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On the Classification of Locally Hamming Distance-Regular Graphs

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Abstract. A distance-regular graph is *locally Hamming* if it is locally isomorphic to a Hamming scheme $H(r,2)$. This paper rediscovers the connection among locally Hamming distance-regular graphs, designs, and multiply transitive permutation groups, through which we classify some of locally Hamming distance-transitive graphs.

§1. Introduction.

By a graph we shall mean a finite undirected graph with no loops and no multiple edges. For a graph G, $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set of G. For a vertex v of a graph G, $N(v)$ denotes the set of adjacent vertices with v. By $\mathbb{F}_2$ we denote the two-element field, and by $H(r)$ we denote the r-dimensional Hamming scheme for $r \geq 1$; that is, $H(r)$ is such a graph that its vertex set is the vector space $\mathbb{F}_2^r$ and $u, v \in \mathbb{F}_2^r$ are adjacent if and only if the Hamming distance $d(u,v) = 1$; i.e., $\#\{i \mid u_i \neq v_i\} = 1$ where $u = (u_1, \ldots, u_r)$ and $v = (v_1, \ldots, v_r)$. The graph obtained from $H(r)$ by identifying antipodal points is called a *folded Hamming sheme*(or a *folded Hamming cube* in [**3**, p.140]). The graph $H(3)$ is called a *cube*, and the induced subgraph obtained by removing one vertex together with the three incident edges from a cube is called a *tulip.* A tulip contains exactly three vertices with degree two, and these vertices are called *petals.* The unique vertex which is not adjacent to any petals is called the *root* of the tulip. A *locally Hamming graph G* is a connected graph such that

(1) G has no triangle,
(2) for any $u, v \in V(G)$ satisfying $d(u,v) = 2$, there exist exactly two vertices adjacent to both u and v,
(3) for any subgraph T of G which is isomorphic to a tulip with petals p, q, r, there exists a vertex $x \in V(G)$ adjacent to all p, q, and r. (The uniqueness of x follows from the condition (2).)

Of course $H(r)$ is an example of locally Hamming graphs, and it was proved that a distance-regular graph with parameters $a_1 = 0$, $a_2 = 0$, $c_2 = 2$, and $c_3 = 3$ is a locally Hamming graph[**2**][**3**, Lemma 4.3.5]. For a generalization to $H(r,q)$ for $q \geq 3$, see [**10**]. (A locally Hamming graph is exactly the same thing with a *rectagraph such that any 3-claw determines a unique 3-cube* in Brouwer's terminology[**3**, p.153].)

Our objective is to classify all locally Hamming distance-regular graphs(LHDRG). A number of authors have contributed to this aim[**2**][**3**][**5**][**10**]. This paper provides an approach to this goal using a universal covering of a LHDRG. Main result is as follows.

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MAIN RESULT. *Let G be an r-regular distance-regular graph with parameters $a_1 = a_2 = 0$ and $c_2 = 2$, $c_3 = 3$, other than $H(r)$. Take the minimum number t such that either $a_t \neq 0$ or $c_t \neq t$ occurs. Put $d = 2t+1$ if $c_t = t$, and put $d = 2t$ otherwise. Then, there exists a t-(r, d, λ) design with $\lambda \leq (r-t)/(d-t)$. If G is distance-transitive, $\mathrm{Aut}(G)$ contains a $\lfloor (d-1)/2 \rfloor$-homogeneous group of degree r acting on the block set of the t-(r, d, λ) design.*

COROLLARY. (See Theorem 3).
Any distance-transitive graph with $a_1 = a_2 = a_3 = 0$ and $c_i = i$ for $i = 1, 2, 3, 4$ is a Hamming scheme or a folded Hamming scheme.

For the case $t = 2, 3$, some nontrivial examples are listed.

§2. The universal covering.

Let G, H be graphs. A mapping $f : V(H) \to V(G)$ is said to be a *covering* if it is surjective and for any $u \in V(H)$, $f_{N(u)}$ is a bijection $N(u) \to N(f(u))$. For a vertex u of G, the set $f^{-1}(u)$ is called the *fiber* on u.

In the previous paper[**9**] we proved the following propositions.

PROPOSITION 1. *Let G be a locally Hamming graph with valency r. Then, there exists a covering $f : H(r) \to G$. Let H be a locally Hamming graph and let $h : H \to G$ be a covering. For any vertex $u \in V(G)$ and for any vertices $x \in f^{-1}(u)$, $y \in h^{-1}(u)$, there exists a unique covering $g : H(r) \to H$ such that $g : x \mapsto y$ and $f = hg$ hold.*

The above covering f is called a *universal covering*. We define the fundamental group $\pi(G, f)$ of the pair G and $f : H(r) \to G$ as

$$\pi(G, f) = \{\gamma \in \mathrm{Aut}(H(r)) | f\gamma = f\}.$$

This definition depends on the choice of f, but unique upto conjugacy in $\mathrm{Aut}(H(r))$. (Note that this definition is not equivalent to the fundamental group of G as a one-dimensional topological object.)

Let Γ be a subgroup of $\mathrm{Aut}(H(r))$. We define the *discreteness* d_Γ of Γ by

$$d_\Gamma = \min\{d(u, \gamma u) \mid \gamma \in \Gamma, \gamma \neq \mathrm{id}, u \in V(H)\}$$

where d denotes the Hamming distance. For the trivial group $\{\mathrm{id}\}$, its discreteness is defined to be ∞. Let Γ be a subgroup of $\mathrm{Aut}(H(r))$ with $d_\Gamma \geq 5$. Then we define the *quotient graph* $H(r)/\Gamma$ as follows. The vertex set is the set of coset of $V(H(r))$ by Γ; i.e., the set $\{\Gamma u | u \in H(r)\}$, where $\Gamma u = \{\gamma u | \gamma \in \Gamma\}$. The two vertices Γu, Γv are adjacent if and only if there exists a vertex $w \in \Gamma u$ such that w is adjacent with v. The obtained graph was proved to be locally Hamming. The canonical mapping $f : H(r) \to H(r)/\Gamma$ defined by $f : u \mapsto \Gamma u$ is a universal covering. (For a detailed proof, see [**9**].)

PROPOSITION 2. *Let G be a locally Hamming graph with valency r, and let $f : H(r) \to G$ be a universal covering. Then, $d_{\pi(G,f)} \geq 5$ and $G \cong H(r)/\pi(G,f)$ hold.*

Thus, the classification problem of LHDRG with valency r is converted to the problem to determine for which subgroup Γ of $\mathrm{Aut}(H(r))$, $H(r)/\Gamma$ is a distance-regular graph.

REMARK 1: Proposition 1 was proved by Brouwer[**2**][**3**, p.153], without emphasis on the universality. Almost equivalent propositions to Propositions 1 and 2 will also be found in the book by Brouwer et. al.[**3**]. What is new in this paper is that we regard a covering f not only as a partition but as a coset by a *group* Γ.

NOTE: In this paper, the letter d always denotes not the diameter but the discreteness.

§3. Designs.

Recall that a t-(v,k,λ) *design* is a family $\mathcal{B}$ of k-element subset of a v-element set X such that for any k-element subset K of X, the number of $B \in \mathcal{B}$ containing K is λ. In Lemmas 3 and 3*, we shall prove a LHDRG produces a design with an additional property.

To begin with, we classify LHDRG using the discreteness of its fundamental group. Let LHDRG_d^r denote the set of LHDRGs with valency r such that the discreteness of its fundamental group is equal to d. Later it will be proved that a distance-regular graph belongs to LHDRG_d^r for some $d \geq 7$ if and only if its parameters satisfy $a_1 = 0$, $a_2 = 0$, $c_2 = 2$, and $c_3 = 3$. Thus, the classification of LHDRG_d^r implies the classification of distance-regular graphs with such parameters. The reason why the cases $d = 5, 6$ are also considered is that some interesting LHDRG with $d = 5, 6$ obtained from Golay codes exist, as shown in Section 5.

The next lemma is useful to shorten some proofs.

LEMMA 1. *Let $H(r)$ be an r-dimensional Hamming scheme and let Γ be a subgroup of $\mathrm{Aut}(H(r))$ with $d_\Gamma \geq 5$. Let $\Gamma x_1, \Gamma x_2, \ldots, \Gamma x_t$ be a walk in H/Γ; i.e., Γx_j is adjacent to Γx_{j+1} for $j = 1, 2, \ldots, t-1$. Then, there exist $x'_2, \ldots, x'_t$ such that $\Gamma x'_j = \Gamma x_j$ for $j = 2, \ldots, t$ and that $x_1, x'_2, \ldots, x'_t$ is a walk in H. Consequently,*

$$d_{H(r)/\Gamma}(\Gamma x, \Gamma y) = d_{H(r)}(x, \Gamma y)$$

holds, where RHS denotes the minimum of $d_{H(r)}(x, \gamma y)$ for all $\gamma \in \Gamma$.

PROOF: Since Γx_1 is adjacent to Γx_2, there exists an $x'_2 \in \Gamma x_2$ adjacent to x_1. Thus, the existence of x'_j is inductively proved. Then there exists a $\gamma \in \Gamma$ such that $x'_t = \gamma y$. Thus, $d_{H(r)/\Gamma}(\Gamma x, \Gamma y) \geq d_{H(r)}(x, \gamma y)$. Converse inequality follows because a path connecting x with γy is mapped to a path connecting Γx to Γy by f. ∎

The behavior of parameters of LHDRG_d^r slightly changes according to the parity of d. For a graph G and its vertex v, $N_t(v)$ denotes the induced subgraph of G by all vertices at distance at most t from v.

LEMMA 2. *Suppose that G belongs to $LHDRG^r_{2t+1}$, let Γ be its fundamental group, and identify G with $H(r)/\Gamma$. Take an arbitrary vertex $v \in V(G)$ and take a vertex u in the fiber $f^{-1}(v)$. Then, f embeds $N_t(u)$ into $N_t(v)$, and $f : V(N_t(u)) \to V(N_t(v))$ is surjective.*

PROOF: Since f is a covering, it is sufficient to prove that f induces a bijection $V(N_t(u)) \to V(N_t(v))$.

Take $\Gamma x \in V(N_t(v))$. Since $d(\Gamma u, \Gamma x) \le t$, Lemma 1 shows that $d(u, \Gamma x) \le t$, and thus there exists a vertex in Γx at distance t from u. This shows surjectivity. For the injectivity, take $x, y \in N_t(u)$ such that $f(x) = f(y)$; i.e., $\Gamma x = \Gamma y$. Then $x = \gamma y$ for some $\gamma \in \Gamma$, and from $d_\Gamma = 2t+1 > t+t \ge d(x,u) + d(u,y) \ge d(x,y) = d(\gamma y, y)$, $\gamma = \mathrm{id}$ follows. Thus $x = y$ holds, and injectivity is proved. ∎

For an r-element set X, we can naturally identify the vertex set of $H(r)$ with the power set $2^X = \mathcal{P}(X)$. The symbol $\emptyset$ denotes the empty set, which is identified with a vertex of $H(r)$. We denote the family of d-element subset of X by $\binom{X}{d}$. We denote by a_i, b_i, c_i the usual parameters of a distance-regular graph G(precise definition will be found in [**1**] or [**3**]).

LEMMA 3. *Let $G = H(r)/\Gamma$ be a graph in $LHDRG^r_{2t+1}$. Then its parameters satisfy equalities $a_i = 0$, $c_i = i$ for $i = 1, 2, \ldots, t-1$ and $a_t = (t+1)\lambda$, $c_t = t$ for a some positive integer λ. Identify the vertex set of $H(r)$ with the power set $\mathcal{P}(X)$ for $X = \{1, 2, \ldots, r\}$. Then, the set $\Gamma\emptyset \cap \binom{X}{2t+1}$ consists of the block set B of a $t-(r, 2t+1, \lambda)$ design, with an additional property that for any $B, B' \in \mathcal{B}$, either $B = B'$ or $\#(B \cap B') \le t$ holds.*

PROOF: Identify $H(r)$ with $\mathcal{P}(X)$. The parameters a_i, b_i, c_i for $i \le t-1$ are determined immediately from Lemma 2 and the fact that f is a covering. We shall fix the vertex $\Gamma\emptyset$ as the one end to calculate the parameters. To determine c_t, take a vertex of G at distance t from $\Gamma\emptyset$. By Lemma 1, this vertex can be written as Γv with $d_{H(r)}(\emptyset, v) = t$; i.e., with $v \in \binom{X}{t}$. Let u be a vertex of $H(r)$ adjacent with v. Then, either $d_{H(r)}(\emptyset, u) = t-1$ or $t+1$ holds. The number of such $u \in N(v)$ that $d_G(\Gamma\emptyset, \Gamma u) = t, t+1, t-1$, respectively, is by definition the number a_t, b_t, c_t. First we determine c_t. Suppose that $d_G(\Gamma\emptyset, \Gamma u) = t-1$ holds. Then, by Lemma 1, $d_{H(r)}(\emptyset, \gamma u) = t-1$ for a $\gamma \in \Gamma$. Consequently, $d_{H(r)}(u, \gamma u) \le d_{H(r)}(u, \emptyset) + d_{H(r)}(\emptyset, \gamma u) \le (t+1) + (t-1) < d_\Gamma$ holds, and γ must be the identity. This implies $d_{H(r)}(\emptyset, u) = t-1$, and conversely, this equality implies $d_G(\Gamma\emptyset, \Gamma u) = t-1$. Thus, c_t coincides with the corresponding parameter of $H(r)$; in other words, $c_t = t$ holds. Next we determine a_t. (In this process, a design arises.) Take a $u \in N(v)$ such that $d_G(\Gamma\emptyset, \Gamma u) = t$. Then, by Lemma 1, $d_{H(r)}(\gamma\emptyset, u) = t$ for a $\gamma \in \Gamma$. If $\gamma = \mathrm{id}$, then $d_{H(r)}(\emptyset, u) = d_{H(r)}(\gamma\emptyset, u)$ is equal to t, and this is a contradiction because this value must be $t-1$ or $t+1$. Since we have $d_{H(r)}(\gamma\emptyset, \emptyset) \le d_{H(r)}(\gamma\emptyset, u) + d_{H(r)}(u, \emptyset) \le t + (t+1) = d_\Gamma$ for $\gamma \ne \mathrm{id}$, all the equalities hold in the above inequality. That is, $d_{H(r)}(\gamma\emptyset, \emptyset) = 2t+1$, $d_{H(r)}(u, \emptyset) = t+1$, and $d_{H(r)}(\gamma\emptyset, u) = t$. By considering u as a $(t+1)$-element set, the latter equality is equivalent to that u is contained in $\gamma\emptyset$. Conversely, if $u \in N(v)$ with $\#(u) = t+1$ is contained in $\gamma\emptyset$ with $\#(\gamma\emptyset) = 2t+1$, then $t \leq d_G(\Gamma\emptyset, \Gamma u) \leq d_{H(r)}(\gamma\emptyset, u) = t$ holds. Thus, a_t is the number of $u \in N(v) \cap \binom{\gamma\emptyset}{t+1}$ for some $\gamma \in \Gamma$ with $\gamma\emptyset \in \binom{X}{2t+1}$.

Suppose that such u is contained in two different $\gamma\emptyset$, $\gamma'\emptyset$ at distance $2t+1$ from $\emptyset$. Since $u \subset \gamma\emptyset \cap \gamma'\emptyset$, $\#(\gamma\emptyset \cap \gamma'\emptyset) \geq \#(u) = t+1$ holds. Then, $2t+1 = d_\Gamma \leq d_{H(r)}(\gamma\emptyset, \gamma'\emptyset) = \#(\gamma\emptyset) + \#(\gamma'\emptyset) - 2\#(\gamma\emptyset \cap \gamma'\emptyset) \leq 2t$, a contradiction. Thus, any such u is contained in at most one $\gamma\emptyset$. Let λ_v be the number of $\gamma\emptyset \in \binom{X}{2t+1}$ containing v. Then, above argument asserts that the number of $u \in N(v) \cap \binom{\gamma\emptyset}{t+1}$ for some $\gamma\emptyset \in \binom{X}{2t+1}$ is exactly $\lambda_v(t+1)$, because the number of such u contained a fixed $\gamma\emptyset$ is $d_{H(r)}(v, \gamma\emptyset) = t+1$. Since G is distance-regular, the value $a_t = \lambda_v(t+1)$ does not depend on the choice of v, and this implies that the set $\{\gamma\emptyset | \#(\gamma\emptyset) = 2t+1, \gamma \in \Gamma\}$ consists of the block set $\mathcal{B}$ of t-$(r, 2t+1, \lambda)$ design with an additional property that for any two different $B, B' \in \mathcal{B}$, $\#(B \cap B') \leq t$ holds. (The last inequality follows from $d_{H(r)}(B, B') \geq 2t+1$ for $B \neq B'$.) ∎

In the case of even discreteness d, the following lemmas hold.

LEMMA 2*. *Suppose that G belongs to $LHDRG^r_{2t}$, let Γ be its fundamental group, and identify G with $H(r)/\Gamma$. Take an arbitrary vertex $v \in V(G)$ and take a vertex u in the fiber $f^{-1}(v)$. Then, $N_{t-1}(v) \cong N_{t-1}(u)$ holds.*

PROOF: This lemma can be proved in exactly the same way as in the proof of Lemma 2, by substituting N_{t-1} for N_t and putting $d_\Gamma = 2t$. ∎

LEMMA 3*. *Let $G = H(r)/\Gamma$ be a graph in $LHDRG^r_{2t}$. Then its parameters satisfy equalities $a_i = 0$, $c_i = i$ for $i = 1, 2, \ldots, t-1$ and $c_t = t(\lambda+1)$ for some positive integer λ. Identify the vertex set of $H(r)$ with the power set $\mathcal{P}(X)$. Then, the set $\Gamma\emptyset \cap \binom{X}{2t}$ consists of the block set $\mathcal{B}$ of a t-$(r, 2t, \lambda)$ design, with an additional property that for any $B, B' \in \mathcal{B}$, either $B = B'$ or $B \cap B' \leq t$ hold.*

PROOF: From Lemma 2*, it is obvious that for $i = 1, 2, \ldots, t-2$, the parameters coincide with the ones of $H(r)$. For a_{t-1}, b_{t-1}, c_{t-1}, take a vertex v of $H(r)$ at distance $t-1$ from $\emptyset$, and let u be one of its adjacent vertex. Then, we see that the number of such u that $d_{H(r)}(u, \Gamma\emptyset) = t-2, t-1, t$, respectively, is c_{t-1}, a_{t-1}, b_{t-1}. The nearest element in $\Gamma\emptyset$ from v is, however, only $\emptyset$, and the other elements in $\Gamma\emptyset$ are at least at distance $t+1$ from v. This implies that a_{t-1}, b_{t-1}, c_{t-1} also coincide with $H(r)$. Take a vertex v of $H(r)$ at distance t from $\emptyset$, and let u be one of its adjacent vertices. Then, the number of such u that $d_{H(r)}(u, \Gamma\emptyset) = t-1$ corresponds to c_t. If $d(u, \Gamma\emptyset) = t-1$ then either $d(\emptyset, u) = t-1$ or $d(u, \gamma\emptyset) = t-1$ holds for some $\gamma \in \Gamma$ with $\gamma\emptyset \in \binom{X}{2t}$. In the latter case, $\#(u) = t+1$ and $u \subset \gamma\emptyset$ holds. Any such u provides an edge of c-type. Such u is easily proved to be contained at most one $\gamma\emptyset \in \binom{X}{2t}$, otherwise $d(\gamma\emptyset, \gamma'\emptyset) < d_\Gamma$ holds for another $\gamma'\emptyset \in \binom{X}{2t}$. Let λ_v be the number of $\gamma\emptyset \in \binom{X}{2t}$ containing v. Then the c-type edges incident with v are exactly ones in the spans between v and such $\gamma\emptyset$ or $\emptyset$. Thus, $c_t = t(\lambda+1)$ for a number λ

independent of the choice of v, since G is assumed to be distance-regular. Clearly $\Gamma\emptyset \cap \binom{X}{2t}$ consists of the block set of a t-$(r, 2t, \lambda)$ design. ■

Thus we have a slogan.

SLOGAN 1. *If all t-(r, d, λ) designs with $d = 2t$ or $d = 2t + 1$ with an additional property that the intersection of any two different blocks B, B' is of size no more than t are classified, then LHDRG will be classified.*

Such nontrivial designs do exist. Some examples are shown in Section 5.

We will use the next criterion in the next section.

CRITERION 1. *Let $\mathcal{B}$ be the block set of a design as in Slogan 1. Then,*

$$\lambda \leq (r - t)/(d - t)$$

holds.

PROOF: Let X be the v-element set. As shown in Lemmas 3 and 3*, every $(t+1)$-element subset of X is contained in at most one block. Since one block contains $\binom{d}{t+1}$ of $(t+1)$-element subsets, the inequality $\binom{r}{t+1} \geq b\binom{d}{t+1}$ follows, where b denotes the cardinality of the block set $\mathcal{B}$. Combining with a well-known identity $\lambda\binom{r}{t} = b\binom{d}{t}$, we have the desired inequality. ■

The next lemma was proved in [**3**, p.153] in a different terminology.

LEMMA 4. *A distance-regular graph G belongs to $LHDRG^r_d$ for some $d \geq 7$ if and only if the parameters of G satisfy $a_1 = 0$, $a_2 = 0$, $c_2 = 2$, and $c_3 = 3$.*

PROOF: The necessity immediately follows from Lemmas 3 and 3*. To prove the sufficiency, let G be a distance-regular graph with the above parameters. It is sufficient to prove that G satisfies the three condition of the definition of locally Hamming, since $d_\Gamma \geq 7$ follows from Lemmas 3 and 3*. The condition (1) follows from $a_1 = 0$. The condition (2) follows from $c_2 = 0$. To prove (3), take a tulip T in G, let p, q, r be its petals, and let v be its root. From the conditions (2) and (1), we have a vertex x in G which is adjacent with both p and q. It is easily checked that x is at distance 3 from v. Since $c_3 = 3$, there exists a vertex y different from both p and q such that a path x, y, z, v of length 3 exists. Then, from the condition (2), there exists a vertex $w \neq y$ such that x, w, z, v is a path. Since $c_3 = 3$, w must coincide with p or q. We may assume that $w = p$. Then, the condition (2) between v and p shows that z is in T and adjacent to p. From the condition (2) between z and x, z must coincide with the vertex in T adjacent with p but nonadjacent with q. Then, $d(q, z) = 3$ follows from the condition $a_1 = a_2 = 0$. Now p, r, v, y are adjacent with z and at distance 2 from q. Since $c_3 = 3$, y must coincide with one of p, r, v, but $p \neq y$ and $d(v, x) = 3$ holds, $y = r$ follows; that is, x is adjacent with p, q, r. ■

Lemmas 3 and 3* show how d_Γ determines the parameters a_i, c_i. Conversely, the parameters a_i, c_i of course determine the d_Γ.

FORMULA 1. *Let G be an r-regular distance-regular graph with parameters $a_1 = a_2 = 0$ and $c_2 = 2$, $c_3 = 3$, other than $H(r)$. Then, the t in Lemmas 3 and 3* is the minimum number k such that either $a_k \neq 0$ or $c_k \neq k$ occurs, and we have*

$$d = \begin{cases} 2t+1 & \text{if } c_t = t \\ 2t & \text{otherwise.} \end{cases}$$

PROOF: By Lemma 4, G is a LHDRG. Then, this formula follows from 3 and 3*. ∎

§4. Multiply transitive groups.

Although the complete classification of designs is not accomplished yet, a similar great problem was settled as a result of the classification of finite simple groups; that is, the classification of multiply transitive groups. A permutation group of degree r is said to be *k-transitive* if the induced action on the set of the ordered k-tuples of the element of X is transitive, and said to be *k-homogeneous* if the induced action on the set of the unordered k-tuples (i.e., on $\binom{X}{k}$) is transitive. Even 2-transitive permutation groups were classified[**4**][**7**]. In this paper we shall use the following two group-theoretical theorems.

THEOREM 1. *All k-transitive permutation groups of degree r with $k \geq 6$ are $\mathcal{A}_r$ and $\mathcal{S}_r$. All k-transitive permutation groups with $k = 5, 4$ except $\mathcal{A}_r$ and $\mathcal{S}_r$ are four Mathieu groups M_{24}, M_{12}, M_{23},and M_{11}. The subscript denotes the degree of the permutation group, and the former two are 5-transitive and the latter two are 4-transitive.*

THEOREM 2. *A k-homogeneous group is $(k-1)$-transitive. For $k \geq 5$, k-homogeneous group is k-transitive. There exist only five 4-homogeneous but not 4-transitive groups of degree more than 5; viz. $PSL_2(5)$, $PGL_2(5)$, $PGL_2(8)$, $P\Gamma L_2(8)$, and $P\Gamma L_2(32)$ with degree 6, 6, 9, 9, 33 respectively.*

Theorem 1 is a consequence of the classification of finite simple group. Theorem 2 is proved in [**6**]. To connect the automorphism group with the fundamental group of G, following lemma is crucial.

LEMMA 5. *Let $G = H(r)/\Gamma$ be a locally Hamming graph. Then,*

$$\mathrm{Aut}(G) \cong N(\Gamma)/\Gamma$$

holds, where $N(\Gamma)$ denotes the normalizer of $\Gamma \in \mathrm{Aut}(H(r))$.

PROOF: Let $f : H(r) \to G = H(r)/\Gamma$ be the canonical covering. For an automorphism $\delta : H(r) \to H(r)$ with $\delta\Gamma\delta^{-1} = \Gamma$, we define $\delta_{/\Gamma} : H(r)/\Gamma \to H(r)/\Gamma$ by $\Gamma x \mapsto \Gamma\delta x$. It is easy to prove that this mapping is a well-defined automorphism, by using $\Gamma\delta x = \delta\Gamma x$. The map $N(\Gamma) \to \mathrm{Aut}(H(r)/\Gamma)$ defined by $\delta \mapsto \delta_{/\Gamma}$ is obviously a homomorphism of groups. This map is proved to be surjective as follows. Take any

$\alpha \in \mathrm{Aut}(H(t)/\Gamma)$. Then, the universality of f asserts that there exists an automorphism $\delta : H(r) \to H(r)$ such that $\alpha f = f\delta$ holds. This implies $\alpha\Gamma x = \Gamma\delta x$ for any x. Thus, if δ is proved to be contained in $N(\Gamma)$, then clearly $\alpha = \delta_{/\Gamma}$ holds and the surjectivity follows, but for any $\gamma \in \Gamma$, we have $f\delta\gamma\delta^{-1} = \alpha f\gamma\delta^{-1} = \alpha f\delta^{-1} = f\delta\delta^{-1} = f$ by definition of Γ, so $\delta\gamma\delta^{-1} \in \Gamma$ holds; in other words, $\delta \in N(\Gamma)$ holds. Thus we have a surjective homomorphism $N(\Gamma) \to \mathrm{Aut}(H(r)/\Gamma)$. It remains to prove that the kernel of this homomorphism is Γ. Take any $\delta \in N(\Gamma)$ such that $\delta_{/\Gamma} = \mathrm{id}$. This implies $\Gamma u = \Gamma\delta u$ for all u; in other words, $fu = f\delta u$ for any u, and $\delta \in \Gamma$ follows from the definition of Γ. Conversely, for any $\gamma \in \Gamma$, $\gamma_{/\Gamma}$ maps any vertex Γx to $\Gamma\gamma x = \Gamma x$, and Γ is proved to be the kernel. ■

LEMMA 6. $\mathrm{Aut}(H(r)) \cong \mathcal{S}_r.\mathrm{Wr}.\mathbb{F}_2$ *holds.*

PROOF: Recall that the wreath product $\mathcal{S}_r.\mathrm{Wr}.\mathbb{F}_2$ is the set

$$\{(\sigma, \mathbf{d}) | \sigma \in \mathcal{S}_r, \mathbf{d} \in \mathbb{F}_2^r\}$$

with multiplication

$$(\sigma, \mathbf{d}) \cdot (\sigma', \mathbf{d}') = (\sigma\sigma', \sigma\mathbf{d}'),$$

where $\sigma \in \mathcal{S}_r$ is considered as a permutation matrix over $\mathbb{F}_2$. We correspond $(\sigma, \mathbf{d})$ to a mapping $H(r) \to H(r)$ defined by $\mathbf{z} \mapsto \sigma\mathbf{z} + \mathbf{d}$. This defines the above isomorphism. Injectivity and surjectivity are easily checked by using Proposition 1. ■

From now on we deal with only locally Hamming distance-*transitive* graphs with discreteness d and valency r(denoted by LHDTG_d^r). Next lemma connects LHDTG with multiply transitive group.

LEMMA 7. *Let $G \cong H(r)/\Gamma$ be a graph contained in $LHDTG_d^r$. Define the point stabilizer*

$$N(\Gamma)_\emptyset := \{\delta \in N(\Gamma) | \delta\emptyset = \emptyset\}.$$

Then, $N(\Gamma)_\emptyset$ can be regarded as a subgroup of $N(\Gamma)/\Gamma \cong \mathrm{Aut}(G)$. Also, $N(\Gamma)_\emptyset$ can be regarded as a subgroup of $\mathcal{S}_r$, and is a k-homogeneous group acting on the r-element subset X for $k = \lfloor (d-1)/2 \rfloor$. Moreover, $N(\Gamma)_\emptyset$ acts on the block set of the design defined in Lemmas 3 and 3; i.e., on $\Gamma\emptyset \cap \binom{X}{d}$.*

PROOF:
Case d=2t+1. In this case we have $k = t$. Take vertices $u, v \in H(r)$ such that $d(\emptyset, u) = d(\emptyset, v) = t$ holds. Since $N_t(\emptyset) \cong N_t(\Gamma\emptyset)$ holds, $d_G(\Gamma\emptyset, \Gamma u) = d_G(\Gamma\emptyset, \Gamma v) = t$ holds. Since $H(r)/\Gamma$ is distance-transitive, there exists a $\delta_{/\Gamma} \in \mathrm{Aut}(H(r)/\Gamma)$ such that $\delta_{/\Gamma}(\Gamma\emptyset) = \Gamma\emptyset$ and $\delta_{/\Gamma}(\Gamma u) = \Gamma v$. We can take a $\delta \in N(\Gamma)$ such that $\delta_{/\Gamma}$ coincides with the one defined in Lemma 4 by the surjectivity of $\delta \mapsto \delta_{/\Gamma}$. Since $\Gamma\delta\emptyset = \delta_{/\Gamma}(\Gamma\emptyset) = \Gamma\emptyset$ holds, a $\gamma \in \Gamma$ satisfies $\gamma\delta\emptyset = \emptyset$, and by retaking $\gamma\delta$ as δ, we may assume that $\delta\emptyset = \emptyset$; i.e., $\delta \in N(\Gamma)_\emptyset$. Now $\Gamma\delta u = \Gamma v$ holds, and since $N_t(\emptyset) \cong N_t(\Gamma\emptyset)$, $\delta u = v$ holds; i.e.,

$N(\Gamma)_\emptyset$ acts on X t-homogeneously. Since Γ has no fixed point, $\Gamma \cap N(\Gamma)_\emptyset = \{\mathrm{id}\}$ holds, and this implies that $N(\Gamma)_\emptyset$ can be embedded into $N(\Gamma)/\Gamma$ as a subgroup. On the other hand, $N(\Gamma)_\emptyset$ can be naturally identified with a subgroup of $\mathcal{S}_r$, since through the identification $\mathrm{Aut}(H(r)) \cong \mathcal{S}_r.\mathrm{Wr}.\mathbb{F}_2$, we have $N(\Gamma)_\emptyset \subset \{(\sigma, 0) | \sigma \in \mathcal{S}_r\} \cong \mathcal{S}_r$. Take an element $x = \gamma\emptyset \in \Gamma\emptyset \cap \binom{X}{2t}$ and an element $g \in N(\Gamma)_\emptyset$. Then, $g = (\sigma, 0)$ and $\sigma x = gx = g\gamma\emptyset = \gamma' g\emptyset = \gamma'\emptyset \in \Gamma\emptyset \cap \binom{X}{2t+1}$ holds for some $\gamma' \in \Gamma$ since $g \in N(\Gamma)$. Thus, we have $gx \in \Gamma\emptyset \cap \binom{X}{2t+1}$, and consequently, $N(\Gamma)_\emptyset$ acts on the block set $\Gamma\emptyset \cap \binom{X}{2t+1}$. For $d = 2t$, the same proof is also valid with modification on only the parameters t and d. ∎

PROOF OF MAIN RESULT: Using Formula 1, we see that the discreteness d is calculated as shown in Main Result. Then, all statements follow from Lemmas 3, 3*, and 7, and Criterion 1. ∎

LEMMA 8. *Let $G = H(r)/\Gamma$, $N(\Gamma)_\emptyset$ be as in Lemma 7. If $N(\Gamma)_\emptyset$ is isomorphic to $\mathcal{A}_r$ or $\mathcal{S}_r$, then G is a Hamming scheme or a folded Hamming scheme.*

PROOF: Let d_Γ be the discreteness of Γ, and let $\mathcal{B}$ be the block set of the t-(r, d, λ) design stated in Lemma 7. If $d_\Gamma = \infty$ then $G = H(r)$, thus we may assume $d_\Gamma < \infty$. Take a block $B \in \mathcal{B}$. Since any cyclic permutation of length 3 is contained in both $\mathcal{A}_r$ and $\mathcal{S}_r$, there exists a $\gamma \in \Gamma$ with $d(B, \gamma B) \leq 2$, unless $B = X$ holds. Since $d_\Gamma \geq 5$, $B = X$ must hold. Then, $d_\Gamma = r$, and the fiber on the vertex $\Gamma\emptyset$ contains exactly two vertices $\emptyset, X \in V(H(r))$. A similar situation occurs on any vertex on G, and consequently, G is obtained from $H(r)$ by identifying two antipodal vertices; i.e., a folded Hamming scheme. ∎

COROLLARY OF LEMMA 8. *All graphs contained in LHDTG$_d^r$ for $d \geq 13$ are the Hamming schemes or the folded Hamming schemes.*

PROOF: For $d \geq 13$, either $d = 2t+1$ with $t \geq 6$ or $d = 2t$ with $t \geq 7$ holds. In each case, $N(\Gamma)_\emptyset$ is at least 6-homogeneous, and the result follows from Theorems 1 and 2 and Lemmas 7 and 8. ∎

Combining a calculation on the admissibility of parameters of a design and Criterion 1, we can improve the lower bound of d.

LEMMA 9. *Corollary of Lemma 8 holds for $d \geq 9$.*

PROOF: Since for any distinct $B, B' \in \mathcal{B}$ $\#(B \cap B') \leq t$ holds, an inequality $2d - t \leq r$ must hold. Thus, $18 \leq 2d \leq r$ follows, and the degree of the corresponding homogeneous group is no less than 18. There are three 4-homogeneous groups satisfying this condition except $\mathcal{A}_r$ and $\mathcal{S}_r$; viz, $\mathrm{P\Gamma L}_2(32)$, M_{23}, and M_{24}.

Let $G = H(r)/\Gamma$ be a graph in LHDTG$_d^r$ which is neither a Hamming nor a folded Hamming scheme. The next table lists all $\lfloor (d-1)/2 \rfloor$-homogeneous groups for $d_\Gamma = 10, 11, 12$. The number s in the column *homogeneousity* shows that the group $N(\Gamma)_\emptyset$ is at least s-homogeneous; i.e., $s = \lfloor (d-1)/2 \rfloor$, and the column $N(\Gamma)_\emptyset$ lists all the

s-homogeneous groups. The column *admissibility* shows the necessary condition on λ deduced from the well-known admissibility condition of a t-(r,d,λ) design; that is, $\binom{d-i}{t-i} | \lambda \binom{r-i}{t-i}$, where $a|b$ denotes that a divides b.

d_Γ	t	Homogeneousity	$N(\Gamma)_\emptyset$	$r = \#(X)$	Criterion 1	Admissibility
12	6	5-homogeneous	M_{24}	24	$\lambda \leq 3$	$42|\lambda$
11	5	5-homogeneous	M_{24}	24	$\lambda \leq 19/6$	$42|\lambda$
10	5	4-homogeneous	M_{24}	24	$\lambda \leq 19/5$	$18|\lambda$
			M_{23}	23	$\lambda \leq 18/5$	$252|\lambda$
			$P\Gamma L_2(32)$	33	$\lambda \leq 28/5$	$504|\lambda$
9	4	4-homogeneous	M_{24}	24	$\lambda \leq 4$	$24|\lambda$
			M_{23}	23	$\lambda \leq 19/5$	$18|\lambda$
			$P\Gamma L_2(32)$	33	$\lambda \leq 29/5$	$21|\lambda$

List 1. Possible parameters for LHDTG_d^r, $9 \leq d \leq 12$.

In each case, Criterion 1 contradicts the admissibility. ∎

THEOREM 3. *Let G be a distance-transitive graph with parameters $a_i = 0$ for $i = 1,2,3$ and $c_i = i$ for $i = 1,2,3,4$. Then, G is a Hamming scheme or a folded Hamming scheme.*

PROOF: Formula 1 implies $t \geq 4$ and $d \geq 9$. Thus, this theorem is a direct consequence of Lemma 9. ∎

§5. Nontrivial Examples.

The next table lists some known locally Hamming distance-regular graphs other than Hamming scheme and folded Hamming scheme. All these examples are cited from [**3**, pp.480–483]. The column *Intersection Array* shows the intersection array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots c_D\}$, where D denotes the diameter. After the array is the reference to [**3**]. The $*$ in the column $N(\Gamma)_\emptyset$ indicates that it is not distance-transitive.

No.	d_Γ	t	r	$N(\Gamma)_\emptyset$	#(V(G))	Intersection Array
1	8	4	24	M_{24}	4096	$\{24,23,22,21;1,2,3,24\}$, Ch.11.3.2
2	8	4	23	M_{23}	4096	$\{23,22,21,20,3,2,1;1,2,3,20,21,22,23\}$, p.362
3	7	3	23	M_{23}	2048	$\{23,22,21;1,2,3\}$, Ch.11.3.4
4	7	3	22	$*$	2048	$\{22,21,20,3,2,1;1,2,3,20,21,22\}$, p.365
5	6	3	22	M_{22}	2048	$\{22,21,20,16,6,2,1;1,2,6,16,20,21,22\}$, p.363
6	6	3	22	M_{22}	1024	$\{22,21,20;1,2,6\}$, Ch.11.3.5
7	5	2	21	$*$	1024	$\{21,20,16,6,2,1;1,2,6,16,20,21\}$, p.365
8	5	2	21	$*$	2048	$\{21,20,16,9,2,1;1,2,3,16,20,21\}$, p.365
9	5	2	21	$P\Gamma L_3(4)$	512	$\{21,20,16;1,2,12\}$, Ch.11.3.6

List 2. Some known LHDRG_d^r with $5 \leq d \leq 8$.

For No.1–6, we shall briefly describe the corresponding designs. For No.1, the corresponding is a 4-(24,8,5) design, which coincides with the block set of the Witt system $S(5,8,24)$. To No.2, a 4-(23,8,4) design corresponds, which has the same number of blocks with the Witt system $S(4,7,23)$. This design is obtained by adding one element to each block in $S(4,7,23)$ so that M_{23} acts on those blocks. To No.3, a 3-(23,7,5) design corresponds, which coincides with the block set of the Witt system $S(4,7,23)$. To No.4, a 3-(22,7,4) design corresponds, which has $2^4 \cdot 11$ blocks. The author doesn't know how to obtain this design from $S(3,6,22)$. To No.5 and No.6, a 3-(22,6,1) design, in other words, the Witt system $S(3,6,22)$ corresponds. This implies that the design does not determine the graph uniquely.

All of the listed graphs are obtained as a coset graph of modified binary Golay codes[**3**, Ch.11.3]. Its fundamental group is a linear subspace in $\mathbb{F}_2^r$ when regarded as a subgroup of $\mathcal{S}_r \ltimes \mathbb{F}_2^r$. This invokes the next conjecture.

CONJECTURE 1. *Let $\Gamma \subset \mathrm{Aut}(H(r))$ be a subgroup with discreteness at least 5. If $H(r)/\Gamma$ is distance-regular, then Γ is a linear subspace in $\mathbb{F}_2^r$ through the identification $\mathcal{S}_r \ltimes \mathbb{F}_2^r$.*

CONJECTURE 2. *Under the same assumption, Γ is a subset of a linear code obtained from binary Golay codes by truncation or shortening.*

Conjecture 2 implies the complete classification of LHDRG. To prove or disprove these conjectures does not seem to be too difficult, at least for the distance-transitive case with $d_\Gamma \geq 7$; i.e., with $a_1 = a_2 = 0$, $c_2 = 2$, $c_3 = 3$, because in this case the point stabilizer is a 2-homogeneous group on which we have much information[**7**].

In this paper, we have not utilized any information on the parameters a_i, b_i, c_i for $i \geq t+1$, so the same proof can also be applied to graphs for which the parameters with subscript $i > t$ can not be defined. Also, we have not used the properties of association schemes at all, so it seems to be possible that one proves a stronger result with easier proof than this paper, using such structures.

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