Fusion in Association Schemes

Akihiro Munemasa
Department of Mathematics
Osaka Kyoiku University

December 17, 1990

Let us begin with a well-known example of cospectral graphs (Two graphs are said to be cospectral if their adjacency matrices have the same spectrum).

The graph $\Gamma_1$ depicted below is called the Shrikhande graph. The vertices with the same label are identified. $\Gamma_1$ is a regular graph of valency 6 with 16 vertices. Its spectrum is $6^12^6(-2)^9$.

\[
\begin{array}{cccccc}
\bullet & 0 & 1 & 2 & 3 & 0 \\
6 & \bullet & \bullet & \bullet & \bullet & 6 \\
5 & \bullet & \bullet & \bullet & \bullet & 5 \\
4 & \bullet & \bullet & \bullet & \bullet & 4 \\
0 & 1 & 2 & 3 & 0 & \bullet
\end{array}
\]

The graph $\Gamma_2$ has $\{0, 1, 2, 3\}^2$ as the set of vertices, and two distinct vertices $(a_1, a_2)$, $(b_1, b_2)$ are adjacent if $a_1 = b_1$ or $a_2 = b_2$. $\Gamma_2$ is called the Hamming graph $H(2, 4)$ or the Latin square graph $L_2(4)$. $\Gamma_2$ has the same spectrum as that of $\Gamma_1$, and Shrikhande [9] showed that $\Gamma_1$ and $\Gamma_2$ are the only graphs with the spectrum $6^12^6(-2)^9$. $\Gamma_1$ and $\Gamma_2$ are not isomorphic. Indeed, $\Gamma_2$ contains $K_4$ as a subgraph, while $\Gamma_1$ does not.

Definition. A strongly regular graph $\Gamma$ with parameters $(k, \lambda, \mu)$ is an undirected graph
such that
\[
\#\{z \in \Gamma | x \sim z, z \sim y\} = \begin{cases} 
  k & \text{if } x = y \\
  \lambda & \text{if } x \sim y \\
  \mu & \text{otherwise}
\end{cases}
\]

The number of vertices is given by \(1 + k + k(k - \lambda - 1)/\mu\). The complement of a strongly regular graph is a strongly regular graph.

Both \(\Gamma_1\) and \(\Gamma_2\) are strongly regular graphs with parameters \((6, 2, 2)\). Let us investigate \(\Gamma_1\) and \(\Gamma_2\) in more detail. The automorphism group of \(\Gamma_1\) is a vertex-transitive, rank 4 permutation group, while the automorphism group of \(\Gamma_2\) is a vertex-transitive, rank 3 permutation group. If we regard \(\text{Aut}\Gamma_i\) as acting on the direct product \(V\Gamma_i \times V\Gamma_i\) of the set of vertices \(V\Gamma_i\), \((i = 1, 2)\), then two nontrivial orbitals of \(\text{Aut}\Gamma_2\) are precisely edges and non-edges. On the other hand, \(\text{Aut}\Gamma_1\) acts transitively on edges, but the set of non-edges is decomposed into two orbitals of \(\text{Aut}\Gamma_1\). Therefore two graphs are group theoretically quite different, however they share the same algebraic properties (parameters, spectrum).

**Definition.** An association scheme of class \(d\) is a pair \((X, \{R_i\}_{0 \leq i \leq d})\) such that
\[
X \times X = R_0 \cup \cdots \cup R_d,
\]
\[
R_0 = \{(x, x) | x \in X\},
\]
\[
(x, y) \in R_i \iff (y, x) \in R_i,
\]
\[
p_{ij}^k = \#\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\} \quad \text{for any } (x, y) \in R_k.
\]

Indeed this is the definition of symmetric association scheme, however in this talk, all association schemes are assumed to be symmetric. For more general and detailed theory of association schemes, see [1].

Note that strongly regular graphs are equivalent to association schemes of class 2, via the correspondence
\[
R_1 = \{\text{edges}\}
\]
\[
R_2 = \{\text{non-edges}\}
\]
The three parameters of the strongly regular graph are \(k = p_{11}^0, \lambda = p_{11}^1, \mu = p_{11}^2\).

An association scheme of class \(d\) can be constructed from a generously transitive permutation group of rank \(d+1\). A permutation group \(G\) on \(X\) is said to be generously transitive if
\[
\forall x, \forall y \in X \ \exists \sigma \in G \text{ such that } x^\sigma = y, y^\sigma = x.
\]
The set of relations \(R_0, R_1, \ldots, R_d\) are defined to be the \(G\)-orbits on \(X \times X\).

Since \(\text{Aut}\Gamma_1\) is a generously transitive permutation group of rank 4, the orbits \(R_0, R_1, R_2, R_3\) defines an association scheme \(\mathcal{X}_1 = (V\Gamma_1, \{R_0, R_1, R_2, R_3\})\) of class 3. We may renumber the relation if necessary and assume \(R_1\) is the set of edges of \(\Gamma_1\).
Since $\Gamma_1$ is a strongly regular graph, $\mathfrak{x}'_1 = (V\Gamma_1, \{R_0, R_1, R_2 \cup R_3\})$ is an association scheme of class 2. We say that $\mathfrak{x}'_1$ is obtained from $\mathfrak{x}_1$ by merging of classes, or by fusion. It is now appropriate to give a formal definition.

**Definition.** Let $\mathfrak{x} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme and let $\{\Lambda_j\}_{0 \leq j \leq d'}$ be a partition of $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$. Let $R_{\Lambda_j} = \bigcup_{i \in \Lambda_j} R_i$. If $(X, \{R_{\Lambda_j}\}_{0 \leq j \leq d'})$ becomes an association scheme, then it is said to be a fusion scheme of $\mathfrak{x}$. $\mathfrak{x}$ is called amorphous if $(X, \{R_{\Lambda_j}\}_{0 \leq j \leq d'})$ is a fusion scheme of $\mathfrak{x}$ for any partition $\{\Lambda_j\}_{0 \leq j \leq d'}$ with $\Lambda_0 = \{\}$.

Any association scheme of class 2 is amorphous. The association scheme $\mathfrak{x}_1$ above is indeed amorphous so that $(V\Gamma_1, \{R_0, R_2, R_1 \cup R_3\})$, $(VT_1, \{R_0, R_3, R_1 \cup R_2\})$ are fusion schemes of $\mathfrak{x}_1$.

An obvious construction of fusion schemes comes from a permutation group and its generously transitive subgroup. Let $\tilde{G} \supset G$ be permutation groups acting on a set $X$, and assume $G$ is generously transitive (hence $\tilde{G}$ is also generously transitive), $\text{rank}\tilde{G} < \text{rank}G$. The $G$-orbits on $X \times X$ are refinement of $\tilde{G}$-orbits on $X \times X$, so one obtains an association scheme and its fusion scheme. From this group theoretic point of view, it is hard to imagine the existence of amorphous association scheme of large number of classes. However, there do exist amorphous association schemes of arbitrary large number of classes.

**Example.** Affine plane scheme (Gol'fand and Klin [4]). Let $S_1, \ldots, S_{n-1}$ be mutually orthogonal Latin squares of order $n$. Define an association scheme with vertices $X = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$,

$$R_1 = \{((j, k), (j, k')) | (j, k), (j, k') \in X, k \neq k'\},$$

$$R_2 = \{((j, k), (j', k)) | (j, k), (j', k) \in X, j \neq j'\},$$

$$R_i = \{((j, k), (j', k')) | S_{i-2}(j, k) = S_{i-2}(j', k'), (j, k) \neq (j', k')\}, \quad 3 \leq i \leq n+1.$$

Then $(X, \{R_i\}_{0 \leq i \leq n+1})$ is an amorphous association scheme.

**Example.** (Baumert–Mills–Ward [3]) Let $p$ be a prime. The cyclotomic scheme of class $e$ on $GF(p^{2e})$ is amorphous if and only if there exists a divisor $r$ of $s$ such that $e$ divides $p^r + 1$. The concept of amorphous association scheme was introduced after the work [3] had been published. Apparently, many combinatorists were unaware of the work of Baumert–Mills–Ward, who classified amorphous cyclotomic schemes in combinatorist’s terminology. The cyclotomic scheme of class $e$ is closely related to the Gauss sum with a multiplicative character of order $e$ as the argument. Such Gauss sums are easily computable precisely when the corresponding cyclotomic scheme is amorphous. See Bannai–Munemasa [2] for details.

Furthermore, amorphous association schemes of class 4 can be derived from Hadamard matrices (Ivanov-Chuvaeva [6]).
Note that a number of strongly regular graphs are obtained from an amorphous association scheme of large number of classes. In particular, each nontrivial relation of an amorphous association scheme defines a strongly regular graph of Latin square type or negative Latin square type.

**Definition.** A strongly regular graph with parameters \((k, \lambda, \mu)\) is said to be of Latin square type if there exist integers \(g, n > 0\) such that \(k = g(n - 1), \lambda = (g - 1)(g - 2) + n - 2, \mu = g(g - 1)\). A strongly regular graph with parameters \((k, \lambda, \mu)\) is said to be of negative Latin square type if there exist integers \(g, n > 0\) such that \(k = g(n + 1), \lambda = (g + 1)(g + 2) - n - 2, \mu = g(g + 1)\).

**Theorem 1** ([7] [5]) Let \(\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})\) be an association scheme of classs \(d\) with \(d \geq 3\). Then \(\mathcal{X}\) is amorphous if and only if either

(i) \((X, R_i)\) is a strongly regular graph of Latin square type for each \(i, 1 \leq i \leq d\)

or

(ii) \((X, R_i)\) is a strongly regular graph of negative Latin square type for each \(i, 1 \leq i \leq d\).

The aim of this talk is to present our recent result [5] which gives a construction of a new family of amorphous association schemes.

**Definition.** Let \(f(x) \in \mathbb{Z}/p\mathbb{Z}[x]\) be a primitive polynomial of degree \(s\), that is, \(GF(p^s) = \mathbb{Z}/p\mathbb{Z}[x]/(f(x))\) and the image of \(x\) is a primitive element of \(GF(p^s)\). There exists \(g(x) \in \mathbb{Z}/p\mathbb{Z}[x]\) such that \(f(x)g(x) = x^{p^s-1} - 1\). By Hensel's Lemma there exist \(F(x), G(x) \in \mathbb{Z}/p^m\mathbb{Z}[x]\) such that

\[ F(x) \equiv f(x) \mod p \]
\[ G(x) \equiv g(x) \mod p \]
\[ F(x)G(x) = x^{p^s-1} - 1 \]

The ring \(R = GR(p^m, s) = \mathbb{Z}/p^m\mathbb{Z}[x]/(F(x))\) is called a Galois ring (see [8]). \(R\) is a local ring with the maximal ideal \(pR\), and \(R/pR \cong GF(p^s)\).

In what follows, we assume \(p^m = 2^2 = 4\). Let \(\xi\) be the image of \(x\) in \(R\), \(\bar{\xi}\) the image of \(\xi\) in \(K\). We have

\[ R^* = \langle \xi \rangle \times \epsilon, \]
\[ |\langle \xi \rangle| = 2^s - 1, |\epsilon| = 2^s \]
\[ \epsilon = 1 + 2R = \{1 + 2\xi^i | i = 0, 1, \ldots, 2^s - 2\} \cup \{1\} \]
and \(T = \{0, 1, \xi, \xi^2, \ldots, \xi^{2^s-2}\}\) is a set of representatives of \(R/2R\). It should be mentioned that \(T - \{0\}\) is a multiplicative subgroup of \(R^*\), but \(T\) is not in general an additive subgroup of \(R\). Since \(R = \{t + 2r | t \in T, r \in R\}\), and \(2R = 2\{t + 2r | t \in T, r \in R\}\)
$= \{2t|t \in T\}$, we see $R = \{t_1 + 2t_2 | t_1, t_2 \in T\}$, and this representation of elements of $R$ is unique.

$$R^* = R - 2R = \{t_2 + 2t_2 | t_1 \neq 0\} = \langle \xi \rangle \times \{1 + 2t_2 | t_2 \in T\}$$

Define a mapping $\iota : K \mapsto R$ by $0 \mapsto 0$, $\xi^i \mapsto \xi^i$, $(0 \leq i \leq 2^s - 2)$. Define $\phi : K \mapsto \epsilon$ by $\phi(\alpha) = 1 + 2\iota(\alpha)$. Since

$$\iota(\alpha + \beta) \equiv \iota(\alpha) + \iota(\beta) \mod 2$$

we have

$$2\iota(\alpha + \beta) = 2\iota(\alpha) + 2\iota(\beta)$$

Therefore $\phi$ is a group isomorphism. Let $H$ be a subgroup of $K$, $K = \bigcup_{i=1}^{r}H + \alpha_i$, $|K : H| = r$. Then

$$\epsilon = \bigcup_{i=1}^{r} \phi(H + \alpha_i)$$

$$= \bigcup_{i=1}^{r} \phi(H) \phi(\alpha_i)$$

Let $\mathcal{X}(H) = (R, \{S_i\}_{0 \leq i \leq r+1})$ be an association scheme defined by

$$(x, y) \in S_0 \Leftrightarrow x - y = 0$$

$$(x, y) \in S_i \Leftrightarrow x - y \in \langle \xi \rangle \phi(H + \alpha_i), \quad 1 \leq i \leq r$$

$$(x, y) \in S_{r+1} \Leftrightarrow x - y \in 2R - \{0\}$$

**Theorem 2** $\mathcal{X}(H)$ is amorphous if and only if $H^\perp$ is totally isotropic with respect to the symmetric bilinear form

$$(\alpha, \beta) = \text{Tr}_{K/\mathbb{F}(2)}(\alpha \beta)$$

If we set $H^\perp$ to be a maximal totally isotropic subspace, then $\mathcal{X}(H)$ is of class $2^{[s/2]} + 1$.

If we set $s = 2$ in the above theorem, $\mathcal{X}(H)$ is the association scheme $\mathcal{X}_1$ which is obtained from the action of $\text{Aut}_\Gamma_1$ described earlier.

The proof of Theorem 2 is best given in terms of Gauss sums and Jacobi sums over $GR(4, s)$. Another tool in the proof is Bannai-Muzichuk criterion for the existence of a fusion scheme. See [5] for details.

A possible direction for further research is to generalize Theorem 2 for an arbitrary characteristic $p^m$. We found a few examples of amorphous association schemes over $GR(p^2, s)$ with $p$ odd, but so far we were unable to obtain a nice criterion as in Theorem 2.
References


