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# A New Generalization of the Friendship Theorem

*Akira Saito*

Department of Mathematics
Nihon University
Sakurajosui 3–25–40
Setagaya-ku, Tokyo 156
JAPAN

A graph $G$ is said to be a friendship graph if for any two vertices of $G$ there is a unique vertex adjacent to both of them. The well-known Friendship Theorem, first proved by Erdős, Rényi and Sós [2], characterizes the friendship graphs.

**Theorem A**(the Friendship Theorem). *A graph $G$ is a friendship graph if and only if $G \simeq K_1 + kK_2$ for some $k \geq 1$.* ∎

The notion of a friendship graph has been generalized in several directions. Recently Heinrich [3] has considered the following generalization of the Friendship Theorem.

**Theorem B**([3]). *Let $G$ be a graph of order at least $k+1$ with the property that for any $k$-subset $S$ of $V(G)$ there is a unique vertex $x \in V(G) - S$ such that $x$ has exactly two neighbors in $S$. (If $k = 2$, then $G$ is a friendship graph.) If $k \geq 3$, then $G$ is a 2-regular graph of order precisely $k+1$.* ∎

In this paper, we consider a further generalization of the property that Heinrich has considered.

For a set $X$ and a nonnegative integer $k$, let $\binom{X}{k} = \{Y : Y \subset X, |Y| = k\}$. For a graph $G$ and its vertex $x$, let $\Gamma_G(x)$ denote the set of vertices adjacent with $x$ in $G$. For $x$, $y \in V(G)$, $\deg_G(x)$ is the degree of $x$ in $G$ and $d_G(x, y)$ is the distance between $x$ and $y$ in $G$. We denote by $\delta(G)$ the minimum degree of $G$. For $X \subset V(G)$ we denote by $G[X]$ the subgraph of $G$ induced by $X$. Notation not defined here can be found in [1]. For a positive integer $m$ and a nonnegative integer $n$ such that $m \geq n$, a graph $G$ is said to have a *property* $P(m, n)$ if $|G| \geq m+1$ and for any subset $S \in \binom{V(G)}{m}$ there exists a unique vertex $x$ in $V(G) - S$ such that $|\Gamma_G(x) \cap S| = n$. The friendship theorem says that a graph

$G$ satisfies $P(2,2)$ if and only if $G \simeq K_1 + kK_2$ for some $k \geq 1$. Theorem B says that $G$ satisfies $P(m,2)$ $(m \geq 3)$ if and only if $|G| = m+1$ and $G$ is 2-regular.

Here we give the characterization of the graphs that satisfy $P(m,n)$ for all the possible values of $m$ and $n$.

**Theorem 1.**

(1) *A graph $G$ satisfies $P(1,1)$ if and only if $G \simeq kK_2$ for some $k$, $k \geq 1$.*

(2) *A graph $G$ satisfies $P(1,0)$ if and only if $G \simeq \overline{kK_2}$ for some $k$, $k \geq 1$.*

(3) *A graph $G$ satisfies $P(2,2)$ if and only if $G \simeq K_1 + kK_2$ for some $k$, $k \geq 1$.*

(4) *A graph $G$ satisfies $P(2,0)$ if and only if $G \simeq \overline{K_1 + kK_2}$ for some $k$, $k \geq 1$.*

(5) *Suppose $(m,n) \notin \{(1,1),(1,0),(2,2),(2,0)\}$. Then $G$ satisfies $P(m,n)$ if and only if $G$ is an $n$-regular graph of order precisely $m+1$.* ∎

From Theorem 1 we have the following immediate corollary.

**Corollary 2.** *If $(m+1)n \equiv 1 \pmod 2$, there are no graphs that satisfy $P(m,n)$.* ∎

Now we prove Theorem 1. It is easy to prove (1). and (3) is exactly the Friendship Theorem.

By the definition of the property $P(m,n)$, we have the following lemma.

**Lemma 3.** *A graph $G$ has a property $P(m,n)$ if and only if its complement $\overline{G}$ has a property $P(m, m-n)$.* ∎

By the above lemma, (2) and (4) follow from (1) and (3), respectively. Also the sufficiency of (5) is trivial. Therefore, we have only to prove the necessity of (5).

By Lemma 3, we may assume $n \leq \frac{1}{2}m$. However, Heinrich's proof of Theorem B works if $1 \leq n \leq \frac{1}{2}m$ and $(m,n) \neq (2,1)$. Therefore, we have only to prove the following two lemmas.

**Lemma 4.** *There are no graphs satisfying $P(2,1)$.*

**Lemma 5.** *Let $m \geq 3$. If a graph $G$ satisfies $P(m,m)$, then $G \simeq K_{m+1}$*

**Proof of Lemma 4.** Assume that there is a graph $G$ which satisfies $P(2,1)$. Then $G$ is not totally disconnected, and $G$ has an edge $e = xy \in E(G)$. Since $G$ satisfies $P(2,1)$, there exists a unique vertex $z \in V(G) - \{x, y\}$ such that $|\Gamma_G(z) \cap \{x, y\}| = 1$. We may assume $yz \in E(G)$ and $xz \notin E(G)$. Then again by the property $P(2,1)$ , there exists a unique vertex $u \in V(G) - \{x, z\}$ such that $|\Gamma_G(u) \cap \{x, z\}| = 1$. Since $|\Gamma_G(y) \cap \{x, z\}| = 2$, $u \neq y$. We may assume $uz \in E(G)$ and $ux \notin E(G)$. Then $|\Gamma_G(x) \cap \{y, z\}| = 1$ and $|\Gamma_G(u) \cap \{y, z\}| \geq 1$. Since $x \neq u$, and $yu \in E(G)$. Then $\Gamma_G(z) \cap \{x, y\} = \Gamma_G(u) \cap \{x, y\} = \{y\}$, contradicting the property $P(2,1)$ of $G$. ∎

**Proof of Lemma 5.** We proceed by induction on $m$. First, we consider the case $m = 3$. It is easy to see that $G$ is connected. We claim that $G$ is a complete graph.

Assume that $G$ is not a complete graph. Since $G$ is connected, $G$ has a pair of two vertices $x$, $y$ such that $d_G(x, y) = 2$. Let $z \in \Gamma_G(x) \cap \Gamma_G(y)$. Since $G$ has property $P(3,3)$, there exists a unique vertex $u \in V(G) - \{x, y, z\}$ suc that $\{x, y, z\} \subset \Gamma_G(u)$, Furthermore, $G$ has a unique vertex $v \in V(G) - \{x, z, u\}$ such that $\{x, z, u\} \subset \Gamma_G(v)$. Since $y \notin \Gamma_G(x)$, $y \neq v$. Then $\{x, y, v\} \subset \Gamma_G(z) \cap \Gamma_G(u)$. This contradicts the property $P(3,3)$ and the claim is proved. It is easy to see that the only complete graph satisfying $P(3,3)$ is $K_4$.

Next, we consider ths case $m \geq 4$.

First, we claim that $\delta(G) \geq m$. Assume $\delta(G) \leq m - 1$ and let $x$ be a vertex of $G$ such that $\deg_G(x) \leq m - 1$. Then there exists $S \in \binom{V(G)}{m}$ such that $\{x\} \cup \Gamma_G(x) \subset S$. However, there is no vertex $v \in V(G) - S$ such that $S \subset \Gamma_G(v)$. This is a contradiction.

For every $x \in V(G)$, let $H_x$ be the subgraph of $G$ induced by $\Gamma_G(x)$, $H_x = G[\Gamma_G(x)]$. By the above claim $|H_x| \geq m$. We claim that $H_x$ satisfies $P(m-1, m-1)$.

Let $S \in \binom{V(H_x)}{m-1}$ and let $S' = S \cup \{x\}$. Since $G$ satisfies $P(m, m)$, there exists a unique vertex $v \in V(G) - S'$ such that $S' \subset \Gamma_G(v)$. Since $x \in S'$, $v \in \Gamma_G(x) = V(H_x)$. Therefore, the claim follows.

By the induction hypothesis, $H_x \simeq K_m$ for all $x \in V(G)$. Then $G \simeq K_{m+1}$ since $G$ is connected. ∎

## References


[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 2nd ed., Wadsworth, Belmont, CA (1986).

[2] P. Erdős, A. Rényi and V. Sós, On a problem of graph theory, *Studia Sci. Math. Hunger.*

**1** (1966) 215–235.

[3] K. Heinrich, Yet another generalization of the friendship theorem, *preprint.*