha^{\text{double}}$ and the theory of them was developed by my student Miss. J. SHEN [23].

6. Admissible curves and their generators.

We are obliged to treat non-convex curves and to seek for something which plays the role of supporting lines to convex curves. They are the *generalized tangent lines* to admissible curves defined below.

Definition 6.1. *An admissible curve Λ and its generalized tangent lines $\ell(\theta), \theta \in \mathrm{T}$, are defined by the following three conditions:*

(1) Λ is a closed continuous curve parametrized by $\theta \in \mathrm{T}$

$$\begin{cases} x = x(\theta) \\ y = y(\theta) \end{cases}$$

(2) Λ is rectifiable, i.e. for any partion Δ of $[0, 2\pi]$

$$\Delta \ : 0 = \theta_0 < \theta_1 < \cdots < \theta_N = 2\pi$$

consider the length of the closed polygon connecting the points $Q_j = (x(\theta_j), y(\theta_j))$ in the natural order.

$$L_\Delta(\Lambda) = \sum_{j=1}^{N} \sqrt{\Big(x((\theta_j) - x((\theta_{j-1})\Big)^2 + \Big(y(\theta_j) - y(\theta_{j-1})\Big)^2}.$$

Then, these quantities are bounded as Δ runs over all partions. The length of Λ is defined by

$$\sup_{\Delta} L_\Delta(\Lambda). \tag{6.1}$$

(3) For every fixed θ, consider the straight line ℓ_θ the equation of which takes the canonical form

$$\ell_\theta \ : x\cos\theta + y\sin\theta = p(\theta). \tag{6.2}$$

Let $e_\theta = (\cos\theta, \sin\theta)$, $\tilde{e}_\theta = (-\sin\theta, \cos\theta)$, e_θ represents the direction perpendicular to the line ℓ_θ, $\tilde{e}_\theta$ represents a direction parallel to ℓ_θ and determines the orientation (or direction) of ℓ_θ. $p(\theta)$ can be considered the distance between the origin O and the oriented line ℓ_θ.

We assume always the inequality

$$p(\theta) > 0 \tag{6.3}$$

holds for all θ. This is a non-trivial restriction on $p(\theta)$.

Further we assume that for every fixed θ_0 the line ℓ_{θ_0} is a generalized tangent line *to the curve Λ at the point $(x(\theta_0), y(\theta_0))$ of Λ. Notice that, since Λ is not C^1*–regular curve *in general, the ordinary tangent lines may not exist. Thus we define the generalized tangent line in the following way: the tangent vector at the point $((x(\theta_0), y(\theta_0))$, if it exists, should be $(\dot{x}(\theta_0), \dot{y}(\theta_0))$. Therefore, we require the condition*

$$\dot{x}(\theta)\cos\theta + \dot{y}(\theta)\sin\theta = 0. \tag{6.4}$$

The derivations $\dot{x}(\theta), \dot{y}(\theta)$ are taken in the sense of $\mathcal{D}'(\mathrm{T})$. Thus, (6.4) should be an equation in an open neighbourhood of the fix value θ_0. Since θ_0 is arbitrary, it is the same to require that the equation (6.4) holds globally.

Remark. Since Λ is rectifiable, the functions $x(\theta), y(\theta)$ are of bounded variation. Therefore, $\dot{x}(\theta), \dot{y}(\theta)$ are Radon measures.

Since the point $(x(\theta), y(\theta))$ is on the line ℓ_θ, we have

$$x(\theta)\cos\theta + y(\theta)\sin\theta = p(\theta). \tag{6.5}$$

This equality shows that $p(\theta)$ is continuous in θ. Differentiating both sides of this equality, we get

$$\dot{p}(\theta) = -x(\theta)\sin\theta + y(\theta)\cos\theta + \dot{x}(\theta)\cos\theta + \dot{y}(\theta)\sin\theta.$$

Using (6.4) we get

$$\dot{p}(\theta) = -x(\theta)\sin\theta + y(\theta)\cos\theta. \tag{6.6}$$

This shows that $\dot{p}(\theta)$ is continuous, i.e. $p \in C^1(\mathrm{T})$. Differentiating again both sides of (6.6), we get

$$\ddot{p}(\theta) = -x(\theta)\cos\theta - y(\theta)\sin\theta - \dot{x}(\theta)\sin\theta + \dot{y}(\theta)\cos\theta.$$

Thus we get

$$p(\theta) + \ddot{p}(\theta) = -\dot{x}(\theta)\sin\theta + \dot{y}(\theta)\cos\theta. \tag{6.7}$$

This shows $p + \ddot{p}$ is a Radon measure on $\mathbf{T}$. From (6.5) and (6.6) we get

$$\begin{pmatrix} p \\ \dot{p} \end{pmatrix} = T(-\theta)\begin{pmatrix} x \\ y \end{pmatrix} \tag{6.8}$$

or

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = T(\theta)\begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix}. \tag{6.9}$$

This is just the parametric representation of Λ, and $p(\theta)$ is its generator (6.4). (6.7) give that

$$\begin{pmatrix} 0 \\ p + \ddot{p} \end{pmatrix} = T(-\theta)\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

or

$$\begin{pmatrix} \dot{x}(\theta) \\ \dot{y}(\theta) \end{pmatrix} = T(\theta)\begin{pmatrix} 0 \\ p + \ddot{p} \end{pmatrix}. \tag{6.10}$$

Therefore $\dot{x}, \dot{y}$ are Radon measures if $p + \ddot{p}$ is such and Λ is rectifiable. Thus we get

Theorem 6.1. *An admissible curve is a curve with a positive C^1 generator $p(\theta)$, such that $p + \ddot{p}$ is a Radon measure, i.e. $p \in \wp$.*

Remark. If the admissible curve Λ has a true tangent line at some point on it, that tangent line should coïncide with some ℓ_θ, i.e. a generalized tangent line.

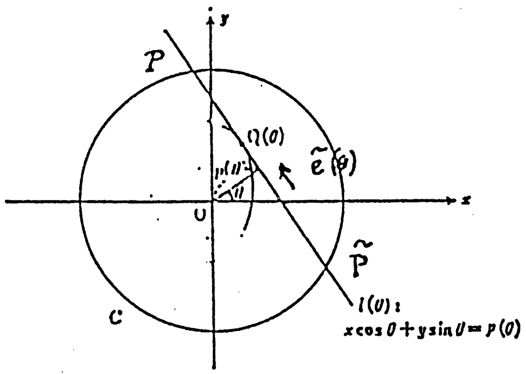
Proposition 6.2. *Λ is an admissible curve contained in C if and only if its generator $p \in \wp$ satisfies*

$$p(\theta)^2 + \dot{p}(\theta)^2 < 1.$$

7. Incoming and outgoing generalized tangent lines of admissible curves (to and from) a point P on the director circle C.

Let Λ be an admissible curve in the director circle C. θ: given, let ℓ_θ be the generalized tangent line directed (or oriented) by the vector $\tilde{e}_\theta = (-\sin\theta, \cos\theta)$.

Fig 7.1



The intersecting points of ℓ_θ and C be P and $\tilde{P}$. P is the target point and $\tilde{P}$ be the source point of the directed segment $\overline{P\tilde{P}}$. For simplicity, this directed segment is also designated by ℓ_θ. We write the coordinates $P = (\cos\varphi, \sin\varphi), \tilde{P} = (\cos\tilde{\varphi}, \sin\tilde{\varphi})$, P and $\tilde{P}$ being on ℓ_θ, the equation of ℓ_θ

$$x\cos\theta + y\sin\theta = p(\theta)$$

leads to the equations

$$\begin{cases} \cos(\varphi - \theta) = p(\theta) \\ \cos(\tilde{\varphi} - \theta) = p(\theta)\,. \end{cases} \tag{7.1}$$

Since $0 < \varphi - \theta < \pi$, $\pi < \tilde{\varphi} - \theta < 2\pi$, we have

$$\begin{cases} \varphi = \theta + \mathrm{ACS}(p(\theta)) \\ \tilde{\varphi} = \theta + \pi - \mathrm{ACS}(p(\theta)) \end{cases} \tag{7.2}$$

where ACS is the *prinicipal value* of the inverse cosine function $\cos^{-1}$. Thus we get

$$\begin{cases} \dot{\varphi} = 1 - \dfrac{\dot{p}}{\sqrt{1-p^2}} & > 0 \\ \dot{\tilde{\varphi}} = 1 + \dfrac{\dot{p}}{\sqrt{1-p^2}} & > 0 \end{cases} \tag{7.3}$$

because $p^2 + \dot{p}^2 < 1$. This result can be stated as

Proposition 7.1. *As the parameter θ goes around in the couter-clock-wise the unit circle* T, *the target point P and the source point $\tilde{P}$ of ℓ_θ both go round C in the couter-clock-wise with speeds greater than a positive constant.*

In other words:

Proposition 7.2. *Given a point $P \in C$, there exist uniquely a pair of values (θ_1, θ_2) such that P is the target point of ℓ_{θ_1} and the source point of ℓ_{θ_2}.*

Definition. *In the above proposition, ℓ_{θ_1} is called the generalized tangent line incoming to P and ℓ_{θ_2} the generalized tangent line outgoing from P.*

8. Curves of constant angle of double type.

Let Λ_1, Λ_2 be two admissible curves in the director circle C. For every point $P \in C$, let $\ell_1(P)$ be the incoming generalized tangent line to P to Λ_1, $\ell_2(P)$ be the outgoing generalized tangent line from P to Λ_2. α being an angle $0 < \alpha < \pi$, suppose that $\ell_1(P)$ and $\ell_2(P)$ span the angle α for all $P \in C$. Then we call the pair (Λ_1, Λ_2) a curve of constant angle α of *double type*. Let p_j be the generator of Λ_j. Then $p = (p_1, p_2)$ is called the generator of the pair (Λ_1, Λ_2).

Proposition 8.1. *For $p = (p_1, p_2)$ to be the generator of a curve of constant angle of double type $\Lambda = (\Lambda_1, \Lambda_2)$, it is necessary and sufficient that*

(i) $p_j(\theta) \in \mathcal{J}_\alpha$, $j = 1.2 \quad (\theta \in \mathrm{T})$

(ii) $p_j(\theta)^2 + \dot{p}_j(\theta)^2 < 1$, $j = 1.2 \quad (0 \in \mathrm{T})$

(iii) $p_2(\theta + \pi - \alpha) = \chi_\alpha(p_1(\theta))$, $\theta \in \mathrm{T}$

(iv) $p_j + \ddot{p}_j \in \mathcal{M}(\mathrm{T})$, $j = 1.2 \quad$ (since $p_j \in \wp$; $\quad j = 1, 2$).

Definition 8.1. *We write $p = (p_1, p_2) \in \wp_\alpha^{double}$ if the conditions of Prop.8.1 are satisfied.*

Definition 8.2. *If $p_1(\theta) \equiv p_2(\theta)$, i.e. $\Lambda_1 = \Lambda_2$, then the curve $\Lambda (= \Lambda_1 = \Lambda_2)$ is called a curve of constant angle α of single type and we write p, instead of (p, p) : Notation $\wp_\alpha^{single}$.*

Definition 8.3. *If $p_2(\theta) = p_1(\theta + \pi)$, we call $\Lambda = (\Lambda_1, \Lambda_2)$ a curve of constant angle α of twin type : Notation $\wp_\alpha^{twin}$.*

Definition 8.4. *If $p = (p_1, p_2) \in \wp_\alpha^{double}$ and $\check{p} = (p_2, p_1) \in \wp_\alpha^{double}$, we say that $p = (p_1, p_2)$ or $\Lambda = (\Lambda_1, \Lambda_2)$ is symmetric double type and write $p \in \wp_\alpha^{double,sym}$.*

Remark. $\wp_\alpha^{twin} \subseteq \wp_\alpha^{double,sym}$.

9. Linearization by $\tilde{\chi}_\alpha$, space $\mathcal{W}_\alpha^{double}$ and the duality.

Since $\wp_\alpha^{\text{single}} \subseteq \wp_\alpha^{\text{double}}, \wp_\alpha^{\text{twin}} \subseteq \wp_\alpha^{\text{double}}$, we explane the situation for the most general case, i.e. the case of double type.

We define the space $\mathcal{W}_\alpha^{\text{double}}$ by

$$\mathcal{W}_\alpha^{\text{double}} = \{w = (w_1, w_2); w_j = \tilde{\chi}_\alpha(p_j), j = 1.2, p = (p_1, p_2) \in \wp_\alpha^{\text{double}}\} \tag{9.1}$$

when $p = (p_1, p_2) \in \wp_\alpha^{\text{double}}$, its dual $\hat{p} = (\hat{p}_1, \hat{p}_2) \in \wp_{\hat{\alpha}}^{double}$ is defined as in the case of twin type

$$\begin{cases} \hat{p}_1(\theta) = \sqrt{1 - p_2(\theta - \frac{\pi}{2})^2} \\ \hat{p}_2(\theta) = \sqrt{1 - p_1(\theta + \frac{\pi}{2})^2} \,. \end{cases} \tag{9.2}$$

And define $\hat{w} = (\hat{w}_1, \hat{w}_2)$ by

$$\hat{w}_j = \tilde{\chi}_{\hat{\alpha}}(\hat{p}_j), \; j = 1, 2. \tag{9.3}$$

Easy calculations show

Theorem 9.1. *$\hat{w}_j$ are calculated as follows:*

$$\begin{cases} \hat{w}_1(\theta) = -w_2(\theta - \frac{\pi}{2}) \\ \hat{w}_2(\theta) = -w_1(\theta + \frac{\pi}{2}) \,. \end{cases} \tag{9.4}$$

And it is clear that $\hat{\hat{w}} = w$.

Remark 1. $\hat{\alpha} = \pi - \alpha, J_{\hat{\alpha}}$ is different from J_α in general. But $I_{\hat{\alpha}} = I_\alpha$ always. This simplifies the situation.

Remark 2. It is easy to see that the union of curves of constant angle of single type and of twin type is also *stable* under the duality map $\hat{}$.

10. Geometry of admissible curves.

Let Λ be an admissible curve. Decompose the open set $\mathbf{R}^2 \setminus \Lambda$ into its connected components. The unbounded component U_∞ is the outside of Λ. The component containing the origin O, denoted by U_0, is the *core* of Λ.

Theorem 10.1. *U_0 is an convex set and it is the interior of the compact convex set Δ which is defined by*

$$\Delta = \bigcap_{\theta \in \mathrm{T}} \Delta(\theta)$$

where $\Delta(\theta)$ is the closed half plane defined by

$$\Delta(\theta) = \{(x, y);\ x \cos\theta + y \sin\theta \leq p(\theta)\},$$

$p(\theta)$ being the generator of Λ.

Remark. $\Lambda^0 = \partial\Delta$ is a closed convex curve, its supporting function is denoted by $p^0(\theta)$.

When p is real analytic, a corner points of Λ^0 is a self-intersection point of Λ. Such points are finite in number.

To each corner point of Λ^0, a swallow tail (possibly of complicated shape) is attached.

Proposition 10.2. *(i) The oriented length of swallow tails are always non-negative.*
(ii) The oriented area of swallow tails are always non-positive.

Remark. For general generator $p \in \wp$, we may approximate by C^∞ generator by regularization. The latter again approximated by analytic generator by using its Fourier expansion, and exploiting a suitable initial finite sum.

11. Non oriented length $|L|[p]$ and non-oriented area $|A|[p]$.

The length of an admissible curve is defined by (6.1). But more maniable new formula is given by

Theorem 11.1. *For a rectifiable closed continuous curve* Λ

$$\begin{cases} x = x(\theta) \\ y = y(\theta) \end{cases}$$

the (non-oriented) length of Λ *can be given by the formula*

$$|L|[\Lambda] = \sup_{\substack{\varphi^2+\psi^2\leq 1 \\ \varphi,\psi\in C(\mathrm{T})}} (\dot{x}[\varphi] + \dot{y}[\psi]). \tag{11.1}$$

Remark. This is a *new formula of arc-length* and it can easily be generalized for a curve in $\mathbf{R}^n$. Here, $C(\mathrm{T})$ can be replaced by $C^\infty(\mathrm{T})$ since the latter is dense in the former.

Using this new formula, we get

Proposition 11.2. *For a admissible curve* Λ *with generator* p*, the length* $|L|[p]$ *is given by the formula*

$$|L|[p] = \|p + \ddot{p}\|_{\mathcal{M}(\mathrm{T})}. \tag{11.2}$$

Remark. This functional $|L|[p]$ is not continuous on $\wp$, but lower semicontinuous. For any $\alpha \in \mathbf{Q}, 0 < \alpha < \pi, |L|[p]$ is unbounded on $\wp_\alpha^{\text{single,twin or double}}$.

Using the results in the preceding section, we define the non-oriented area $|A|[p]$ by

$$|A|[p] = 2A[p^0] - A[p] \tag{11.3}$$

where p^0 is the supporting function of Λ^0 (or of the convex core $\overset{\circ}{\Delta}$).

Remark. This functional $|A|[p]$ is continuous in $\wp$.

By constructing explicitly an extremalizing sequence $p_k \in \wp_\alpha^{\text{single}}$ $k = 1, 2, \cdots$, we have

Theorem 11.3. $\alpha/\pi = \dfrac{m}{n}$ *(irreducible fraction),* $mn = \text{even}, 0 < \alpha \leq \pi/2$. *Then we have*

$$\sup_{p\in\wp_\alpha^{\text{single}}} |A|[p] = \pi. \tag{11.4}$$

Remark. Every curve with generator $p \in \wp_\alpha^{\text{single}}$ is contained in C. But π is the area of C. Thus, this result may be somewhat amazing. In the case $\pi/2 < \alpha < \pi$, the problem remains unsolved.

12. Local regularity of solutions to simultaneous relations.

To simplify the situations, we first limit ourselves to the case of single type. Let $u(\theta)$ be an unknown function. In order for u to be the supporting function of a convex curve of constant angle α with respect to the director circle C (its radius being 1), it is necessary and sufficient to satisfy the following four conditions (Theorem 3.1).

(1°) $u(\theta)$ is a continuous function with period 2π

(2°) $u(\theta) \in J_\alpha$ for all $\theta \in \mathrm{R}$

(3°) $u(\theta + \pi - \alpha) = \chi_\alpha(u(\theta))$

(4°) $u + \ddot{u} \geq 0$.

Then, what can we say about the local regularity of u?

Theorem 12.1. *If a distribution $u \in \mathcal{D}'(\mathrm{R})$ satisfies the simultaneous relations (1°), (2°), (3°), (4°), then $u \in C^1(\mathrm{R})$ and $\ddot{u} \in L^\infty(\mathrm{R})$. This result is the best possible one as the local regularity of u for general angle α.*

Remark 1. The same results hold for the twin type curves if they are convex.

Remark 2. Thus, there exist strictly convex curves whose radius of curvature is everywhere essentially discontinuous.

For non-convex curves of constant angle, the local regularity is weaker. The characterizing conditions are the following five ones.

(1*) $u(\theta)$ is a C^1-function with period 2π

(2*) $u(\theta) \in J_\alpha$ for all $\theta \in \mathrm{R}$

(3*) $u(\theta)^2 + \dot{u}(\theta)^2 < 1$

(4*) $u(\theta + \pi - \alpha) = \chi_\alpha(u(\theta))$

(5*) $u + \ddot{u} \in \mathcal{M}(\mathrm{R})$

Then, what we can say?

Theorem 12.2. *If an element $u \in \mathcal{D}'(\mathrm{R})$ satisfies the simultaneous relations* (1*), (2*), (3*), (4*), (5*), *then* $u \in C^1(\mathrm{R})$ *and* $\ddot{u}$ *is an atom-free Radon measure. This result is the best possible one as the local regularity of* u *for general angle* α.

Remark 1. "atom-free" means that every one point set $\{\theta\}$ is of measure 0 with respect to $\ddot{u}$.

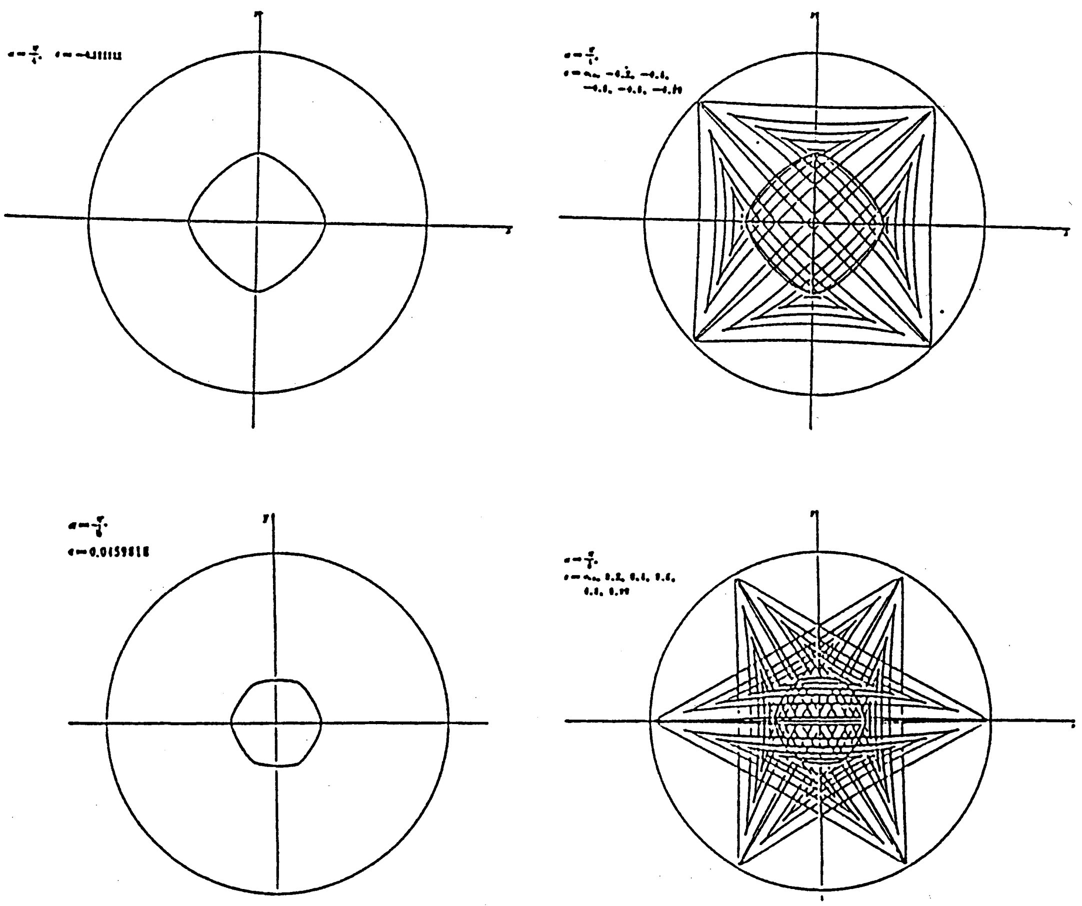
Remark 2. The same results hold for the twin type curves if they are non-convex.

Remark 3. Thus, there exist non-convex curves whose oriented radius of curvature is an unbounded L^1 function, a continuous singular measure or their sum.

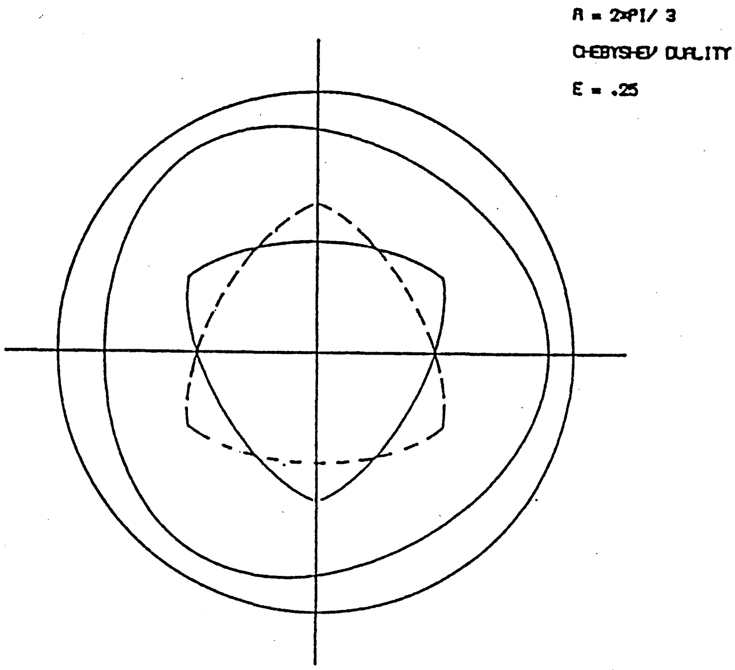
13. Examples.

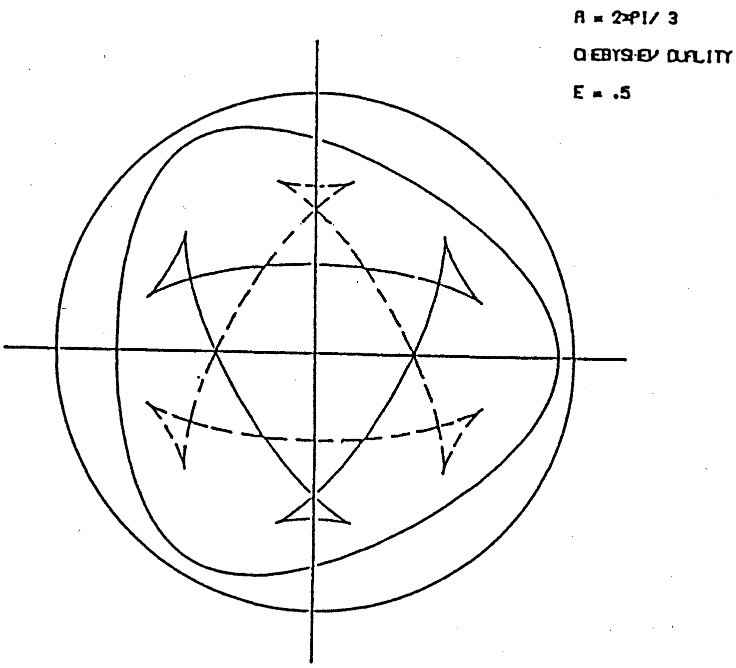
Because our subjects are special kind of plane curves, it would not be good for us to state only theoretical facts without giving examples. Thus we give in this section two types of examples. .

(1) We give here a few examples of convex and non-convex curves of constant angle α for $\alpha = \dfrac{\pi}{4}$ and for $\alpha = \dfrac{\pi}{6}$. The convex ones are embeded in the family of non-convex ones.



(2) Here we give two examples of a curve of constant angle $\alpha = \frac{2}{3}\pi$ of single type and its dual curves of constant angle $\alpha = \frac{1}{3}\pi$ of twin type of convex and nonconvex type.





References

[1] Matsuura, S., I~ VII, Mathematical Seminar (1981, August ~ 1982, Feb.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[2] _____, VIII~ XI, Mathematical Seminar (1982, July ~ 1982, Oct.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[3] _____, XII, Mathematical Seminar (1982, Dec.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[4] _____, XIII, Mathematical Seminar (1983, Feb.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[5] _____, XVI, Mathematical Seminar (1983, Apr.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[6] _____, XV ~ XVI, Mathematical Seminar (1983, June ~ 1983, July) (Nihon Hyōron sha Co. Ltd) in Japanese.

[7] _____, XVII ~ XVIII, Mathematical Seminar (1983, Sept. ~ 1983, Oct.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[8] _____, XIX ~ XXIII, Mathematical Seminar (1984, Jan. ~ 1984, May)(Nihon Hyōronsha Co. Ltd) in Japanese.

[9] _____, XXIV ~ XXVII, Mathematical Seminar (1984, Aug. ~ 1984, Nov.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[10] _____, XXVIII, Mathematical Seminar (1985, Jan.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[11] _____, XXIX ~ XXXIV, Mathematical Seminar (1985, Mar. ~1985, Aug.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[12] _____, XXXV ~ XXXVI, Mathematical Seminar (1985, Oct. ~1985, Nov.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[13] _____, XXXVII ~ XXXX, Mathematical Seminar (1986, Jan. ~1986, Apr.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[14] _____, XXXXI ~ XXXXVII, Mathematical Seminar (1986, June ~1986, Dec) (Nihon Hyōronsha Co. Ltd) in Japanese.

[15] ______, XXXXVIII ~ XXXXIX, Mathematical Seminar (1987, Feb.~1987, Mar.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[16] ______, Mathematical Seminar (1987, Apr.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[17] Matsuura, S. and Kasahara, K., Mathematical Seminar (1988, Mar.) (Nihon Hyōronsha Co. Ltd) in Japanese.

[18] Matsuura, S., Theory of non-convex curves of constant angle (22 expositions), Notes taken by M. Kametani (1990).

[19] Blaschke, W., Über eine Ellipsen eigenschaft und über gewisse Eilinien, Ark. der Maths und Phy. III, Reihe 26 (1917).

[20] Schwartz, L., Théorie des distributions, Hermann, Paris (1966).

[21] Hörmander, L., The analysis of linear partial differential operators I (2nd ed.), Springer-Verlag (1990).

[22] Bourbaki, N., Espaces vectoriels topologiques I,II, Hermann, Paris (1966).

[23] Shen, J., Non-convex curves of constant angle of double type (Master Thesis) (1991).

[24] Yosida, K., Functional Analysis, Springer-Verlag (1968).