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# ON THE SUBSCHEMES OF THE JOHNSON SCHEME J(v,d)

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Abstract.

For two association schemes $\chi$ and $\chi'$, defined on the same set, we call $\chi'$ a subscheme of $\chi$ if each relation of $\chi'$ is a union of some relations of $\chi$. In this paper we prove that the Johnson scheme J(v,d) has no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$ . This slightly improves the earlier result of Muzichuk that the conclusion holds if $v\geq 3d+4$ .

## 1.Introduction

Let $\chi=(X,\{R_i\}_{0\leq i\leq d})$ and $\chi'=(X,\{R'_i\}_{0\leq i\leq d'})$ be two association schemes defined on the same set X. We say that $\chi'$ is a subscheme of $\chi$ if each relation $R'_i$ is a union of some relations $R_i$'s.

The purpose of this paper is to prove that the Johnson scheme J(v,d) have no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$ .For a subscheme $\chi'$ of $\chi$, we have a partition $\tau=\{T_0=\{0\},T_1\cdots,T_{d'}\}$ of the index set $\{0,1,\cdots,d\}$ such that

$$R'_i=\bigcup_{\alpha\in T_i} R_\alpha \quad \text{( using the adjacency matrices, } A'_i=\sum_{\alpha\in T_i} A_\alpha \text{ ) .}$$

Clearly $\chi_0=(X,\{R_0,\bigcup_{i>0} R_i\})$ and $\chi$ are subschemes of $\chi$. We call these subschemes trivial subschemes. Clearly if the association scheme $\chi$ is commutative then it's subscheme $\chi'$ is commutative, and if $\chi$ is symmetric then $\chi'$ is symmetric,too.

Throughout the whole paper we shall consider only symmetric association schemes. If we look at the Bose-Mesner algebra $\mathfrak{a}$ and $\mathfrak{a}'$, it immediately holds that $\chi'$ is a subscheme of $\chi$ if and only if $\mathfrak{a}' \subset \mathfrak{a}$ . Then we also have the partition $\pi=\{\pi_0=\{0\},\pi_1,\cdots,\pi_{d'}\}$ of the index set $\{0,1,\cdots,d\}$ such that

$$E'_j=\sum_{\beta\in\pi_j} E_\beta \ .$$

Now we calculate the entries of the first and second eigenmatrices of $\chi'$ . ( For the notation of association scheme, see [2],[4])

$$|X|E'_j=\sum_{0\le i\le d'} q'_j(i)A'_i \ . \qquad \cdots (1,1)$$

$$\text{The LHS of } (1,1) = |X|\sum_{\beta\in\pi_j} E_\beta$$

$$=\sum_{0\le\alpha\le d}\Big(\sum_{\beta\in\pi_j} q_\beta(\alpha)\Big)A_\alpha \ .$$

$$\text{The RHS of } (1,1) = \sum_{0\le i\le d'}\sum_{\alpha\in T_i} q'_j(i)A_\alpha \ .$$

Therefore compairing the both sides, we get $q'_j(i)=\sum_{\beta\in\pi_j} q_\beta(\alpha) \quad (\alpha\in T_i)$.

Dually we get $p'_i(j)=\sum_{\alpha\in T_i} p_\alpha(\beta) \quad (\beta\in\pi_j)$.

With the above notation ,we get the following lemma.

<u>Lemma 1.</u> Let $\chi'$ be a subscheme of $\chi$, and $\tau,\pi$ be the partitions of $\chi'$. The indices $i,j$ are glued in $\tau$ ( namely $R_i$ and $R_j$ are in a same realtion $R'$ of $\chi'$ ) ( dually in $\pi$ ) if and only if for each $0\le k\le d'$

$$\sum_{\beta\in\pi_k} q_\beta(i)=\sum_{\beta\in\pi_k} q_\beta(j) \ \Big(\text{ dually } \sum_{\alpha\in T_k} p_\alpha(i)=\sum_{\alpha\in T_k} p_\alpha(j)\Big) \ .$$

( cf. [1],[4] )

Using this lemma, we get the following corollary.

<u>Corollary.</u> Let $\chi=(X,\{R_i\}_{0\le i\le d})$ be an association scheme and

$\chi'=(X,\{R'_i\}_{0\le i\le d'})$ be a non-trivial subscheme of $\chi$ .
If there exist $0< i_0, j_0 \le d$ such that
$q_j(i_0)>q_j(i)$ ( $0<i\neq i_0$ , $0<j\neq j_0$ ) , then
$R_{i_0}\in\{R'_i\}_{0\le i\le d'}$ .

Proof Since $\chi'$ is non-trivial, there exists a part $\pi$ ($\neq\{0\}$) of the partition $\pi$ which dose not contain $j_0$ . Then

$$\sum_{j\in\pi} q_j(i_0)> \sum_{j\in\pi} q_j(i) \qquad ( 0<i\neq i_0 ) .$$

With Lemma 1 , we have $R_{i_0}\in\{R'_i\}_{0\le i\le d'}$ . ( Q.E.D.)

We now state our main result. Before that, we mention a brief history of this probrem ( according to Muzichuk [4] ) . Let $J(v,d)$ be the Johnson scheme of class d ( see [2],[3],for the details about the Johnson scheme ) . As for the study of subscheme of $J(v,d)$ or on the enumeration of subschemes of the Johnson scheme, first L.A.Kaluznin M.H.Klin proved that there exists a function $f(d)$ such that Jonhson scheme $J(v,d)$ does not have non-trivial subscheme for $v\ge f(d)$. M.E.Muzichuk proved that the same conclusion is true if $v\ge 3d+4$ . The purpose of this paper is to prove the following Theorem.

Theorem A The Johnson scheme $J(v,d)$ has no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$ .

This slightly improves the result of Muzichuk [4], because our condition becomes $v\ge 3d$ for $d\ge 6$ .

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on that of Muzichuk [4] . The auther thanks M.E.Muzichuk for making the preprint of [4] available to the auther before publication.

## 2. Proof of the Theorem A

For the Johnson scheme $J(v,d)$, we have

$$p_i(j)=\sum_{0\le\nu\le j}(-1)^{\nu}\binom{j}{\nu}\binom{d-j}{i-\nu}\binom{v-d-j}{i-\nu}\ ,\quad k_i=\binom{d}{i}\binom{v-d}{i}\ .\qquad (\text{ see } [2],[3],[4])$$

In order to prove Theorem A , we need the following lemma.

<u>Lemma 2.</u> For $J(v,d)$ with $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$ , we have

$$q_j(1)>|q_j(i)| \qquad (\ 1<i\le d, 0<j<d\ )\ .$$

First we show that Thorem A is easily obtained for Lemma 2 .

<u>Proof of Theorem A from Lemma 2.</u>

Let $\chi'=(X,\{R_i\}_{0\le i\le d'})$ be a non-trivial subscheme of $J(v,d)$ . Then by Corollary, we have $R_1\in\{R'_i\}_{0\le i\le d'}$ . i.e. we have $\{1\}\in\tau$ . If the indices $i,j$ are glued in $\pi$, with Lemma1 , we have

$p_1(i)=p_1(j)$ . Since

$p_1(i)=i^2-(v+1)i+d(v-d)$ , $p_1(j)=j^2-(v+1)j+d(v-d)$ , we have

$i=j$ . ( Q.E.D.)

Now we prove Lemma 2 with the inductive formula of the Johnson scheme:

$$p_i^{v,d}(j)=\begin{cases} p_i^{v-2,d-1}(j-1)-p_{i-1}^{v-2,d-1}(j-1) & (\ 0<i<d\ ) \\ -p_{d-1}^{v-2,d-1}(j-1) & (\ i=d\ )\ . \end{cases}$$

From now on, if necessary, we let $p_i^{v,d}(j)$ and $k_i(v,d)$ be the entries of the first eigenmatrix P and the valencies of $J(v,d)$ respectively. Concerning Lemma 2 , we check at the special value of $j=d-1$ by the following Proposition.

<u>Proposition.</u> For $J(v,d)$ with $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$ , we have

$$q_{d-1}(1)>|q_{d-1}(i)| \qquad (\ 1<i\leq d\ )\ .$$

<u>Proof</u> Since $q_j(i)/m_j=p_i(j)/k_i$, we only have to show that

$$|p_i(d-1)|/k_i<p_1(d-1)/k_1 \qquad (\ 1<i\leq d\ )\ .$$

With the direct calculation, we have

$$p_i(d-1)/k_i=(-1)^{i-1}\ \frac{(v-2d+2)i-d}{d\binom{v-d}{i}}\ .\ \text{Then}$$

$$d(v-d-i)\binom{v-d}{i}(|p_i(d-1)|/k_i-|p_{i+1}(d-1)|/k_{i+1})$$
$$=-2(v-2d+2)i^2+(v^2-3dv+2d^2+4d-4)i-(d+1)v+d^2+3d-2\ .\ \cdots(*)$$

This reaches it's minimum when i=1 or d-1, i.e.

$$(*)\geq \min\{v^2-(4d+3)v+3d^2+11d-10,(d-1)(v-2d+1)\{v-(3d+\frac{2}{d-1})\}\}$$

Since $v^2-(4d+3)v+3d^2+11d-10>0$ is equal to $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$ ,

we have

1)If $v\geq 3d+\frac{2}{d-1}$ (i.e.$v\geq 3d+1$) ,then we have $(*)\geq 0$ . Then

$$p_1(d-1)/k_1>|p_2(d-1)|/k_2\geq \cdots\cdots \geq|p_d(d-1)|/k_d\ .$$

2)In the case of v=3d ( $d\geq 6$ ), we have

$$|p_i(d-1)|/k_i\leq \max\{\ p_1(d-1)/k_1\ ,\ |p_d(d-1)|/k_d\ \}$$
$$=\max\{\ 1/d^2,\ (d+1)/\binom{2d}{d}\ \}\ .\ \text{Since}$$

$$\frac{d+1}{\binom{2d}{d}}d^2=\frac{d^2(d+1)!}{2d(2d-1)\cdots(d+1)}$$
$$<\frac{5\times4\times3}{(2d-1)(2d-2)}\leq\frac{6}{11}\ ,\ \text{we have}$$

$$|p_i(d-1)|/k_i<p_1(d-1)/k_1 \qquad (\ i>1\ )\ . \qquad (\ \text{Q.E.D.})$$

<u>Proof of Lemma 2</u>

By Proposition, we only have to show that

$$|p_i(j)|/k_i<p_1(j)/k_1 \qquad (\ 1<i\leq d,\ 0<j<d-1\ )\ .$$

1)d=3

With the direct calculation, we have

$$p_1(1)/k_1=\frac{2v-9}{3(v-3)}\ ,\ p_2(1)/k_2=\frac{v-9}{3(v-3)}\ ,\ p_3(1)/k_3=\frac{-3}{v-3}\ .$$

Since $v>10$ , we have

$$|p_i(1)|/k_i < |p_1(1)|/k_1 \qquad (\ i=2,3\ )\ .$$

2) $d>3$

We consider the following two cases, Case 2.1; $1<i<d$ , $0<j<d-1$ and Case 2.2; $i=d$ , $0<j<d-1$ ,separately.

<u>Case 2.1</u> $1<i<d$ , $0<j<d-1$

With the inductive formula, we have

$$p_i^{v,d}(j) = p_i^{v-2,d-1}(j-1) - p_{i-1}^{v-2,d-1}(j-1)\ .$$

Since $v-2>2(d-1)+\frac{3}{2}+\sqrt{\{(d-1)-\frac{5}{2}\}^2+6}$ , we have

$$|p_i^{v-2,d-1}(j-1)| \le p_1^{v-2,d-1}(j-1)\frac{k_i(v-2,d-1)}{k_1(v-2,d-1)}\ ,$$

$$|p_{i-1}^{v-2,d-1}(j-1)| \le p_1^{v-2,d-1}(j-1)\frac{k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)}\ ,\ \text{and}$$

$$|p_i^{v,d}(j)|/k_i(v,d) \le \frac{k_i(v-2,d-1)+k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)k_i(v,d)}p_1^{v-2,d-1}(j-1)\ .$$

We show that

$$\frac{k_i(v-2,d-1)+k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)k_i(v,d)}p_1^{v-2,d-1}(j-1)k_1(v,d)/p_1^{v,d}(j)<1 \quad (*)$$

With the direct calculation, we have

$$\text{The LHS of } (*) = \frac{2i^2-vi+d(v-d)}{(d-1)(v-d-1)}\left(1+\frac{1}{j^2-(v+1)j+d(v-d)}\right)\ .$$

This reaches its maxmum when $i=2$ , $j=d-2$ . Therefore

$$\text{The LHS of}(*) \le \left(1-\frac{v-7}{(d-1)(v-d-1)}\right)\left(1+\frac{1}{2v-5d+6}\right)\ .$$

Since $\frac{1}{2v-5d+6} \le \frac{1}{d+6} < \frac{v-7}{(d-1)(v-d-1)}$ , we have the LHS of $(*)<1$.

<u>Case 2.2</u> $i=d$ , $0<j<d-1$

Since $p_d^{v,d}(j) = -p_{d-1}^{v-2,d-1}(j-1)$ , we have

$$|p_d^{v,d}(j)|/k_d(v,d) \le \frac{k_{d-1}(v-2,d-1)}{k_1(v-2,d-1)k_d(v,d)}p_1^{v-2,d-1}(j-1)\ .$$

Now we get

$$\frac{k_{d-1}(v-2,d-1)}{k_1(v-2,d-1)k_d(v,d)}p_1^{v-2,d-1}(j-1)k_1(v,d)/p_1^{v,d}(j)$$

$$= \frac{d^2}{(d-1)(v-d-1)}\left(1+\frac{1}{j^2-(v+1)j+d(v-d)}\right)$$

$$\leq \left(1-\frac{v-7}{(d-1)(v-d-1)}\right)\left(1+\frac{1}{2v-5d+6}\right) < 1 .$$

This completes the proof of Lemma 2, hence of Theorem A.

Added in proof

Recently we improve the Theorem A in this paper as follows;

The Johnson scheme $J(v,d)$ has no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{7}{2})^2+6}$ .

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