

PROJECTIVE-PLANAR GRAPHS WITH EVEN DUALS

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1. Introduction

As is well-known, a graph is bipartite if and only if it contains no odd cycle. So if we embed a bipartite graph G on a closed surface F^2 , then the boundary walk of each face has even length. We call such a face simply an *even face*. In this case, the dual of G embedded in F^2 is *even*, that is, each vertex of the dual has even degree.

Also it is well-known that a planar graph has an embedding with only even faces in the sphere or the plane if and only if it is bipartite. For any eulerian map on the sphere or the plane is 2-colorable. However, the necessity does not hold for other surfaces in general since an odd cycle might be hidden as one that is not null-homotopic in the surface. In this paper, we discuss those graphs which have even duals in closed surfaces in Section 2 and characterize such projective-planar graphs in Section 3.

To describe our main theorem, we define the *canonical bipartite covering*, denoted by $B(G)$, for any graph G as follows: When G is not bipartite, we prepare two vertices v_1 and v_2 for each vertex $v \in V(G)$ and add two edges v_1u_2 and v_2u_1 if there is an edge $vu \in E(G)$. The resulting graph is $B(G)$ in this case and it is a bipartite graph with partite sets $\{v_1 : v \in V(G)\}$ and $\{v_2 : v \in V(G)\}$. When G is bipartite, we set $B(G) = G$.

THEOREM 1. *Let G be a connected projective-planar graph but nonplanar. Then G has a projective-planar embedding with only even faces if and only if either G is bipartite or $B(G)$ is planar.*

For example, consider the *Möbius ladder* O_n which is defined as a cycle $v_1v_2 \cdots v_{2n}$ of length $2n$ with n diagonals $v_i v_{i+n}$. It is easy to see that O_n is bipartite if and only if n is odd. When n is even, that is, when O_n is not bipartite, $B(O_n)$ is isomorphic to $C_{2n} \times K_2$ and is planar. Thus, O_n can be embeddable in the projective plane so that it has only even faces by Theorem 1.

Actually O_n has two inequivalent projective-planar embeddings given in Figure 1. (Each pair of antipodal points on the boundary circle should be identified to get the projective plane.) The left one has an even dual while the right one does not if n is even.

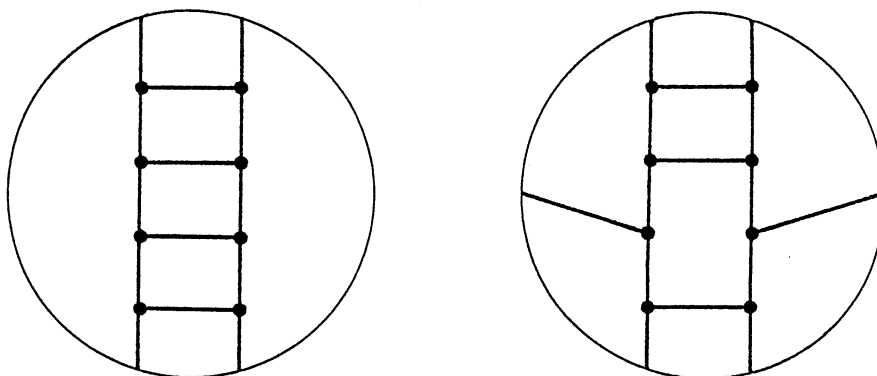


Figure 1: Möbius ladders

As this example shows, a graph might have a projective-planar embedding without an even dual even if $B(G)$ is planar. We can however show the uniqueness of even duals, as follows:

THEOREM 2. *A 3-connected projective-planar graph G , not planar, has a unique even dual in the projective plane if $B(G)$ is planar.*

This can be said to correspond to the well-known fact that every 3-connected planar graph has a unique dual in the sphere, proved by Whitney [6]. As is pointed out in [1], his result is equivalent to that every 3-connected planar graph is uniquely and faithfully embeddable in the sphere. The uniqueness and the faithfulness of planar embeddings will play an important role to prove our theorems via the notion of covering spaces as well as in proofs of the results given in [2] and [4].

2. General Observations

A connected graph \tilde{G} is called an (n -fold) *covering* of a connected graph G if there is an (n -to-one) surjection $p : V(\tilde{G}) \rightarrow V(G)$, called its *projection*, which induces bijections between neighborhoods of corresponding vertices. Such a projection $p : V(\tilde{G}) \rightarrow V(G)$ extends naturally to a map between edge sets by $p(uv) = p(u)p(v)$. We denote simply $p : \tilde{G} \rightarrow G$ to express these two projections and often regard it as a local homeomorphism from \tilde{G} to G in the topological sense. Although the covering spaces of graphs can be defined combinatorially as above, we shall attempt here to discuss covering spaces of graphs in terms of algebraic topology. (See standard text books of algebraic topology, for example [5], if you want to know the general arguments on covering spaces.)

By the classification of covering spaces, there is a one-to-one correspondence between the equivalence classes of coverings of G and the conjugacy classes of subgroups of its fundamental group $\pi_1(G)$. If a covering $p : \tilde{G} \rightarrow G$ is associated with a subgroup H , then the fold number of p is equal to the number of cosets of H in $\pi_1(G)$. In particular, if H is normal in $\pi_1(G)$, then the quotient group $\pi_1(G)/H$ acts on \tilde{G} so that any two vertices in \tilde{G} which project to the same vertex belong to the same orbit. Such a covering is called a

regular covering and the group $\pi_1(G)/N$ acting on \tilde{G} is called the *covering transformation group* of $p: \tilde{G} \rightarrow G$.

The *canonical bipartite covering* $B(G)$ can be defined topologically as the one associated with the subgroup N in $\pi_1(G)$ which consists of all the closed walks in G of even length. Such a subgroup N is normal and has index 2 in $\pi_1(G)$ if G is not bipartite while $N = \pi_1(G)$ if G is bipartite. Thus, $B(G)$ is a 1- or 2-fold covering of G . In general, a closed walk W in G can be lifted to a closed walk in \tilde{G} if and only if the homotopy class of W belongs to the subgroup H which \tilde{G} is associated with; otherwise, we would obtain a walk in \tilde{G} with distinct ends, tracing edges of W via p . This fact implies that $B(G)$ contains no odd cycle and hence $B(G)$ is bipartite.

The following lemma shows the most important property of $B(G)$ and has been proved in [3] by two methods, combinatorial or topological.

LEMMA 3. *Every bipartite covering $p: \tilde{G} \rightarrow G$ factors through the canonical bipartite covering $B(G)$ of G . That is, there is a covering projection $q: \tilde{G} \rightarrow B(G)$ with $p = bq$, where $b: B(G) \rightarrow G$ is the covering projection from $B(G)$ to G . ■*

Notice that G has a unique 2-fold bipartite covering, which is nothing but $B(G)$, when G is not bipartite; if $p: \tilde{G} \rightarrow G$ is a 2-fold bipartite covering, then $q: \tilde{G} \rightarrow B(G)$ has to be an isomorphism between them.

The following theorem generalizes Theorem 1 but shows only the necessity:

THEOREM 4. *Let G be a connected graph 2-cell embedded in a closed surface F^2 . If the dual of G in F^2 is even, then either G is bipartite or $B(G)$ is embeddable in a 2-fold covering space of F^2 .*

Proof. Let $f: G \rightarrow F^2$ be the embedding of G into F^2 . Then we have the following short exact sequence induced by the surjective homomorphism $f_*: \pi_1(G) \rightarrow \pi_1(F^2)$:

$$1 \longrightarrow \ker f_* \longrightarrow \pi_1(G) \longrightarrow \pi_1(F^2) \longrightarrow 1$$

The kernel $\ker f$ can be regarded to be generated by the boundary walks of faces of $f(G)$. (More precisely, they need headers and tails which connect them to the base point of $\pi_1(G)$.)

Let N be the normal subgroup of $\pi_1(G)$ consisting of only closed walks of even length. Then the dual of this embedding is even if and only if

$$\ker f_* \subset N.$$

Suppose that G is not bipartite. Then the normal subgroup N has index 2 in $\pi_1(G)$. If $f_*(N) = \pi_1(F^2)$, then any odd cycle $f(C)$ of $f(G)$ is homotopic to a closed walk $f(W)$ of even length on F^2 and hence $f(C)f(W)^{-1}$ is null-homotopic walk of odd length. This implies that CW^{-1} belongs to $\ker f$ and hence it can be decomposed into a product of face boundary walks, at least one of which should have odd length.

Therefore, if the dual of $f(G)$ is even, then $f_*(N)$ does not coincide with $\pi_1(F^2)$ and has index 2 in $\pi_1(F^2)$. So we can take the 2-fold covering $p: \tilde{F}^2 \rightarrow F^2$ associated with $f_*(N)$. Then the lift $\tilde{G} = p^{-1}(f(G))$ of $f(G)$ covers $f(G)$ and also G doubly. By the

definition of N , only closed walks of even length can be lifted to \tilde{G} . This implies that the 2-fold covering \tilde{G} is bipartite and hence it has to be isomorphic to $B(G)$ by Lemma 3. Thus, $B(G)$ is embeddable in \tilde{F}^2 . ■

If \tilde{F}^2 covers F^2 n -fold, then $\chi(\tilde{F}^2) = n\chi(F^2)$, where $\chi(*)$ denotes the Euler number and any nonorientable surface cannot cover an orientable one. Thus, any two 2-fold covering spaces of an orientable closed surface are homeomorphic to each other. For example, any 2-fold covering of a torus is also a torus. So the above theorem can read for a toroidal graph G that if G has an even dual in the torus, then either G is bipartite or $B(G)$ is toroidal. On the other hand, a nonorientable closed surface has exactly two 2-fold covering spaces, orientable or nonorientable, unless it is the projective plane. For example, the torus and the Klein bottle cover the Klein bottle doubly while the only sphere covers the projective plane. These facts imply that the condition of $B(G)$ in Theorem 4 is not so ambiguous.

Examples: The canonical bipartite covering $B(K_n)$ of the complete graph K_n is isomorphic to the complete bipartite graph $K_{n,n}$ with a perfect matching deleted. For example, $B(K_4)$ is planar and K_4 has a projective-planar embedding with three rectangular faces, and hence with an even dual but not simple. The complete graph K_5 is projective-planar but $B(K_5)$ is nonplanar since it contains $K_{3,3}$ with one edge subdivided by two vertices. Thus, K_5 does not have any even dual in the projective plane while it has an even dual in the torus, isomorphic to K_5 itself, and hence $B(K_5)$ is toroidal. For $n \geq 6$, $B(K_n)$ is not toroidal since it contains two disjoint $K_{3,3}$'s. Thus, neither K_6 nor K_7 have even duals in the torus although they are embeddable there.

3. Projective-Planar Case

There are two important notions to discuss on projective-planar graphs, as the arguments in [2] and [4] suggest; the first is to connect projective-planar embeddings and planar coverings and the second is the uniqueness and faithfulness of 3-connected planar graphs, defined first in [1], as follows. Two embeddings $f, g : G \rightarrow F^2$ are said to be *equivalent* here if there exist an automorphism $\sigma : G \rightarrow G$ and a homeomorphism $h : F^2 \rightarrow F^2$ with $hf = g\sigma$. A graph G is said to be *uniquely embeddable* in F^2 if G has only one equivalence class of embeddings into F^2 . On the other hand, A *faithful embedding* $f : G \rightarrow F^2$ is such a one that for any automorphism $\sigma : G \rightarrow G$, there is a homeomorphism $h : F^2 \rightarrow F^2$ with $hf = f\sigma$, and G is said to be *faithfully embeddable* in F^2 if G has such an embedding.

As is well-known, every 3-connected planar graph has a unique dual in the sphere and it can be said in our terminology that it is uniquely and faithfully embeddable in the sphere. Suppose that a given planar covering \tilde{G} is regular and 3-connected. Then we can embed \tilde{G} in the sphere S^2 so that its covering transformation group Γ extends to a group of auto-homeomorphisms on S^2 since it is faithfully embeddable, and the orbit space S^2/Γ should be the projective plane, where G is embedded.

By the same way, we can get an even dual of G in the projective plane if $B(G)$ is planar and 3-connected. The assumption of $B(G)$ being 3-connected is however too strong to be

valid in general. We need only that its covering transformations extend, but not all the automorphisms of $B(G)$.

The following three lemmas will be used to prove the extendability of covering transformations.

LEMMA 5. *Let G be a 2-connected graph and \tilde{G} a 2-fold planar covering of G . Then \tilde{G} can be embedded in the sphere S^2 so that its covering transformation of order 2 extends to an involution on S^2 .*

Proof. We use induction on the number of vertices of G . Let $p : \tilde{G} \rightarrow G$ be the covering projection and $\tau : \tilde{G} \rightarrow \tilde{G}$ the covering transformation. It is easy to see that τ extends to the antipodal map or the rotation around some axis through π when G has a few vertices.

Suppose that G has more vertices. When \tilde{G} is 3-connected, \tilde{G} is uniquely and faithfully embeddable in the sphere and hence τ extends automatically for any spherical embedding of \tilde{G} . On the other hand, when \tilde{G} is not 3-connected, we can reduce it to the case that G has fewer vertices, as follows.

Let $U \subset V(\tilde{G})$ be a 1- or 2-cut which \tilde{G} has to have. First suppose that $p(U)$ consists of only one vertex of G . Since G is 2-connected, any two vertices in $V(G) - p(U)$ can be joined by a path which does not meet $p(U)$. This implies that $\tilde{G} - U$ consists of precisely two components and hence that \tilde{G} decomposes to two subgraphs H and $\tau(H)$ with $H \cap \tau(H) = U = \tau(U)$. If U were a 1-cut, then the cut vertex in U would be a fixed point of τ , contrary to τ being a free involution. Thus, U consists of two vertices which τ exchanges. In this case, τ extends to an involution on S^2 equivalent to either the antipodal map or the rotation around axis through π , depending on the embeddings of H and $\tau(H)$.

Now suppose that $p(U)$ consists of two vertices of G . Since G has no cut vertex by the above arguments, U has to contain two vertices u, v and $\tau(U) \cap U = \emptyset$. Then G decomposes into three subgraphs K, H and $\tau(H)$ such that $\tau(K) = K, K \cap H = U$ and $H \cap \tau(H) = \emptyset$ if we choose U so that one of components of $\tilde{G} - U$ is minimal with respect to inclusion among all the 2-cuts of \tilde{G} .

Add new edges uv and $\tau(u)\tau(v)$ to K if they do not exist in K . Then τ induces a free involution τ_K over the resulting graph $K' (= K + \{uv, \tau(u)\tau(v)\})$. Now G decomposes into two subgraphs $p(K)$ and $p(H)$ which meet in $p(U)$ and $G' = p(K) + p(U)p(H)$ is 2-connected. Since K' covers G' doubly, K' can be embedded in S^2 so that τ_K extends to an involution $h : S^2 \rightarrow S^2$. If we embed H along the arc uv and $\tau(H)$ along $\tau(u)\tau(v)$, then h will be an extension of τ .

In either case, we have got a spherical embedding of \tilde{G} with τ extendable and hence our induction completes. ■

LEMMA 6. *Let G be a connected nonplanar graph and let \tilde{G} be a 2-fold planar covering of G embedded in the sphere. Then no two vertices of \tilde{G} which project to the same vertex are incident to a common face.*

Proof. Let $p : \tilde{G} \rightarrow G$ be the covering projection and $\tau : \tilde{G} \rightarrow \tilde{G}$ the covering transformation of order 2. Since G is nonplanar, G contains a subdivision K of either K_5 or $K_{3,3}$ and $\tilde{K} = p^{-1}(K)$ is a 2-fold planar covering of K with transformation $\tau|_{\tilde{K}}$.

By Lemma 4 in [2], \tilde{K} is a subdivision of a 3-connected planar graph. Since every 3-connected planar graph is faithfully embeddable in the sphere S^2 , $\tau|_{\tilde{K}}$ extends to an involution $h : S^2 \rightarrow S^2$, which is equivalent to the antipodal map. It is not however assured whether the whole of τ extends or not.

Suppose that for some vertex $v \in V(\tilde{G})$, both v and $\tau(v)$ are incident to a common face. Then they are contained in one face A of \tilde{K} . This implies that either $\tau(A) = A$ or $\tau(A) \cap A = \{v, \tau(v)\}$. In the former case, the face A would contain a fixed point of h , contrary to h being a free involution. In the latter case, \tilde{K} splits at $\{v, \tau(v)\}$, which contradicts that K is a subdivision of a 3-connected graph. Therefore, v and $\tau(v)$ lie on the boundaries of distinct faces. ■

LEMMA 7. *Let G be a connected graph with blocks B_0, B_1, \dots, B_n and let $p : \tilde{G} \rightarrow G$ be a 2-fold planar covering of G . If G is projective-planar but nonplanar, then precisely one of blocks, say B_0 is nonplanar and the others can be lifted to \tilde{G} isomorphically.*

Proof. It is clear that at least one of B_i 's, say B_0 , is nonplanar. If we embed $G = B_0 \cup B_1 \cup \dots \cup B_n$ in the projective plane, then the other blocks B_i ($i \geq 1$) is embedded within faces of B_0 , which are homeomorphic to 2-cells. Thus, they are planar.

Suppose that B_1 could not be lifted isomorphically. Then $p^{-1}(B_1)$ is a 2-fold covering of B_1 , which is connected. Let P be a shortest path connecting B_0 to B_1 with ends $u \in V(B_0)$ and $v \in V(B_1)$ and let P' and P'' be two distinct lifts of P in \tilde{G} . Then the connected subgraph $P' \cup p^{-1}(B_1) \cup P''$ has to be contained in a face of the 2-fold planar covering $p^{-1}(B_0)$ and hence the two lifts of u are incident to the face together, contrary to Lemma 6. ■

The following theorem is just what we need to prove Theorem 1:

THEOREM 8. *Let G be a connected graph which is projective-planar but nonplanar and let $p : \tilde{G} \rightarrow G$ be a 2-fold planar covering of G with transformation $\tau : \tilde{G} \rightarrow \tilde{G}$ of order 2. Then \tilde{G} can be embedded in the sphere S^2 so that τ extends to an involution on S^2 .*

Proof. After unifying some blocks, we can assume that G decomposes into a 2-connected nonplanar subgraph B_0 and several mutually disjoint connected planar subgraphs B_1, \dots, B_n each of which meets B_0 in a cut vertex v_i . By Lemma 7, \tilde{G} can be obtained as the union $\tilde{B}_0 \cup B'_1 \cup B''_1 \cup \dots \cup B'_n \cup B''_n$, where \tilde{B}_0 is a 2-fold planar covering of B_0 and B'_i and B''_i are disjoint copies of B_i ($i \geq 1$).

By Lemma 5, \tilde{B}_0 can be embedded in S^2 so that $\tau|_{\tilde{B}_0}$ extends to an involution $h : S^2 \rightarrow S^2$ since B_0 is 2-connected. By Lemma 6, the two lifts v'_i and v''_i of v_i are incident to distinct faces A and $h(A)$. First embed B'_i in A and next embed B''_i onto $h(B'_i)$ for each i , so that $h|_{\tilde{G}}$ realizes τ . ■

Note that the assumption of G being nonplanar cannot be omitted from Theorem 8. For example, let H_n be the graph obtained as a connected planar graph H with n disjoint triangles added to a face boundary at one vertex each. Then H_n has a 2-fold planar covering \tilde{H}_n which can be obtained as two copies of H joined by n hexagons. Its covering transformation $\tau : \tilde{H}_n \rightarrow \tilde{H}_n$ fixes each hexagon setwise. To realize this action on the n

hexagons, we have to place them as concentric circles on the sphere, so that the rotation around the center through π acts on them. If $n \geq 3$, then we cannot add two H 's to get a planar embedding of \tilde{H} . Since H_n is not 2-connected, this example also shows that G has to be 2-connected in Lemma 5.

Suppose that H_n had a projective-planar embedding with an even dual. Then the n triangles would be essential cycles, but any two essential cycles have to meet each other in the projective plane. Thus, H_n with $n \geq 2$ has no even dual in the projective plane even if $B(H_n) = \tilde{H}_n$ is planar when H is bipartite. That is, such an H_n is a counterexample to Theorem 1 with the nonplanarity of G omitted.

Proof of Theorem 1. The necessity follows from Theorem 4 since the only 2-fold covering space of the projective plane is the sphere.

To show the sufficiency, we suppose that G is not bipartite and that $B(G)$ is planar. Then $B(G)$ is a 2-fold covering of G and has the covering translation $\tau : B(G) \rightarrow B(G)$ of order 2. By Theorem 8, $B(G)$ can be embedded in the sphere S^2 so that τ extends to a free involution $h : S^2 \rightarrow S^2$, equivalent to the antipodal map of S^2 ; otherwise, G would be planar.

The orbit space S^2/h is homeomorphic to the projective plane P^2 and contains a projective planar embedding of G . If h fixed a face setwise, there would be a fixed point within the face by Brouwer's fixed point theorem, contrary to h being free. Thus, h carries each face of $B(G)$ onto another different one and the covering projection $p : S^2 \rightarrow P^2 (= S^2/h)$ sends it onto a face of G homeomorphically. Since $B(G)$ is bipartite, $B(G)$ has only even faces and hence so does G . ■

Proof of Theorem 2. Let $f : G \rightarrow P^2$ be any projective-planar embedding with an even dual and let $p : S^2 \rightarrow P^2$ be the 2-fold covering of the projective plane P^2 . Then $p^{-1}(f(G))$ is a 2-fold covering of G and also is embedded in the sphere S^2 with an even dual. This implies that $p^{-1}(f(G))$ is bipartite and has to be isomorphic to $B(G)$ by Lemma 3.

Negami has shown in [2] that there is a bijection between the equivalence classes of projective-planar embeddings and the isomorphism classes of 2-fold planar coverings of G . Thus, f is equivalent to the one derived from a planar embedding of $B(G)$ in the proof of Theorem 1, which implies that any two even duals are isomorphic to each other via $B(G)$. ■

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