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A Harnack Inequality for solutions of difference elliptic-partial differential equations

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Abstract. We establish a Harnack inequality for solutions of difference elliptic-partial differential equations with bounded and measurable coefficients. To do it, we need to consider local estimates which are analogue to, but more complicated than those for elliptic and parabolic equations.

1.Introduction

In treating the regularity problem for solutions of elliptic and parabolic equations, in particular of nonlinear ones, we need to consider the corresponding linear equations with only measurable coefficients. Hölder continuity of bounded weak solutions to equations with bounded and measurable coefficients was obtained in the paper [8], [9], [10], [11] and [12]. So-called Harnack inequality was also established for solutions of elliptic and parabolic equations with only measurable coefficients by J.Moser(refer to [10], [11]).

It is our aim to derive a Harnack inequality uniformly with respect to an approximation for solutions of difference elliptic-partial differential equations with only bounded and measurable coefficients. Originally such local estimates for solutions of difference elliptic-partial equations was studied by N.Kikuchi([4]), who has shown that Hölder estimates for bounded weak solutions of equations of this type hold independently of an approximation number. In order to obtain uniform estimates with respect to an approximating, we need to distinguish the calculations according to the relation between the size of a local cube and a mesh h. Namely one has to make an estimation, analoguely to parabolic equations if a local cube is large in comparison with a mesh h, and otherwise, to elliptic equations. This treatment seems to be crusial and characteristic in working for difference elliptic-partial differential equations. We also think that a time-discrete approximation of the evolution equations will play an essential role in constructing Morse flows for a functional in the calculus of variations (refer to [1] and [5]) and then such estimates represented in this paper will be fundamental and useful(see [7]). Let Ω be a bounded open set in Euclidean space $R^m, m \geq 2$, u be a function: $\Omega \to R$ and $Du = (D_1u, D_2u, \ldots, D_mu)$, $D_\alpha u = \partial u/\partial x^\alpha$ $(1 \leq \alpha \leq m)$ be the gradient of u. Let T be a positive number arbitrarily given and set $Q = (0,T) \times \Omega$. We use the usual Lebesgue space $L_p(\Omega)$, Sobolev spaces; $W_p^k(\Omega) = W_p^k(\Omega, R)$, $\overset{\circ}{W}{}_p^k(\Omega) = \overset{\circ}{W}{}_p^k(\Omega, R)$, $V_2(Q) = L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); W_2^1(\Omega))$ and $\overset{\circ}{V}_2(Q) = L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); \overset{\circ}{W}{}_2^1(\Omega))$.

For a positive integer N, $N \geq 2$, we put $h = T/N$ and $t_n = nh$ $(0 \leq n \leq N)$. Let u_0 be a function belonging to $W_2^1(\Omega)$. We shall be concerned with a family of linear elliptic partial differential equations:

$$\frac{u_n - u_{n-1}}{h} = D_\alpha(a_n^{\alpha\beta}(x) D_\beta u_n). \qquad (1 \leq n \leq N) \tag{1.1}$$

In the summation convention over repeated indices, the Greek indices run from 1 to m. The coefficients $a_n^{\alpha\beta}(\cdot)$ $(1 \leq \alpha, \beta \leq m)$ $(1 \leq n \leq N)$ are measurable functions defined in Ω satisfying the relation with positive constants λ and μ :

$$\mu|\xi|^2 \geq a_n^{\alpha\beta}(x)\xi^\alpha\xi^\beta \geq \lambda|\xi|^2 \qquad \text{for } \xi = (\xi^\alpha) \in R^m, 1 \leq n \leq N \text{ and any } \quad x \in \Omega. \tag{1.2}$$

We mean a family of weak solutions of (1.1) with an initial datum u_0 by a family $\{u_n\}$ $(1 \le n \le N)$ of functions $u_n \in W_2^1(\Omega)$ which satisfy

$$\int_\Omega \frac{u_n - u_{n-1}}{h}\varphi dx + \int_\Omega a_n^{\alpha\beta} D_\beta u_n D_\alpha \varphi dx = 0 \quad \text{for any } \varphi = (\varphi^i) \in \overset{\circ}{W}{}_2^1(\Omega). \tag{1.3}$$

For a family $\{u_n\}(1 \le n \le N)$ satisfying $u_n \in W_2^1(\Omega)$, we define a function $u_h(t,\cdot)$: $t \in [0,T] \to u_h(t,\cdot) \in W_2^1(\Omega)$ as follows:

$$\begin{aligned} &u_h(0,\cdot) = u_0(\cdot), \\ &u_h(t,\cdot) = u_n(\cdot) \qquad \text{for } t_{n-1} < t \le t_n \qquad (1 \le n \le N). \end{aligned} \tag{1.4}$$

If $\{u_n\}(1 \le n \le N)$ is a family of weak solutions of (1.1) with an intial datum u_0, then we call u_h, defined by (1.4), a weak solution of (1.1). Also $a^{\alpha\beta}(t,\cdot)$ is defined for $t \in (0,T]$ as follows:

$$a^{\alpha\beta}(t,\cdot) = a_n^{\alpha\beta}(\cdot), \quad \text{for} \quad t_{n-1} < t \le t_n \ (1 \le n \le N). \tag{1.5}$$

If u_h is a weak solution of (1.1), then we deduce from (1.3) and the definitions (1.4) and (1.5) that u_h satisfies the identity

$$\int_\Omega \frac{u_h(t,\cdot) - u_h(t-h,\cdot)}{h}\varphi(\cdot)dx + \int_\Omega a^{\alpha\beta}(t,\cdot)D_\beta u_h(t,\cdot)D_\alpha\varphi(\cdot)dx = 0 \tag{1.6}$$

for any $\varphi = (\varphi^i) \in \overset{\circ}{W}{}_2^1(\Omega)$ and all $t \in (0,T]$.

Here we recall some standard notations: For a point $z_0 = (t_0, x_0) \in Q$, we put

$$\begin{aligned} &B_r(x_0) = \{x \in R^m : |x^\alpha - x_0^\alpha| < r \ (1 \le \alpha \le m)\}, \\ &C_{r,\tau}(z_0) = \{t \in R : |t - t_0| < \tau\} \times B_r(x_0), \\ &C_{r,\tau}^+(z_0) = \{t \in R : t_0 - \tau < t < t_0\} \times B_r(x_0), \\ &C_{r,\tau}^-(z_0) = \{t \in R : t_0 < t < t_0 + \tau\} \times B_r(x_0). \end{aligned} \tag{1.7}$$

These domains are referred as "cubes". For simplicity we shall use abbreviations:

$$C_r(z_0) = C_{r,r^2}(z_0),\ C_r^+(z_0) = C_{r,r^2}^+(z_0),\ C_r^-(z_0) = C_{r,r^2}^-(z_0).$$

In the above notations, the centre x_0 and z_0 will be abbreviated when no confusion may arise. For $z_i = (t_i, x_i)$ $(i = 1,2)$, we introduce the parabolic metric

$$\delta(z_1,z_2) = max\{|t_1 - t_2|^{1/2}, |x_1^\alpha - x_2^\alpha|(1 \le \alpha \le m)\} \tag{1.8}$$

For a measurable set A in R^k, we denote the k-dimensional measure of A by $|A|$ and for a measurable function f, we shall put

$$\bar{f}_A = \frac{1}{|A|}\int_A f(z)dz. \tag{1.9}$$

For a positive number l we denote by $[l]$ the greatest non-negative integer not greater than l and by $\tilde{n}_l$ the greatest non-negative integer less than l^2/h. The same letter γ will be used to denote different constants depending on the same parameters of arguments.

Now let N_0 be a positive integer satisfying

$$N_0 > \frac{\log(1+\frac{m}{2})}{\log(1+\frac{2}{m})}$$

and h_0 be an arbitrarily given positive number sufficiently small. From now on we take N sufficiently large *i.e.*,

$$N \geq \max\{N_0, T/h_0\}.$$

We also define a cube $\widetilde{Q_{h_0}}$ as follows:

$$\widetilde{\Omega}_{h_0} = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) > \sqrt{N_0 h_0}\} \quad \widetilde{Q_{h_0}} = (N_0 h_0,\ T) \times \widetilde{\Omega}_{h_0}.$$

Now we shall describe our main results:

Theorem 1.1.(Weak Harnack inequality of parabolic version). *Let u_h be a weak solution of*(1.1)*. If u_h is nonnegative in a cube $C_r^+(t_{n_0}, x_0) \subset Q$ with $r^2 > h$, then, for any p; $0 < p < 1+\frac{2}{m}$, there exists a positive constant γ depending only on λ, μ and m, p such that,*

$$\left(\frac{1}{|D_{\frac{1}{2}}^-|}\iint_{D_{\frac{1}{2}}^-} (u_h)^p dxdt\right)^{\frac{1}{p}} \leq \gamma \inf_{D_{\frac{1}{2}}^+} u_h \tag{1.10}$$

holds where

$$D_{\frac{1}{2}}^- = (t_{n_0 - \widetilde{n}_r},\ t_{n_0 - \widetilde{n}_r} + \frac{1}{8}\widetilde{n}_r h) \times B_{\frac{1}{2}\sqrt{\widetilde{n}_r h}}(x_0),$$
$$D_{\frac{1}{2}}^+ = (t_{n_0} - \frac{1}{8}\widetilde{n}_r h,\ t_{n_0}) \times B_{\frac{1}{2}\sqrt{\widetilde{n}_r h}}(x_0).$$

Theorem 1.2 (Weak Harnack inequality of elliptic version). *Let u_h be a weak solution of* (1.1) *satisfying*

$$\iint_Q (u_h)^2 dxdt \leq \gamma_1$$

with a uniform constant γ_1. If $u_n \geq 0 (N_0 \leq n \leq N)$ in $B_{2r}(x_0) \subset \Omega$ with $r^2 \leq h$, then, for any p; $0 < p < \frac{m}{m-2}$, there exist positive constants $\gamma$and $\alpha; 0 < \alpha < 1$ depending only on λ, μ, m and γ_1, $dist(x_0, \partial\Omega)$ such that

$$\left(\frac{1}{|B_{\frac{r}{2}}|}\int_{B_{\frac{r}{2}}(x_0)} (u_n)^p dx\right)^{1/p} \leq \gamma\left[\inf_{B_r(x_0)} u_n + r^\alpha\right] \tag{1.11}$$

holds.

Theorem 1.3(Local boundedness of solutions). *Let u_h be a weak solution of* (1.1) *satisfying*

$$\iint_Q (u_h)^2 dxdt \leq \gamma_1$$

with a uniform constant γ_1. *Then, for all* $(\bar t,\bar x)\subset \widetilde{Q_{h_0}}$ *with* $d=\frac{1}{4}min\{|\bar t-N_0h_0|^{\frac{1}{2}}, dist(\bar x,\partial\Omega)\}$ *and any* $p>1$, *there exist positive constants* γ *and* $\alpha; 0<\alpha<1$ *depending only on* λ,μ, γ_1 *and* p,d *such that, setting* $u_h^{\pm}=\max\{\pm u_h,0\}$

$$\sup_{C^+_{r/2}(t_{n_0},x_0)} u_h^{\pm}\le\gamma\left[\left(\frac{1}{|C_r^+|}\iint_{C_r^+(t_{n_0},x_0)}(u_h^{\pm})^p dxdt\right)^{1/p}+r^{\alpha}\right] \tag{1.12}$$

holds for any $(t_{n_0},x_0)\in C^+_{d/2}(\bar t,\bar x)$ *and all* $0<r<d/2$.

We would like to emphasize that the above theorems hold uniformly with respect to h and u_h.

This paper is arranged in the following: In Section2 we shall derive so-called Caccioppoli inequality for $u_h^{\frac{p}{2}}(p\neq -1)$. Here we need to use a cut-off function with respect to time-variable t, which was introduced in the paper[4], [7]. Section3 is devoted to an estimate for supu_h. It seems impossible to obtain the boundedness of solution of (1.1) by Moser's iteration only. In order to obtain the boundedness of solution of (1.1), we exploit DeGirgi's iterative technique. In Section4 we estimate logu_h, which is most important and difficult estimate in all parts. In Section5 we shall prove Theorem1.1, 1.2 and 1.3. Here we also obtain Hölder estimates for weak solutions of (1.1).

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2.Estimates for u^p

Lemma2.1.(Caccioppoli type inequality analogue to Moser's ones). *Let* u_h *be a weak solution of* (1.1) *and us take* $C^-_{\rho,\tau}(t_{n_0},x_0)$, $C^+_{\rho,\tau}(t_{n_0},x_0)\subset Q$ *arbitrarily. Then there exists a positive constant* γ *depending only on* $\lambda\mu$ *and* m *such that, if* u_h *is nonnegative in* $C^+_{\rho,\tau}(t_{n_0},x_0)$ *and* $u_{n_0-[\tau/h]-1}\ge 0$ *in* $B_\rho(x_0)$, *then*

$$\begin{aligned}\operatorname*{Sup}_{t_{n_0}-\tau(1-\sigma_2)\le t\le t_{n_0}}\int_{B_{\rho(1-\sigma_1)}(x_0)}(u_h+\varepsilon)^p(t,\cdot)dx+\iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)}\left|D(u_h+\varepsilon)^{p/2}\right|^2dxdt\\ \le\gamma\Big((\sigma_1\rho)^{-2}+\sigma_2\tau)^{-1}\Big)\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}(u_h+\varepsilon)^pdxdt\end{aligned} \tag{2.1}$$

holds for any $p<0$, *all* $\sigma_1,\sigma_2\in(0,1)$ *and any* $\varepsilon>0$. *If* u_h *is nonnegative in* $C^-_{\rho,\tau}(t_{n_0},x_0)$ *and* $u_{n_0}\ge 0$ *in* $B_\rho(x_0)$,

$$\begin{aligned}\operatorname*{Sup}_{t_{n_0}\le t\le t_{n_0}+\tau(1-\sigma_2)}\int_{B_{\rho(1-\sigma_1)}(x_0)}(u_h+\varepsilon)^p(t,\cdot)dx+\iint_{C^-_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)}\left|D(u_h+\varepsilon)^{p/2}\right|^2dxdt\\ \le\gamma\frac{1}{(1-p)^2}\Big((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\Big)\iint_{C^-_{\rho,\tau}(t_{n_0},x_0)}(u_h+\varepsilon)^pdxdt\end{aligned} \tag{2.2}$$

holds for any $p; 0<p<1$, *all* $\sigma_1,\sigma_2\in(0,1)$ *and any* $\varepsilon>0$.

REMARK. *For* $p=0$, *the above estimates are trivial.*

Proof.In the arguments we omit writing a center point or vertex of cubes; B_ρ, $C^+_{\rho,\tau}=C^+_{\rho,\tau}(t_{n_0},x_0)$ for simplicity.

We demonstrate only the proof of (2.1). Let $\eta \in C_0^\infty(B_\rho(x_0))$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{\rho(1-\sigma_1)}(x_0)$ and $|D\eta| \leq 2(\sigma_1\rho)^{-1}$. Also we take some appropriate cut-off function $\sigma(t)$ defined on $[t_{n_0-\tau}, t_{n_0}]$, of which the definition is given later. We remark that, since $u_h(t,\cdot)$ is nonnegative in $C^+_{\rho,\tau}(t_{n_0}, x_0)$, $(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)$ is admissible for $p<0, \varepsilon>0$ as a test function in the identity (1.6) in $C^+_{\rho,\tau}(t_{n_0}, x_0)$. Taking a function $(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)$ for $\varepsilon>0$ as a test-function in the identify (1.6) and integrating the resultant inequality with respect to time variable t in $(t_{n_0}-\tau, t_{n_0}]$, we have

$$\iint_{C^+_{\rho,\tau}} \frac{u_h(t,\cdot)-u_h(t-h,\cdot)}{h}(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt$$
$$+\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta u_h(t,\cdot)D_\alpha\Big[(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\Big]\sigma(t)dxdt = 0.$$

Namely

$$\int_{C^+_{\rho,\tau}} \frac{u_h(t,\cdot)+\varepsilon-(u_h(t-h,\cdot)+\varepsilon}{h}(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt$$
$$+\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta(u_h(t,\cdot)+\varepsilon)D_\alpha\Big[(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\Big]\sigma(t)dxdt = 0.$$

From now on let's put $v(t,\cdot) := u_h(t,\cdot)+\varepsilon$, so that the above inequality becomes

$$\begin{aligned}&\iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)-v(t-h,\cdot)}{h}(v(t,\cdot))^{p-1}\eta^2(\cdot)\sigma(t)dxdt\\ &+\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v(t,\cdot)D_\alpha\Big[(v(t,\cdot))^{p-1}\eta^2(\cdot)\Big]\sigma(t)dxdt = 0.\end{aligned} \tag{2.3}$$

Now we make estimates of each term in (2.3). To do it, we shall distinguish our proof into two cases:

Case $1, \sigma_2\tau > 3h$ and Case $2, \sigma_2\tau \leq 3h$

Firstly we consider Case1. Then we take $\sigma(t)$ as follows (see [4] or [7]):

$$\sigma(t) = \sigma_n \quad \text{for} \quad t_{n-1} < t \leq t_n (1 \leq n \leq N)$$

$$\sigma_n = \begin{cases} 1, & \text{for} \quad n_0 - [(1-\sigma_2)\tau/h] \leq n \leq n_0, \\ \frac{n-n_0+[\tau/h]-1}{[\tau/h]-1-[(1-\sigma_2)\tau/h]} & \text{for} \quad n_0 - [\tau/h]+1 \leq n \leq n_0 - [(1-\sigma_2)\tau/h] \\ 0, & \text{for} \quad n \leq n_0 - [\tau/h]. \end{cases} \tag{2.4}$$

(Quotient term of (2.3))Using Young's inequality and noting that $p<1$, we have

$$(v(t,\cdot)-v(t-h,\cdot))(v(t,\cdot))^{p-1} \leq (v^p(t,\cdot)-v^p(t-h,\cdot))/p,$$

so that

$$\iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)-v(t-h,\cdot)}{h}(v(t,\cdot))^{p-1}\eta^2(\cdot)\sigma(t)dxdt \leq \iint_{C^+_{\rho,\tau}} \frac{v^p(t,\cdot)-v^p(t-h,\cdot)}{ph}\eta^2(\cdot)\sigma(t)dxdt.$$

Furthermore, noting the definition of u and σ, it follows that

(Quotient term of (2.3))

$$
\begin{aligned}
&=\frac{1}{p}\sum_{n=n_0-[(1-\sigma_2)\tau/h]+1}^{n_0}\int_{B_\rho}\left(v_n^p-v_{n-1}^p\right)\eta^2dx+\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}\left(v_n^p-v_{n-1}^p\right)\sigma_n\eta^2dx\\
&=\frac{1}{p}\int_{B_\rho}v_{n_0}^p\eta^2dx-\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\eta^2dx+\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}\left(v_n^p-v_{n-1}^p\right)\sigma_n\eta^2dx.
\end{aligned}\tag{2.5}
$$

Since $\sigma_{n_0-[\tau/h]+1}=0$, we have, for the third term of the right hand in (2.5),

(The third term of (2.5))

$$
\begin{aligned}
&\leq\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}\left(v_n^p\sigma_n-v_{n-1}^p\sigma_{n-1}\right)\eta^2dx-\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}(\sigma_n-\sigma_{n-1})\int_{B_\rho}v_{n-1}^p\eta^2dx\\
&=\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\eta^2dx-\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\left(\sigma_n-\sigma_{n-1}\right)\int_{B_\rho}v_{n-1}^p\eta^2dx.
\end{aligned}
$$

Here noting the estimations: $\sigma_n-\sigma_{n-1}\leq 3h/\sigma_2\tau$, we obtain the following calculations:

$$
\begin{aligned}
&\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\eta^2dx-\frac{3}{p}(\sigma_2\tau)^{-1}h\sum_{n=n_0-[\tau/h]+1}^{n=n_0-[(1-\sigma_2)\tau/h]-1}\int_{B_\rho}v_n^p\eta^2dx\\
&\leq\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\eta^2dx-\frac{3}{p}(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}v^p\eta^2dxdt.
\end{aligned}\tag{2.6}
$$

Substituting (2.6) into (2.5) gives that

$$
\text{(Quotient term of (2.3))}\geq\frac{1}{p}\int_{B_\rho}v_{n_0}^p\eta^2dx-\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C_{\rho,\tau}^+}v^p\eta^2dxdt.\tag{2.7}
$$

Next we shall deal with the term including spatial derivatives.

(the estimation for spatial derivative's term of (2.3)) Noting that $p<1$ and using Young's inequality, we have

(Spatial derivative's term of (2.3))

$$
\begin{aligned}
&=\frac{4(p-1)}{p^2}\iint_{C_{\rho,\tau}^+}a^{\alpha\beta}D_\beta v^{p/2}D_\alpha v^{p/2}\eta^2\sigma dxdt+\frac{4}{p}\iint_{C_{\rho,\tau}^+}a^{\alpha\beta}D_\beta v^{p/2}v^{p/2}\eta D_\alpha\eta\sigma dxdt\\
&\leq\frac{4\lambda(p-1)}{p^2}\iint_{C_{\rho,\tau}^+}\left|Dv^{p/2}\right|^2\eta^2\sigma dxdt+\frac{4}{p}\iint_{C_{\rho,\tau}^+}a^{\alpha\beta}D_\beta v^{p/2}v^{p/2}\eta D_\alpha\eta\sigma dxdt\\
&\leq\left(\frac{4(p-1)}{p^2}\lambda-\frac{2\varepsilon}{|p|}\mu\right)\iint_{C_{\rho,\tau}}|Dv^{p/2}|^2\eta^2\sigma dxdt-\frac{2\mu}{|p|\varepsilon}\iint_{C_{\rho,\tau}^+}(v^{\frac{p}{2}})^2|D\eta|^2\sigma dxdt.
\end{aligned}
$$

Here,taking $\varepsilon:=-\frac{\lambda(p-1)}{\mu|p|}(>0)$, we obtain

(Spatial derivative's term of (2.3))

$$
\geq\frac{\lambda 2(p-1)}{p^2}\iint_{C_{\rho,\tau}^+}\left|Dv^{p/2}\right|^2\eta^2\sigma dxdt+\frac{2\mu^2}{\lambda(p-1)}\iint_{C_{\rho,\tau}^+}v^p|D\eta|^2\sigma dxdt.\tag{2.8}
$$

Combining (2.8) with (2.7) gives that

$$\frac{1}{p}\int_{B_\rho} v_{n_0}^p \eta^2 dx - \frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2 dxdt$$
$$+\frac{2(p-1)\lambda}{p^2}\iint_{C^+_{\rho,\tau}} \left|Dv^{p/2}\right|^2 \eta^2\sigma dxdt - \frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma dxdt \geq 0.$$

From this inequality, it follows that

$$\frac{1}{p}\int_{B_\rho} v_{n_0}^p\eta^2 dx \geq \frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2 dxdt + \frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma dxdt, \tag{2.9}$$

$$\frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} |Dv^{p/2}|^2\eta^2\sigma dxdt \geq \frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2 dxdt + \frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma dxdt. \tag{2.10}$$

Dividing the both sides of (2.9) and(2.10) by $\frac{1}{p}(<0)$ and $\frac{2\lambda(p-1)}{p^2}(<0)$ respectively, we obtain

$$\int_{B_\rho} v_{n_0}^p\eta^2 dx \leq 3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2 dxdt + \frac{2\mu^2 p}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma dxdt, \tag{2.11}$$

$$\iint_{C^+_{\rho,\tau}} |Dv^{p/2}|^2\eta^2\sigma dxdt \leq \frac{3p(\sigma_2\tau)^{-1}}{2\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p\eta^2 dxdt + \left(\frac{\mu p}{\lambda(p-1)}\right)^2\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma dxdt. \tag{2.12}$$

Noting that $p-1<p<0$, (2.11)and (2.12) become, respectively

$$\int_{B_\rho} v_{n_0}^p\eta^2 dx \leq \max\left(3, \frac{8\mu^2}{\lambda}\right)\left((\sigma_2\tau)^{-1} + (\sigma_1\rho)^{-2}\right)\iint_{C^+_{\rho,\tau}} v^p dxdt, \tag{2.13}$$

$$\iint_{C^+_{\rho,\tau}} |Dv^{p/2}|^2\eta^2\sigma dxdt \leq \max\left(\frac{3}{2\lambda}, \frac{4\mu^2}{\lambda}\right)\left((\sigma_2\tau)^{-1} + (\sigma_1\rho)^{-2}\right)\iint_{C^+_{\rho,\tau}} v^p dxdt. \tag{2.14}$$

Estimating similarly as (2.13), we obtain, for $n; n_0 - [(1-\sigma_2)\tau/h] \leq n \leq n_0$

$$\int_{B_\rho} v_n^p\eta^2 dx \leq \max\left(3, \frac{8\mu^2}{\lambda}\right)\left((\sigma_2\tau)^{-1} + (\sigma_1\rho)^{-2}\right)\iint_{C^+_{\rho,\tau}} v^p dxdt, \tag{2.15}$$

Thus we have

$$\underset{t_{n_0}-(1-\sigma_2)\tau\leq t\leq t_{n_0}}{Sup}\int_{B_\rho} v^p(t,\cdot)\eta^2(\cdot)dx \leq \max\left(3, \frac{8\mu^2}{\lambda}\right)\left((\sigma_2\tau)^{-1} + (\sigma_1\rho)^{-2}\right)\iint_{C^+_{\rho,\tau}} v^p dxdt. \tag{2.16}$$

Next, we shall consider the Case 2. Then we put $\sigma(t)$ as $\sigma \equiv 1$ on $[t_{n_0}-\tau, t_{n_0}]$, so that we have (2.3) with $\sigma \equiv 1$. Let's remark that since u_h is nonnegative in $C^+_{\rho,\tau}(t_{n_0}, x_0)$ and $u_{n_0-[\tau/h]-1} \geq 0$ in $B_\rho(x_0)$, $v = u_h + \varepsilon$ also is nonnegative in $C^+_{\rho,\tau}(t_{n_0}, x_0)$ and $v_{n_0-[\tau/h]-1} = u_{n_0-[\tau/h]-1} + \varepsilon \geq 0$ in $B_\rho(x_0)$. Thus

$$\iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)-v(t-h,\cdot)}{h}(v(t,\cdot))^{p-1}\eta^2(\cdot)\sigma(t)dxdt$$
$$=\iint_{C^+_{\rho,\tau}} \frac{v^p(t,\cdot)-v^{p-1}(t,\cdot)v(t-h,\cdot)}{h}\eta^2(\cdot)dxdt \leq \frac{1}{h}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\eta^2(\cdot)dx,$$

so that we obtain from (2.3)

$$\frac{1}{h}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\eta^2(\cdot)dxdt + \frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha,\beta}D_\beta v^{p/2}D_\alpha v^{p/2}\eta^2 dxdt$$
$$+\frac{4}{p}\iint_{C^+_{\rho,\tau}} a^{\alpha,\beta}D_\beta v^{p/2}v^{p/2}\eta D_\alpha\eta dxdt \geq 0.$$

Noticing that $p<1$ and Young's inequality, we have

$$\frac{1}{h}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\eta^2(\cdot)dxdt$$
$$+\left(\frac{4(p-1)\lambda}{p^2}+\frac{4\mu\varepsilon}{2|p|}\right)\iint_{C^+_{\rho,\tau}}\left|Dv^{p/2}\right|^2\eta^2dxdt + \frac{2\mu}{|p|\varepsilon}\iint_{C^+_{\rho,\tau}}(v^{\frac{p}{2}})^2|D_\alpha\eta|^2dxdt \geq 0. \tag{2.17}$$

Putting $\varepsilon = -\frac{(p-1)\lambda}{|p|\mu}(>0)$ in (2.17) and noting that $\sigma_2\tau < 3h$ give that

$$\frac{3}{\sigma_2\tau}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\eta^2(\cdot)dxdt+\frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}\left|Dv^{p/2}\right|^2\eta^2dxdt+\frac{2\mu^2}{\lambda(1-p)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2dxdt \geq 0. \tag{2.18}$$

Namely we have

$$\frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}\left|Dv^{p/2}\right|^2\eta^2dxdt \leq \frac{3}{\sigma_2\tau}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\eta^2(\cdot)dxdt + \frac{2\mu^2}{\lambda(1-p)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2dxdt.$$

Dividing the both side of this inequality by $\frac{2\lambda(1-p)}{p^2}(>0)$, we have

$$\iint_{C^+_{\rho,\tau}}\left|Dv^{p/2}\right|^2\eta^2dxdt \leq \max\left(\frac{3}{2\lambda},\frac{4\mu^2}{\lambda^2}\right)\left((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\right)\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)dxdt. \tag{2.19}$$

From now on we shall estimate the quantity: $\int_{B_{\rho(1-\sigma_1)}} v^p(t,\cdot)dx$ for $t_{n_0}-(1-\sigma_2)\tau < t < t_{n_0}$. To do this, it is sufficient to estimate the quantity: $\int_{B_{\rho(1-\sigma_1)}} v^p_n(\cdot)dx$ for $n_0 - [(1-\sigma_2)\tau/h] \leq n \leq n_0$. Since $t_{n_0-[(1-\sigma_2)\tau/h]+1} \leq t_n - h < t_n \leq t_{n_0}$ for $n_0-[(1-\sigma_2)\tau/h]+1 \leq n \leq n_0$, so that

$$\int_{B_{\rho(1-\sigma_1)}} v^p_n dx = h/h\int_{B_{\rho(1-\sigma_1)}} v^p_n dx \leq 3(\sigma_2\tau)^{-1}h\int_{B_{\rho(1-\sigma_1)}} v^p_n dx$$
$$=3(\sigma_2\tau)^{-1}\int_{t_n-h}^{t_n}\int_{B_{\rho(1-\sigma_1)}} v^p dxdt \leq 3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_{\rho(1-\sigma_1)}} v^p dxdt$$

For $n_0-[(1-\sigma_2)\tau/h] = n$, we must consider two cases; If $n_0-[(1-\sigma_2)\tau/h] > n_0-[\tau/h]$ *i.e.* $n_0-[(1-\sigma_2)\tau/h] \geq n_0-[\tau/h]+1$, then $t_{n_0-[(1-\sigma_2)\tau/h]} \geq t_{n_0-[\tau/h]}$, so that

$$\int_{B_{\rho(1-\sigma_2)}} v^p_{n_0-[(1-\sigma_2)\tau/h]}dx = h/h\int_{B_{\rho(1-\sigma_2)}} v^p_{n_0-[(1-\sigma_2)\tau/h]}dx$$
$$\leq 3(\sigma_2\tau)^{-1}h\int_{B_{\rho(1-\sigma_2)}} v^p_{n_0-[(1-\sigma_2)\tau/h]}dx = 3(\sigma_2\tau)^{-1}\int_{t_{n_0-[(1-\sigma_2)\tau/h]-1}}^{t_{n_0-[(1-\sigma_2)\tau/h]}}\int_{B_{\rho(1-\sigma_2)}} v^p dxdt$$
$$\leq 3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_{\rho(1-\sigma_2)}} v^p dxdt$$

If $n_0 - [(1-\sigma_2)\tau/h] = n_0 - [\tau/h]$, from that

$$\begin{aligned}&t_{n_0-[\tau/h]} - (t_{n_0}-\tau) = t_{n_0-[(1-\sigma_2)\tau/h]} - (t_{n_0}-\tau)\\ =&t_{n_0} - [(1-\sigma_2)\tau/h]h - t_{n_0} + \tau = -[(1-\sigma_2)\tau/h]h + \tau\\ \geq& -(1-\sigma_2)\tau + \tau = \sigma_2\tau,\end{aligned}$$

we have the following calculations:

$$\begin{aligned}&\int_{B_{\rho(1-\sigma_2)}} v^p_{n_0-[(1-\sigma_2)\tau/h]}dx = \frac{t_{n_0-[(1-\sigma_2)\tau/h]} - (t_{n_0}-\tau)}{t_{n_0-[(1-\sigma_2)\tau/h]} - (t_{n_0}-\tau)}\int_{B_{\rho(1-\sigma_2)}} v^p_{n_0-[(1-\sigma_2)\tau/h]}dx\\ \leq&(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0-[(1-\sigma_2)\tau/h]}}\int_{B_{\rho(1-\sigma_2)}} v^p_{n_0-[(1-\sigma_2)\tau/h]}dx = (\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0-[\tau/h]}}\int_{B_{\rho(1-\sigma_2)}} v^p dxdt\\ \leq&(\sigma_2\tau)^{-1}\int_{t_\tau-\tau}^{t_{n_0}}\int_{B_{\rho(1-\sigma_2)}} v^p dxdt\end{aligned}$$

As a result we have, for n; $n_0 - [(1-\sigma_2)\tau/h] \leq n \leq n_0$

$$\int_{B_{\rho(1-\sigma_1)}} v_n^p dx \leq 3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p dxdt. \tag{2.20}$$

Lemma2.2. *Let u_h be a weak solution of(1.1). If $u_h \geq 0$ in $C^+_{\rho_0,\tau_0}(t_{n_0},x_0) \subset Q$ and $u_{n_0-[\tau_0/h]-1} \geq 0$ in $B_{\rho_0}(x_0)$, then, for $p<0$, there exists a positive constant γ depending only on λ,μ,m and p such that,*

$$\left(\frac{1}{|C^+_{\rho_0,\tau_0}|}\iint_{C^+_{\rho_0,\tau_0}(t_{n_0},x_0)} u_h^p(t,x)dtdx\right)^{\frac{1}{p}} \leq \gamma(2+\rho_0^{-2}\tau_0)^{-p^{-1}(2+\frac{m}{2})} \inf_{(t,x)\in C^+_{\rho_0/2,\tau_0/2}(t_{n_0},x_0)} u_h(t,x) \tag{2.39}$$

If $u_h \geq 0$ in $C^-_{\rho_0,\tau_0}(t_{n'_0},x'_0) \subset Q$ and $u_{n_0} \geq 0$ in $B_{\rho_0}(x_0)$, then, for any $p,q; 0<q<p<1+2/m$, there exists a positive constant γ depending only on λ,μ,m and p such that,

$$\begin{aligned}&\left(\frac{1}{|C^-_{\rho_0/2,\tau_0/2}|}\iint_{C^-_{\rho_0/2,\tau_0/2}(t_{n'_0},x'_0)} u_h^p(t,x)dtdx\right)^{\frac{1}{p}}\\ &\leq\gamma\left(\frac{1}{1-p}\right)^{\frac{m+2}{p}}\left(\frac{1}{|C^-_{\rho_0,\tau_0}|}\iint_{C^-_{\rho_0,\tau_0}(t_{n'_0},x'_0)} u_h^q(t,x)dtdx\right)^{\frac{1}{q}}\end{aligned} \tag{2.40}$$

Proof.The proof is proceeded similarly as in [9]. Here we remark only the following. Making a changing of variables:

$$\begin{cases} x - x_0 = \rho_0 y,\\ t - t_{n_0} = \rho_0^2 s\end{cases} \tag{2.41}$$

and putting

$$\widetilde{u}_h(s,y) = u_h(t_{n_0}+\rho_0^2 s,\ x_0+\rho_0 y),$$

we find that u satisfies the identity: For any s; $-\rho_0^2\tau_0 \leq s \leq 0$ and for all $\varphi = (\varphi^i) \in \overset{\circ}{W}{}^1_2(B_1)$

$$\int_{B_1}\frac{\widetilde{u}_h(s,\cdot) - \widetilde{u}_h(s-h/\rho_0^2,\cdot)}{h/\rho_0^2}\varphi dy + \int_{B_1}\widetilde{a}^{\alpha\beta}(s,\cdot)D_\beta\widetilde{u}_h(s,\cdot)D_\alpha\varphi dy = 0. \tag{2.42}$$

Thus, from noticing that $\widetilde{u}_{-[\tau_0/h]-1} \geq 0$ in B_1 and calculating similarly as (2.1) it follows that

$$\begin{aligned}
&\operatorname*{Sup}_{0\geq t\geq \tilde{\tau}(1-\sigma_2)} \int_{B_{\tilde{\rho}(1-\sigma_1)}(0)} (\tilde{u}_h+\varepsilon)^p(t,\cdot)dy + \iint_{C^+_{\tilde{\rho}(1-\sigma_1),\tilde{\tau}(1-\sigma_2)}(0)} |D(\tilde{u}_h+\varepsilon)^{\frac{p}{2}}|^2 dyds \\
&\leq \gamma\Big((\sigma_1\tilde{\rho})^{-2}+(\sigma_2\tilde{\tau})^{-1}\Big)\iint_{C^+_{\tilde{\rho},\tilde{\tau}}(0)} (\tilde{u}_h+\varepsilon)^2 dyds
\end{aligned} \tag{2.43}$$

holds for $0<\tilde{\rho}<1$, $0<\tilde{\tau}<\theta=\rho_0^{-2}\tau_0$, $\sigma_1,\sigma_2\in(0.1)$, all $p<0$ and for any $\varepsilon>0$.

Lemma2.3. *Let u_h be a weak solution of (1.1). For any $p; 1<p\leq m+2$, then there exists a constant γ depending only on λ,μ,m and p such that, setting $v_h=\max\{\pm u_h,0\}$,*

$$\begin{aligned}
&\operatorname*{Sup}_{t_{n_0}-\tau(1-\sigma_2)\leq t\leq t_{n_0}} \int_{B_{\rho(1-\sigma_1)}(x_0)} v_h^p(t,\cdot)dx + \iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)} \left|Dv_h^{p/2}\right|^2 dxdt \\
&\leq \gamma\frac{p}{p-1}\Big(1+\frac{p}{p-1}\Big)\Big\{\Big((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\Big)\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} v_h^p dxdt \\
&+(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} |u_h|^p(t-h,\cdot)dxdt\Big\}.
\end{aligned} \tag{2.44}$$

holds for any $C^+_{\rho,\tau}(t_{n_0},x_0)\subset\widetilde{Q_{h_0}}$, all $\sigma_1,\sigma_2\in(0,1)$.

Proof.Let $\eta\in C_0^\infty(B_\rho(x_0))$ satisfying $\eta=1$ on $B_{\rho(1-\sigma_1)}(x_0)$, $|D\eta|\leq 2/\sigma_1\rho$ and $\sigma(\cdot)$ be some function defined on $[t_{n_0}-\tau,t_{n_0}]$, of which the definition is given later. At first we consider a case of $1<p\leq 2$. Then we remark that $(u_h^\pm(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)$, $\varepsilon>0$ is belonging to $\overset{\circ}{W}{}_2^1(B_\rho)$ for $t\in[t_{n_0}-\tau,t_{n_0}]$. Testing the identity (1.6) by a function $(u_h^\pm(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)$, and integrating the resultant equality with respect to time variable t in $(t_{n_0}-\tau,t_{n_0}]$, we have

$$\begin{aligned}
&\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} \frac{\pm u_h(t,\cdot)-\pm u_h(t-h,\cdot)}{h}(u_h^\pm(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt \\
&+\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} a^{\alpha\beta}(t,\cdot)D_\beta(\pm u_h(t,\cdot))D_\alpha\Big[(u_h^\pm(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\Big]\sigma(t)dxdt=0.
\end{aligned} \tag{2.45}$$

Now we put

$$v=u_h^\pm$$

and omit a center or vertex of a cube for simplicity. We shall estimate each term of (2.45) in the following manner:

$$\begin{aligned}
&(\text{Quotient term of (2.45)}) \\
&=\iint_{C^+_{\rho,\tau}\cap\{v>0\}} \frac{\pm u_h(t,\cdot)-\pm u_h(t-h,\cdot)}{h}(v(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt \\
&+\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}} \frac{\pm u_h(t,\cdot)-\pm u_h(t-h,\cdot)}{h}(v(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt \\
&\geq\iint_{C^+_{\rho,\tau}\cap\{v>0\}} \frac{\pm u_h(t,\cdot)+\varepsilon-(\pm u_h(t-h,\cdot)+\varepsilon)}{h}(v(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt \\
&+\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}} \frac{\pm u_h(t,\cdot)-\pm u_h(t-h,\cdot)}{h}(\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt.
\end{aligned} \tag{2.46}$$

Here we use the fact that, in a set $\{\pm u_h > 0\}$

$$\pm u_h(t,\cdot) - (\pm u_h(t,\cdot)) \geq v(t,\cdot) - v(t-h,\cdot) = v(t,\cdot) + \varepsilon - (v(t-h,\cdot)+\varepsilon).$$

For the spatial derivative term, we have

$$\begin{aligned}
&(\text{Spatial derivative term of (2.45)})\\
=&(p-1)\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}(v(t,\cdot)+\varepsilon)^{p-2}D_\alpha(v(t,\cdot)+\varepsilon)^{p/2}\eta^2\sigma dxdt\\
&+2\iint_{C^+_{\rho,\tau}\cap\{v>0\}} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}(v(t,\cdot)+\varepsilon)^{p-1}\eta D_\alpha\eta\sigma dxdt\\
&+2\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}} a^{\alpha\beta}D_\beta(v(t,\cdot))^{p/2}\varepsilon^{p-1}\eta D_\alpha\eta\sigma dxdt.
\end{aligned}\tag{2.47}$$

Combining the above estimates (2.46) and (2.47) gives that

$$\begin{aligned}
&\iint_{C^+_{\rho,\tau}\cap\{v>0\}} \frac{v(t,\cdot)+\varepsilon-(v(t-h,\cdot)+\varepsilon)}{h}(v(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt\\
&+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}D_\alpha(v(t,\cdot)+\varepsilon)^{p/2}\eta^2\sigma dxdt\\
&+2\iint_{C^+_{\rho,\tau}\cap\{v>0\}} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}(v(t,\cdot)+\varepsilon)^{p-1}\eta D_\alpha\eta\sigma dxdt\\
&+\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}} \frac{\pm u_h(t,\cdot)-\pm u_h(t-h,\cdot)}{h}\eta^2(\cdot)\sigma(t)dxdt\\
&+2\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}} a^{\alpha\beta}D_\beta(v(t,\cdot))^{p/2}\eta D_\alpha\eta\sigma dxdt \leq 0.
\end{aligned}\tag{2.48}$$

Adding (2.48) by

$$\begin{aligned}
&\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}} \frac{v(t,\cdot)+\varepsilon-(v(t-h,\cdot)+\varepsilon)}{h}(v(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt\\
=&\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}} \frac{v(t-h,\cdot)}{h}\varepsilon^{p-1}\eta^2\sigma(t)dxdt \leq 0
\end{aligned}$$

and noting that

$$\begin{aligned}
&\iint_{C^+_{\rho,\tau}\cap\{v>0\}} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}(v(t,\cdot)+\varepsilon)^{p-1}\eta D_\alpha\eta\sigma dxdt\\
=&\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}(v(t,\cdot)+\varepsilon)^{p-1}\eta D_\alpha\eta\sigma dxdt,
\end{aligned}\tag{2.49}$$

we obtain

$$\begin{aligned}
&\iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)+\varepsilon-(v(t-h,\cdot)+\varepsilon)}{h}(v(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)dxdt\\
&+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}D_\alpha(v(t,\cdot)+\varepsilon)^{p/2}\eta^2\sigma dxdt\\
&+\frac{4}{p}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} a^{\alpha\beta}D_\beta(v(t,\cdot)+\varepsilon)^{p/2}(v(t,\cdot)+\varepsilon)^{p/2}\eta D_\alpha\eta\sigma dxdt\\
&+\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{v\leq 0\}}\left[\frac{v(t,\cdot)-v(t-h,\cdot)}{h}\eta^2(\cdot)\sigma(t)+2a^{\alpha\beta}D_\beta(v(t,\cdot))\eta D_\alpha\eta\sigma\right]dxdt \leq 0.
\end{aligned}$$

If $\sigma_2\tau > 3h$, then we are able to proceed the the calculations similarly as a case of $p < 0$ in the proof of Lemma2.1. Now we take $\sigma(t)$ as a cut-off function defined in (2.4) in the proof of Lemma2.1, so that we conclude that, for $n; n_0 - [(1-\sigma_2)\tau/h] \le n \le n_0$

$$\begin{aligned}&\int_{B_\rho}(v_n+\varepsilon)^p\eta^2dx+\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{v\le 0\}}\Big[\frac{v(t,\cdot)-v(t-h,\cdot)}{h}\eta^2(\cdot)\sigma(t)\\&+2a^{\alpha\beta}D_\beta v(t,\cdot)\eta D_\alpha\eta\sigma\Big]dxdt\le\max\Big(3,\frac{8\mu^2}{\lambda}\Big)\Big((\sigma_2\tau)^{-1}+(\sigma_1\rho)^{-2}\Big)\iint_{C^+_{\rho,\tau}}(v+\varepsilon)^pdxdt\end{aligned}\tag{2.50}$$

and that

$$\begin{aligned}&\iint_{C^+_{\rho,\tau}}|D(v+\varepsilon)^{p/2}|^2\eta^2\sigma dxdt\\&+\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{v\le 0\}}\Big[\frac{v(t,\cdot)-v(t-h,\cdot)}{h}\eta^2(\cdot)\sigma(t)+2a^{\alpha\beta}D_\beta(v(t,\cdot))\eta D_\alpha\eta\sigma\Big]dxdt\\&\le\max\Big(3,\frac{8\mu^2}{\lambda}\Big)\Big((\sigma_2\tau)^{-1}+(\sigma_1\rho)^{-2}\Big)\iint_{C^+_{\rho,\tau}}(v+\varepsilon)^pdxdt.\end{aligned}\tag{2.51}$$

If $\sigma_2\tau \le 3h$, let's take $\sigma\equiv 1$ on $[t_{n_0}-\tau, t_{n_0}]$, so that we have the inequality which is obtained from putting $\sigma\equiv 1$ in (2.45). For the quotient term, using Young's inequality and noting that $(\sigma_2\tau)^{-1}\le 3h^{-1}$, we have

$$\begin{aligned}&\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}\frac{\pm u_h(t,\cdot)+\varepsilon-(\pm u(t-h,\cdot)+\varepsilon)}{h}(u^\pm(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)dxdt\\&\ge\frac{1}{p}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}\frac{(v(t,\cdot)+\varepsilon)^p-|\pm u(t-h,\cdot)+\varepsilon|^p}{h}\eta^2(\cdot)dxdt\\&\ge-\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}|v(t-h,\cdot)+\varepsilon|^p\eta^2(\cdot)dxdt.\end{aligned}$$

Making calculations similarly as (2.51), we have

$$\begin{aligned}&\iint_{C^+_{\rho,\tau}(t_{n_0}x_0)}|D(v+\varepsilon)^{p/2}|^2\eta^2\sigma dxdt\\&+\frac{p^2}{2\lambda(p-1)}\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{\pm u_h\le 0\}}\Big[\frac{v(t,\cdot)-v(t-h,\cdot)}{h}\eta^2(\cdot)\sigma(t)+2a^{\alpha\beta}D_\beta(v(t,\cdot))\eta D_\alpha\eta\sigma\Big]dxdt\\&\le\frac{\mu^2p^2}{\lambda^2(p-1)^2}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}(v+\varepsilon)^p|D\eta|^2dxdt+\frac{3p(\sigma_2\tau)^{-1}}{2\lambda(p-1)}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}|v(t-h,\cdot)+\varepsilon|^p\eta^2dxdt.\end{aligned}\tag{2.52}$$

Also we remark that the calculation of getting (2.20) is justified in this case since $v+\varepsilon = u^\pm_h+\varepsilon\ge 0$.

Finally tending ε to 0 in (2.50), (2.51) and (2.52) and noting Fatou's lemma, we obtain (2.44) for $1<p\le 2$.

Next we deal with a case of $p>2$. Then we remark that $[(u^\pm_h(t,\cdot))^{(M)}]^{p-1}\eta^2(\cdot)\sigma(t)$, $M>0$ is admissible as a test function in the identity(1.1) for any $t\in[t_{n_0}-\tau, t_{n_0}]$, where $v^{(M)}$ is defined as follows:

$$v^{(M)}=\begin{cases}M, & v\ge M\\ v, & v<M,\end{cases}$$

$\eta(\cdot)$ is the same function as in a case of $1 < p \le 2$ and $\sigma(t)$ is some function on $[t_{n_0} - \tau, t_{n_0}]$ given later. Taking a function $\varphi = [(u_h^{\pm}(t.\cdot))^{(M)}]^{p-1}\ \eta^2(\cdot)\sigma(t)$ in the identity (1.6) and integrating the resultant inequality with respect to t in $(t_{n_0} - \tau, t_{n_0})$, we have

$$\begin{aligned}&\iint_{C^+_{\rho,\tau}} \frac{\pm u_h(t,\cdot) - \pm u_h(t-h,\cdot)}{h}[(u_h^{\pm}(t,\cdot))^{(M)}]^{p-1}\eta^2(\cdot)\sigma(t)dxdt\\ &+\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta(\pm u_h)(t,\cdot)D_\alpha\Big[[(u_h^{\pm}(t,\cdot))^{(M)}]^{p-1}\eta^2(\cdot)\Big]\sigma(t)dxdt = 0.\end{aligned} \tag{2.53}$$

Similarly as in a case of $1 < p \le 2$, let's put $v = u_h^{\pm}$. We shall estimate each term of (2.53). Firstly we consider Case1 : $\sigma_2\tau > 3h$. Then we put $\sigma(t)$ the same function as in (2.4). Noting the definition of σ, we have

$$\begin{aligned}&(\text{Quotient term of (2.53)})\\ =&\iint_{C^+_{\rho,\tau}} \frac{\pm u(t,\cdot) - \pm u(t-h,\cdot)}{h}[v^{(M)}(t,\cdot)]^{p-1}\eta^2(\cdot)\sigma(t)dxdt\\ =&h\sum_{n=n_0-[\tau/h]+2}^{n_0}\int_{B_\rho(x_0)} \frac{\pm u_n - \pm u_{n-1}}{h}[(v^{(M)}]^{p-1}\eta^2(\cdot)\sigma_n dx\\ =&\sum_{n=n_0-[\tau/h]+2}^{n_0}\int_{B_\rho(x_0)} (\pm u_n - \pm u_{n-1})[v^{(M)}]^{p-1}\eta^2(\cdot)\sigma_n dx\end{aligned}$$

Here, noting that

$$\begin{aligned}&(\pm u_n \mp u_{n-1})[(u_n^{\pm})^{(M)}]^{p-1} \le (u_n^{\pm} - u_{n-1}^{\pm})[(v_n)^{(M)}]^{p-1}\\ =&v[(v_n)^{(M)}]^{p-1} - v_{n-1}[(v_{n-1})^{(M)}]^{p-1} - v_{n-1}([(v_n)^{(M)}]^{p-1} - [(v_{n-1})^{(M)}]^{p-1})\\ \le&v_n[(v_n)^{(M)}]^{p-1} - v_{n-1}[(v_{n-1})^{(M)}]^{p-1} - (v_{n-1})^{(M)}([(v_n)^{(M)}]^{p-1} - [(v_{n-1})^{(M)}]^{p-1}),\end{aligned}$$

we obtain

$$\begin{aligned}&(\text{Quotient term of (2.53)})\\ \ge&\sum_{n=n_0-[\tau/h]+2}^{n_0}\int_{B_\rho(x_0)} \Big(v_n[(v_n)^{(M)}]^{p-1} - v_{n-1}[(v_{n-1})^{(M)}]^{p-1}\Big)\sigma_n\eta^2 dx\\ -&\sum_{n=n_0-[\tau/h]+2}^{n_0}\int_{B_\rho(x_0)} (v_{n-1})^{(M)}([(v_n)^{(M)}]^{p-1} - [(v_{n-1})^{(M)}]^{p-1})\sigma_n\eta^2 dx.\end{aligned} \tag{2.54}$$

We deal with the first term of (2.54).

$$\begin{aligned}&(\text{First term of (2.54)})\\ =&\sum_{n=n_0-[(1-\sigma_2)\tau/h]+1}^{n_0}\int_{B_\rho} \Big(v_n(v_n^{(M)})^{p-1} - v_{n-1}(v_{n-1}^{(M)})^{p-1}\Big)\eta^2 dx\\ +&\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho} \Big(v_n(v_n^{(M)})^{p-1} - v_{n-1}(v_{n-1}^{(M)})^{p-1}\Big)\sigma_n\eta^2 dx\\ =&\int_{B_\rho(x_0)} v_{n_0}(v_{n_0}^{(M)})^{p-1}\eta^2 dx - \int_{B_\rho(x_0)} v_{n_0-[(1-\sigma_2)\tau/h]}(v^{(M)}_{n_0-[(1-\sigma_2)\tau/h]})^{p-1}\eta^2 dx\\ +&\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho} \Big(v_n(v_n^{(M)})^{p-1}\sigma_n - v_{n-1}(v_{n-1}^{(M)})^{p-1}\sigma_{n-1}\Big)\eta^2 dx\\ -&\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}(\sigma_n - \sigma_{n-1})\int_{B_\rho} v_{n-1}(v_{n-1}^{(M)})^{p-1}\eta^2 dx\end{aligned} \tag{2.55}$$

Noting the definition of σ_n and that $\sigma_n - \sigma_{n-1} \le 3h/\sigma_2\tau$, we obtain, from (2.55)

$$
\begin{aligned}
&(\text{First term of } (2.54)) \\
&\geq \int_{B_\rho(x_0)} v_{n_0}(v_{n_0}^{(M)})^{p-1}\eta^2\,dx - 3(\sigma_2\tau)^{-1}h\sum\nolimits_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho} v_{n-1}(v_{n-1}^{(M)})^{p-1}\eta^2\,dx \\
&\geq \int_{B_\rho(x_0)} v_{n_0}(v_{n_0}^{(M)})^{p-1}\eta^2\,dx - 3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho} v(t-h,\cdot)(v^{(M)})^{p-1}(t-h,\cdot)\eta^2\,dxdt.
\end{aligned} \tag{2.56}
$$

Next we make a estimate for the second term of (2.54). By Young's inequality, we have

$$
\begin{aligned}
&(\text{Second term of } (2.54)) \geq -\frac{p-1}{p}\sum\nolimits_{n=n_0-[\tau/h]+2}^{n_0}\int_{B_\rho(x_0)}\Big((v_n^{(M)})^p - (v_{n-1}^{(M)})^p\Big)\sigma_n\eta^2\,dx \\
&= -\frac{p-1}{p}\sum\nolimits_{n=n_0-[(1-\sigma_2)\tau/h]+1}^{n_0}\int_{B_\rho}\Big((v_n^{(M)})^p - (v_{n-1}^{(M)})^p\Big)\eta^2\,dx \\
&\qquad -\frac{p-1}{p}\sum\nolimits_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}\Big((v_n^{(M)})^p - (v_{n-1}^{(M)})^p\Big)\sigma_n\eta^2\,dx.
\end{aligned}
$$

Here, noting the identity:

$$
(a_n - a_{n-1})b_n = a_nb_n - a_{n-1}b_{n-1} - a_{n-1}(b_n - b_{n-1}),
$$

we have calculations:

$$
\begin{aligned}
&(\text{Second term of } (2.54)) \\
&= -\frac{p-1}{p}\int_{B_\rho(x_0)}(v_{n_0}^{(M)})^p\eta^2\,dx - \frac{p-1}{p}\int_{B_\rho(x_0)}(v_{n_0-(1-\sigma_2)\tau/h]}^{(M)})^p\eta^2\,dx \\
&\quad -\frac{p-1}{p}\sum\nolimits_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}\Big((v_n^{(M)})^p\sigma_n - (v_{n-1}^{(M)})^p\sigma_{n-1}\Big)\eta^2\,dx \\
&\quad -\frac{p-1}{p}\sum\nolimits_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}(\sigma_n - \sigma_{n-1})\int_{B_\rho}(v_{n-1}^{(M)})^p\eta^2\,dx.
\end{aligned}
$$

Moreover we recall that $\sigma_{n_0-[\tau/h]+1} = 0$ and that $\sigma_n - \sigma_{n-1} \le 3(\sigma_2\tau)^{-1}$, so that we have

$$
\begin{aligned}
&(\text{Second term of } (2.54)) \\
&\geq -\frac{p-1}{p}\int_{B_\rho(x_0)}(v_{n_0}^{(M)})^p\eta^2\,dx - \frac{p-1}{p}3(\sigma_2\tau)^{-1}\sum\nolimits_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}h\int_{B_\rho}(v_{n-1}^{(M)})^p\eta^2\,dx \\
&\geq -\frac{p-1}{p}\int_{B_\rho(x_0)}(v_{n_0}^{(M)})^p\eta^2\,dx - \frac{p-1}{p}3(\sigma_2\tau)^{-1}\int_{t_{n_0}}^{t_{n_0}-\tau}\int_{B_\rho}(v^{(M)})^p(t,\cdot)\eta^2\,dxdt.
\end{aligned} \tag{2.57}
$$

Substituting (2.56) and (2.57) into (2.54) gives that

$$
\begin{aligned}
&(\text{Quotient term of } (2.53)) \\
&\geq \int_{B_\rho(x_0)} v_{n_0}(v_{n_0}^{(M)})^{p-1}\eta^2\,dx - 3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho} v_{n-1}(v_{n-1}^{(M)})^{p-1}\eta^2\,dxdt \\
&\quad -\frac{p-1}{p}\int_{B_\rho(x_0)}(v_{n_0}^{(M)})^p\eta^2\,dx - 3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(v_{n-1}^{(M)})^p\eta^2\,dxdt
\end{aligned} \tag{2.58}
$$

From now on we treat the spatial derivatives term:

$$
\begin{aligned}
&(\text{Spatial derivatives term of (2.53)})\\
=&(p-1)\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta(\pm u(t,\cdot))^{(M)}[(\pm u(t,\cdot))^{(M)}]^{p-2}D_\alpha v^{(M)}(t,\cdot)\eta^2(\cdot)\sigma(t)dxdt\\
&+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v(t,\cdot)(v^{(M)})^{p-1}(t,\cdot)\eta D_\alpha\eta(\cdot)\sigma(t)dxdt\\
=&\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta(v^{(M)})^{\frac{p}{2}}D_\alpha(v^{(M)})^{\frac{p}{2}}\eta^2\sigma dxdt\\
&+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v(v^{(M)})^{p-1}\eta D_\alpha\eta\sigma dxdt.
\end{aligned}
\tag{2.59}
$$

Combining (2.58) with (2.59), we have

$$
\begin{aligned}
&\int_{B_\rho(x_0)} v_{n_0}(v^{(M)}_{n_0})^{p-1}\eta^2dx-3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho} v(t,\cdot)(v^{(M)})^{p-1}(t,\cdot)\eta^2dxdt\\
&-\frac{p-1}{p}\int_{B_\rho(x_0)}(v^{(M)}_{n_0})^p\eta^2dx-\frac{3(p-1)}{p}(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(v^{(M)})^p(t,\cdot)\eta^2dxdt\\
&+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta(v^{(M)})^{p/2}D_\alpha(v^{(M)})^{p/2}\eta^2(\cdot)\sigma(t)dxdt\\
&+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v(v^{(M)})^{p-1}\eta D_\alpha\eta(\cdot)\sigma(t)dxdt\le 0
\end{aligned}
\tag{2.60}
$$

Here we remark that the above estimates getting (2.60) is valid if changing n_0 by n; $n_0-[(1-\sigma_2)\tau/h]\le n\le n_0$, so that we have, for t; $t_{n_0}-(1-\sigma_2)\tau\le t\le t_{n_0}$

$$
\begin{aligned}
&\int_{B_\rho(x_0)} v(t,\cdot)(v^{(M)})^{p-1}(t,\cdot)\eta^2dx+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}D_\beta(v^{(M)})^{p/2}D_\alpha(v^{(M)})^{p/2}\eta^2\sigma dxdt\\
&+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v(v^{(M)})^{p-1}\eta D_\alpha\eta\sigma dxdt\le 3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho} v(v^{(M)})^{p-1}\eta^2dxdt\\
&+\frac{p-1}{p}\int_{B_\rho(x_0)}(v^{(M)})^p(t,\cdot)\eta^2dx+\frac{3(p-1)}{p}(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(v^{(M)})^p(t,\cdot)\eta^2(\cdot)dxdt
\end{aligned}
$$

Case2. Now we shall deal with a case of $\sigma_2\tau\le 3h$. Let's put $\sigma(t)$ as $\sigma\equiv 1$ on $[t_{n_0}-\tau,t_{n_0}]$, so that we obtain (2.45) with setting $\sigma\equiv 1$. For the quotient term we make estimate as follows:

$$
\begin{aligned}
(\text{Quotient term})\ge&\iint_{C^+_{\rho,\tau}}\frac{v^{(M)}(t,\cdot)-(\pm u_h)(t-h,\cdot)}{h}[v^{(M)}]^{p-1}(t,\cdot)\eta^2(\cdot)dxdt\\
=&\iint_{C^+_{\rho,\tau}}\frac{[v^{(M)}]^p(t,\cdot)-(\pm u_h)(t-h,\cdot)[v^{(M)}]^{p-1}(t,\cdot)}{h}\eta^2(\cdot)dxdt.
\end{aligned}
$$

Then Young's inequality yields that

$$
\begin{aligned}
&(\text{Quotient term})\\
\ge&\frac{1}{p}\iint_{C^+_{\rho,\tau}}\frac{[v^{(M)}]^p(t.\cdot)-|u_h|^p(t-h,\cdot)}{h}\eta^2(\cdot)dxdt\ge-\frac{1}{ph}\iint_{C^+_{\rho,\tau}}|u_h|^p(t-h,\cdot)\eta^2(\cdot)dxdt.
\end{aligned}
\tag{2.61}
$$

For the spatial derivatives term we have (2.59). We also recall that (2.20) holds for $v^{(M)}$ in this case. Thus we deduce from (2.20), (2.59) and (2.61) that, for $t; t_{n_0} - \tau(1-\sigma_2) \le t \le t_{n_0}$

$$\begin{aligned}&\int_{B_\rho(x_0)} (v^{(M)})^p(t,\cdot)\eta^2 dx - 3(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}(v^{(M)})^p dxdt\\ &+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta(v^{(M)})^{\frac{p}{2}}D_\alpha(v^{(M)})^{\frac{p}{2}}\eta^2(\cdot)\sigma(t)dxdt\\ &+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v(v^{(M)})^{p-1}\eta D_\alpha\eta(\cdot)\sigma(t)dxdt - \frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}|u_h|^p(t-h,\cdot)\eta^2(\cdot)dxdt \le 0\end{aligned} \tag{2.62}$$

As a result we obtain that (2.60), (2.62) is valid in a case of $\sigma_2\tau > 3h$ and $\sigma_2\tau \le 3h$ respectively.

Now, noticing that, by Young's inequality

$$\left|\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}D_\beta v(v^{(M)})^{p-1}\eta D_\alpha\eta dxdt\right| \le \frac{1}{2}\mu\iint_{C^+_{\rho,\tau}}|Dv|^2\eta^2dxdt + \frac{1}{2}\mu\iint_{C^+_{\rho,\tau}}(v^{(M)})^{p-1}|D\eta|^2dxdt, \tag{2.63}$$

we are able to pass M to the limit in (2.60) and (2.62) if $p=2$. From it, we obtain that, for any $t; t_{n_0} - (1-\sigma_2)\tau \le t \le t_{n_0}$

$$\begin{aligned}&\frac{1}{2}\int_{B_\rho} v^2(t,\cdot)\eta^2dx + \frac{\lambda}{2}\iint_{C^+_{\rho,\tau}}|Dv|^2\eta^2(\cdot)\sigma(t)dxdt\\ \le& 3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^2\eta^2dxdt + \frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}|u_h|^2(t-h,\cdot)\eta(\cdot)dxdt + \frac{2\mu^2}{\lambda}\iint_{C^+_{\rho,\tau}} v^2|D\eta|^2dxdt.\end{aligned} \tag{2.64}$$

Then Sobolev's type inequality(see [9],p76) implies that

$$v \in L^{2(1+\frac{2}{m})}_{\text{loc}}.$$

Noting (2.63) again, we find it justified to pass M to the limit in (2.60) and (2.62) for $p; 2 < p \le 2(1+\frac{2}{m})$ respectively. Repeating the above procedure inductively(see the proof of Lemma2.2), we deduce that, for any $t; t_{n_0} - (1-\sigma_2)\tau \le t \le t_{n_0}$ and all $p;\ 2 < p \le m+2$

$$\begin{aligned}&\frac{1}{p}\int_{B_\rho} v^p(t,\cdot)\eta^2dx - 3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2dxdt + \frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v^{\frac{p}{2}}D_\alpha v^{\frac{p}{2}}\eta^2\sigma dxdt\\ &+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v v^{p-1}\eta D_\alpha\eta\sigma dxdt - \frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}|u_h|^p(t-h,\cdot)\eta(\cdot)dxdt \le 0.\end{aligned} \tag{2.65}$$

As a result we conclude from (2.65) that, for any $p;\ 2 < p \le m+2$

$$\begin{aligned}&\frac{1}{p}\int_{B_\rho} v^p(t,\cdot)\eta^2dx + \frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}|Dv^{\frac{p}{2}}|^2\eta^2(\cdot)\sigma(t)dxdt\\ \le&\frac{3}{\sigma_2\tau}\iint_{C^+_{\rho,\tau}} v^p\eta^2dxdt + \frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2dxdt + \frac{3}{p\sigma_2\tau}\iint_{C^+_{\rho,\tau}}|u_h|^p(t-h,\cdot)\eta(\cdot)dxdt\end{aligned}$$

for any $C^+_{\rho,\tau}(t_{n_0}, x_0) \subset \widetilde{Q_{h_0}}$.

3.Bounds for weak solutions.

Now we describe the boundedness of weak solutions of (1.1). Firstly we shall note Caccioppoli inequality to DeGiorgie's ones, but omit the proof(refer to [4]).

Lemma3.1.(Caccioppoli type inequality analogue to DeGiorgie's ones). *Let u_h be a weak solution of (1.1). Then, there exists a positive constant γ independent of h and u_h such that, setting $v_h = \pm u_h$,*

$$\begin{aligned}
&\mathop{Sup}_{t_{n_0}-\tau(1-\sigma_2)\le t\le t_{n_0}} \int_{B_{\rho(1-\sigma_1)}(x_0)} (v_h-k)^{+p}(t,\cdot)dx + \iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)} \left|D(v_h-k)^{+\frac{p}{2}}\right|^2 dxdt \\
&\le \gamma\Big((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\Big)\iint_{C^-_{\rho,\tau}} (v_h-k)^{+p}dxdt + \frac{1}{p}(\sigma_2\tau)^{-1}\Big(\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} |v_h|^q dxdt\Big)^{\frac{p}{q}} \\
&\quad\times |C^+_{\rho,\tau}(t_{n_0},x_0)\cap\{w_h>k\}|^{1-\frac{p}{q}} \\
&\quad \text{with some} \quad q>(m+2)p/2
\end{aligned} \tag{3.1}$$

holds for any $k\ge 0$, $\sigma_1,\sigma_2\in(0.1)$, $C^+_{\rho,\tau}(t_{n_0},x_0)\subset Q$ and all $p; 1<p\le 2$.

By exploiting Lemma3.1 and carrying out the iterative procedure similarily as in [8],$p105$ (and remark the proof of Lemma2.2), we obtain the boundedness of weak solutions of (1.1).

Lemma3.2 (A LOCAL BOUNDEDNESS OF u_h). *Let u_h be a weak solution of (1.1). Then there exists a positive constant γ independent of h and u_h such that, setting $v_h = \pm u_h$*

$$\begin{aligned}
\mathop{Sup}_{C^+_{\rho_0/2,\tau_0/2}(t_{n_0},x_0)} v_h \le \gamma\Bigg\{&\Big(\frac{1}{|C^+_{\rho_0,\tau_0}|}\iint_{C^+_{\rho_0,\tau_0}(t_{n_0},x_0)}(v_h)^p dxdt\Big)^{\frac{1}{p}}\Big(1+{\tau_0}^{-\frac{1}{2}}\rho_0\Big)^{\frac{1}{p}} \\
&+\Big(\frac{1}{|C^+_{\rho_0,\tau_0}|}\iint_{C^+_{\rho_0,\tau_0}(t_{n_0},x_0)}(v_h)^q dxdt\Big)^{\frac{1}{q}}\Bigg\} \\
&\text{with some } q>p(m+2)/2
\end{aligned} \tag{3.15}$$

holds for $C^+_{\rho_0,\tau_0}(t_{n_0},x_0)\subset \widetilde{Q_{h_0}}$ and any $p;\ 1<p\le 2$.

4. Estimates for logu_h

We shall need the following lemmata. For the proof we can refer to [6],[11].

Lemma4.1. *(John-Nirenberg estimate of elliptic version) Let u be integrable in a cube B_0 and assume that there is a constant κ such that, for every parallel subcube $B\subset B_0$, we have*

$$\frac{1}{|B|}\int_B |u-\overline{u}_B|dx \le \kappa$$

Then, setting

$$S_\sigma := \{x\in B_0 : |u-\overline{u}_{B_0}|\ge\sigma\},$$

there exist positive constants a,α depending only on m such that

$$|S_\sigma|\le e^{\alpha a}e^{-\alpha\sigma\kappa^{-1}}|B_0|. \tag{4.1}$$

holds for $\sigma > 0$.

Lemma4.2.(John-Nirenberg estimate of parabolic version) Let u be a integrable function in C_R for which

$$\frac{1}{|C_r^+||C_r^-|}\iint_{(t',x')\in C_r^+}\iint_{(t,x)\in C_r^-}\varphi(u(t',x')-u(t,x))dtdxdt'dx' \leq \gamma$$

holds for all pairs C_r^+ and C_r^- in C_R, where $\varphi(s) := \begin{cases} \sqrt{s}, & s > 0, \\ 0, & s \leq 0. \end{cases}$ Then there exist positive constants ξ and γ independent of u such that

$$\frac{1}{|D_R^+||D_R^-|}\iint_{(t',x')\in D_R^+}\iint_{(t,x)\in D_R^-}\Psi(u(t',x')-u(t,x))dtdxdt'dx' \leq 1, \tag{4.2}$$

where $\Psi(s) := \gamma^{-1}e^{\xi s}$.

Now we shall give the fundamental estimate for $-\mathrm{log}u_n$ $(1 \leq n \leq N)$.

Lemma4.3. *Let* u_h *be a weak solution of* (1.1) *and us take a cube* $B_{2\rho}(x_0) \subset \Omega$ *arbitrarily. Then there exists a constant* γ *independent of* h *and* u_h *such that, if* $u_n, u_{n-1}(2 \leq n \leq N)$ *is nonnegative in* $B_{2\rho}(x_0)$ *and setting* $v_n = -\mathrm{log}u_n(1 \leq n \leq N)$,

$$\frac{1}{|B_r|}\int_{B_r(y)}|v_n - \overline{v_n}_{B_r(y)}|dx \leq \gamma\left(\frac{16\mu^2}{\lambda^2}+\frac{2\rho^2}{\lambda h}\right)^{\frac{1}{2}} \tag{4.3}$$

holds for any $r \leq \rho$ *and* $y \in B_\rho(x_0)$.

Proof.We take a domain $B_r(x) \subset B_{2\rho}(x_0)$ arbitrarily and fix it. Now, testing the identity (1.3) by a function: $(u_n)^{-1}\eta^2$ for $\eta \in C_0^\infty(B_{2r})$, $\eta = 1$ on B_r and $|D\eta|^2 \leq 4r^{-2}$, we have

$$\begin{aligned}\frac{1}{h}\int_{B_{2r}}\left(1-\frac{u_{n-1}(x)}{u_n(x)}\right)\eta^2dx - \int_{B_{2r}}a_n^{\alpha\beta}D_\beta\mathrm{log}u_nD_\alpha\mathrm{log}u_n\eta^2dx \\ +2\int_{B_{2r}}a_n^{\alpha\beta}D_\beta\mathrm{log}u_n\eta D_\alpha\eta dx = 0.\end{aligned} \tag{4.4}$$

Noting the nonnegativity of $\frac{1}{h}\int_{B_{2r}}\frac{u_{n-1}(x)}{u_n(x)}\eta^2dx$, we have the following calculations:

$$\begin{aligned}\lambda\int_{B_{2r}}|D\mathrm{log}u_n|^2\eta^2dx \leq& \int_{B_{2r}}a_n^{\alpha\beta}D_\beta\mathrm{log}u_nD_\alpha\mathrm{log}u_n\eta^2dx + \frac{1}{h}\int_{B_{2r}}\eta^2dx \\ \leq& \varepsilon\mu\int_{B_{2r}}|D\mathrm{log}u_n|^2\eta^2dx + \frac{\mu}{\varepsilon}\int_{B_{2r}}|D\eta|^2dx + \frac{1}{h}\int_{B_{2r}}\eta^2dx.\end{aligned} \tag{4.5}$$

From using that $|D\eta| \leq 2r^{-1}$ and taking $\varepsilon = \frac{\lambda}{2\mu}$ in (4.5), it follows that

$$\frac{1}{|B_{2r}|}\int_{B_{2r}}|D\mathrm{log}u_n|^2\eta^2dx \leq \frac{2}{\lambda}\left(\frac{8\mu^2}{\lambda}+\frac{1}{h}\right). \tag{4.6}$$

Adopting Hölder and Poincáre inequality for (4.6) gives that

$$
\begin{aligned}
&\frac{1}{|B_r|}\int_{B_r}|v_n-\frac{1}{|B_r|}\int_{B_r}v_n|^2dx \le \left(\frac{1}{|B_r|}\int_{B_r}|v_n-\frac{1}{|B_r|}\int_{B_r}v_n|dx\right)^{\frac{1}{2}} \\
\le&|B_r|^{-\frac{1}{2}}\left\{\gamma r^2|B_{2r}|\times\left(\frac{8\mu^2}{\lambda}\frac{1}{r^2}+\frac{1}{h}\right)\frac{2}{\lambda}\right\}^{\frac{1}{2}}=\gamma\left\{\left(\frac{8\mu^2}{\lambda}+\frac{r^2}{h}\right)\frac{2}{\lambda}\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{4.7}
$$

Therefore we have shown Lemma4.3.

Remark. u_n^{-1} is not admissible as a test function in the identity (1.3). However, by testing the identity by $(u_n+\varepsilon)^{-1}\eta^2$, calculating similarly as above and tending ε to 0 in the resultant inequality, we have (4.3).

Lemma4.4. *Let u_h be a weak solution of* (1.1) *and us take a cube $B_{2\rho}(x_0)\subset\Omega$ arbitrarily. Then there exist positive constants a,α independent of h and u_h (depending only on m) such that, if $u_n,u_{n-1}(2\le n\le N)$ is nonnegative in $B_{2\rho}(x_0)$ and setting $v_n=-\log u_n(1\le n\le N)$, $\kappa=\kappa(\rho)=\gamma\left(\frac{16\mu^2}{\lambda^2}+\frac{2\rho^2}{\lambda h}\right)^{\frac{1}{2}}$,*

$$
|\{x\in B_\rho(x_0):|v_n(x)-\overline{v_n}_{B_\rho}|>\sigma\}|\le e^{\alpha a}e^{-\alpha\sigma\kappa^{-1}}|B_\rho| \tag{4.8}
$$

holds.

Proof.Since $u_n,u_{n-1}\ge 0$ in $B_{2\rho}(x_0)$, from Lemma4.3, it follows that (4.3) holds for any $B_r\subset B_\rho(x_0)$. Thus, by applying Lemma4.1 for u_n in $B_\rho(x_0)$, we immediately obtain (4.8).

Lemma4.5. *Let u_h be a weak solution of* (1.1). *Then there exists a constant γ independent of h and u_h such that, if u_h is nonnegative in $C_R^+(\bar t,\bar x)\subset Q$ and $u_{[(\bar t-R^2)/h]}\ge 0$ in $B_R(\bar x)$ then, setting $v=-\log u_h$,*

$$
\frac{1}{|C_r^+||C_r^-|}\iint_{(t',x')\in C_r^+}\iint_{(t,x)\in C_r^-}\varphi(v(t',x')-v(t,x))dtdxdt'dx'\le C \tag{4.9}
$$

holds for all pairs C_r^+ and C_r^- in $C_R^+(\bar t,\bar x)$ where $\varphi(s):=\begin{cases}\sqrt{s}, & s>0,\\ 0, & s\le 0.\end{cases}$

Lemma4.6. *Suppose that the same assumption as Lemma4.5 is satisfied. Then there exist positive constants ξ and γ independent of h and u_h such that*

$$
\frac{1}{|D_R^+|}\iint_{D_R^+}u^{-\xi}dtdx\frac{1}{|D_R^-|}\iint_{D_R^-}u^{\xi}dt'dx'\le\gamma. \tag{4.10}
$$

Proof of Lemma4.6. Now suppose that the assertion of Lemma4.5 is valid. Then, by adopting Lemma4.2 for $-\log u_h$ in $C_R^+(\bar t,\bar x)$, we immediately obtain the assertion.

From now on we shall prove Lemma4.5.

Proof of Lemma4.5. Now let's take cubes $C_{r,\tau}^+(t_0,x_0)$ and $C_{r,\tau}^-(t_0,x_0)$ in $C_R^+(\bar t,\bar x)$ arbitrarily and fix them. In the following arguments we use some notations. Here we gather them.

On $B_{2r}(x_0)$,

$$v_n = -\log u_n, \quad v(t,\cdot) = v_n(\cdot), \quad (n-1)h < t \le nh \ (1 \le n \le N)$$

For any $\sigma \ge 0$ and any n; $[(\bar{t} - R^2)/h] \le n \le [\bar{t}/h] + 1$, on $B_{2r}(x_0)$

$$\begin{aligned}
&\widetilde{v_n} = v_n - \int_{B_{2r}} v_0 \eta^2 dy \Big/ \int_{B_{2r}} \eta^2 dy, \quad \widetilde{V_n} = \int_{B_{2r}} \widetilde{v_n} \eta^2 dy \Big/ \int_{B_{2r}} \eta^2 dy \\
&w_n = \widetilde{v_n} - \gamma_2 (n - [t_0/h] - 1)h/r^2, \quad \gamma_2 = 4\mu^2 |B_2|/\lambda |B_1|, \\
&w(t,\cdot) = w_n(\cdot), \quad (n-1)h < t \le nh, \\
&W_n = \widetilde{V_n} - \gamma_2 (n - [t_0/h] - 1)h/r^2, \\
&W(t) = W_n, \quad (n-1)h < t \le nh, \\
&B_\sigma^n = \{x \in B_{2r}(x_0) : w_n(x) > \sigma\}, \\
&B_\sigma(t) = \{x \in B_{2r}(x_0) : w(t,x) > \sigma\} \quad \text{for} \quad t_{[(\bar{t}-R^2)/h]} < t \le t_{[\bar{t}/h]+1}.
\end{aligned} \tag{4.11}$$

First we prove the following:

Claim. *There exists a positive constant γ_1 depending only on m, λ such that*

$$\int_{t_{[t_0/h]+1}}^{t_{[(t_0+r)/h]+1}} |B_\sigma(t)| dt \le \gamma_1 r^2 |B_r| \sigma^{-1}$$

holds for any $\sigma > 0$.

Proof of "Claim". We remark that, since $u_h \ge 0$ in $C_R^+(\bar{t}, \bar{x})$ and $u_{[(\bar{t}-R^2)/h]} \ge 0$ in $B_R(\bar{x})$,

$$u_n \ge 0 \quad \text{in} \quad B_R(\bar{x}) \quad (n;\ [(\bar{t} - R^2)/h] \le n \le [\bar{t}/h] + 1).$$

Testing the identity (1.3) by $\varphi = u_n^{-1}\eta^2$, $\eta \in C_0^\infty(B_{2r}(x_0))$, $|D\eta| \le 2/r$, we have

$$\int_{B_{2r}(x_0)} \frac{u_n - u_{n-1}}{h} u_n^{-1} \eta^2 dx + \int_{B_{2r}(x_0)} a_n^{\alpha\beta} D_\beta u_n D_\alpha \Big\{ u_n^{-1} \eta^2 \Big\} dx = 0. \tag{4.12}$$

Now we make a estimate of each term of (4.12).

(Quotient term of (4.12)) Let's remark that

$$(u_n - u_{n-1})u_n^{-1} = 1 - u_{n-1}u_n^{-1} \le -\log u_{n-1}u_n^{-1} = -\log u_{n-1} - \log u_n^{-1} = \log u_n - \log u_{n-1},$$

which implies that

$$\begin{aligned}
&(\text{Quotient term of (4.12)}) \\
&\le \int_{B_{2r}} \frac{\log u_n - \log u_{n-1}}{h} \eta^2 dx = \frac{-\int_{B_{2r}} (-\log u_n \eta^2) dx + \int_{B_{2r}} (-\log u_{n-1} \eta^2) dx}{h}.
\end{aligned} \tag{4.13}$$

Next we treat the term including spatial derivatives of (4.12).

(Spatial derivative's term of (4.12)) Noting that

$$D_\alpha \Big\{ u_n^{-1} \eta^2 \Big\} = -u_n^{-2} D_\alpha u_n \eta^2 + u_n^{-1} 2\eta D_\alpha \eta,$$

we have

$$
\begin{aligned}
&(\text{Spatial derivative's term of (4.12)})\\
&= \int_{B_{2r}} a_n^{\alpha\beta} D_\beta u_n D_\alpha u_n^{-1} \eta^2 dx\\
&= -\int_{B_{2r}} a_n^{\alpha\beta} D_\beta u_n (u_n)^{-2} D_\alpha u_n \eta^2 dx + 2\int_{B_{2r}} a_n^{\alpha\beta} D_\beta u_n u_n^{-1} \eta D_\alpha \eta dx\\
&= -\int_{B_{2r}} a_n^{\alpha\beta} D_\beta \mathrm{log} u_n D_\alpha \mathrm{log} u_n \eta^2 dx + 2\int_{B_{2r}} a_n^{\alpha\beta} D_\beta \mathrm{log} u_n \eta D_\alpha \eta dx.
\end{aligned}
\tag{4.14}
$$

Combining (4.13) with (4.14), we obtain

$$
\begin{aligned}
&\frac{-\int_{B_{2r}}(-\mathrm{log} u_n \eta^2 dx) + \int_{B_{2r}}(-\mathrm{log} u_{n-1} \eta^2 dx)}{h}\\
&-\int_{B_{2r}} a_n^{\alpha\beta} D_\beta \mathrm{log} u_n D_\alpha \mathrm{log} u_n \eta^2 dx + 2\int_{B_{2r}} a_n^{\alpha\beta} D_\beta \mathrm{log} u_n \eta D_\alpha \eta dx \geq 0,
\end{aligned}
$$

namely,

$$
\begin{aligned}
&\frac{\int_{B_{2r}}(-\mathrm{log} u_n \eta^2 dx) - \int_{B_{2r}}(-\mathrm{log} u_{n-1} \eta^2 dx)}{h}\\
&+\int_{B_{2r}} a_n^{\alpha\beta} D_\beta(-\mathrm{log} u_n) D_\alpha(-\mathrm{log} u_n) \eta^2 dx + 2\int_{B_{2r}} a_n^{\alpha\beta} D_\beta(-\mathrm{log} u_n) \eta D_\alpha \eta dx \leq 0.
\end{aligned}
\tag{4.15}
$$

Here noting $v_n := -\mathrm{log} u_n$, the inequality (4.15) is rewritten in the form:

$$
\begin{aligned}
&\frac{\int_{B_{2r}} v_n \eta^2 dx - \int_{B_{2r}} v_{n-1} \eta^2 dx}{h}\\
&+\int_{B_{2r}} a_n^{\alpha\beta} D_\beta v_n D_\alpha v_n \eta^2 dx + 2\int_{B_{2r}} a_n^{\alpha\beta} D_\beta v_n \eta D_\alpha \eta dx \leq 0.
\end{aligned}
\tag{4.16}
$$

Using the ellipticity condition (1.2) and Young's inequality gives that

$$
\int_{B_{2r}} a_n^{\alpha\beta} D_\beta v_n D_\alpha v_n \eta^2 dx \geq \lambda \int_{B_{2r}} |Dv_n|^2 \eta^2 dx
$$

and

$$
\left| 2\int_{B_{2r}} a_n^{\alpha\beta} D_\beta v_n D_\alpha \eta dx \right| \leq \varepsilon\mu \int_{B_{2r}} |Dv_n|^2 \eta^2 dx + \frac{\mu}{\varepsilon} \int_{B_{2r}} |D\eta|^2 dx.
$$

Taking $\varepsilon = \frac{\lambda}{2\mu}$ in the above inequality and substituting the resultant inequality into (4.16), we have, for $n; [(\bar{t} - R^2)/h] + 1 \leq n \leq [\bar{t}/h] + 1$

$$
\frac{\int_{B_{2r}} v_n \eta^2 dx - \int_{B_{2r}} v_{n-1} \eta^2 dx}{h} + \frac{\lambda}{2}\int_{B_{2r}} |Dv_n|^2 \eta^2 dx \leq \frac{2\mu^2}{\lambda} \int_{B_{2r}} |D\eta|^2 dx. \tag{4.17}
$$

Here, let's recall the inequality of Poincáre type. For the proof, we refer to *J. Moser's* paper [10].

Lemma4.6. *There exists an uniform constant γ such that*

$$\int_{B_{2r}}\left(v-\frac{\int_{B_{2r}}v\eta^2dy}{\int_{B_{2r}}\eta^2dy}\right)^2dx=\min_k\int_{B_{2r}}(v-k)^2\eta^2dx\le\gamma(4m)^2r^2\int_{B_{2r}}|Dv|^2\eta^2dx \tag{4.18}$$

for $v\in W_2^1(B_{2r})$.

Now, adopting (4.18) for (4.17), we have, for $n;[(\bar{t}-R^2)/h]+1\le n\le[\bar{t}/h]+1$

$$\begin{aligned}&\frac{\int_{B_{2r}}v_n\eta^2dx-\int_{B_{2r}}v_{n-1}\eta^2dx}{h}+\frac{\lambda}{2\gamma(4m)^2}r^{-2}\int_{B_{2r}}\left(v_n-\frac{\int_{B_{2r}}v_n\eta^2dy}{\int_{B_{2r}}\eta^2dy}\right)^2dx\\ \le&\frac{2\mu^2}{\lambda}\int_{B_{2r}}|D\eta|^2dx\le\frac{4\mu^2}{\lambda}r^{-2}|B_{2r}|.\end{aligned} \tag{4.19}$$

Dividing the both side of (4.19) by $r^{-2}\int_{B_{2r}}\eta^2dy$, taking η as $\eta=1$ in $B_r(x_0)$ and noting that $\int_{B_{2r}}\eta^2dx\le|B_{2r}|3^m$, we have, for $n;[(\bar{t}-R^2)/h]+1\le n\le[\bar{t}/h]+1$

$$\begin{aligned}&\frac{\int_{B_{2r}(x_0)}v_n\eta^2dx-\int_{B_{2r}(x_0)}v_{n-1}\eta^2dy}{h/r^2}\\ &+\frac{\lambda}{2\gamma(4m)^2}\frac{1}{3^m|B_r|}\int_{B_{2r}(x_0)}\left(v_n-\frac{\int_{B_{2r}(x_0)}v_n\eta^2dy}{\int_{B_{2r}(x_0)}\eta^2dy}\right)^2dx\le\frac{2\mu^2}{\lambda}\frac{|B_{2r}|}{|B_r|}.\end{aligned} \tag{4.20}$$

Now we notice that, from the definition $\widetilde{V_n}$; (4.11)

$$\widetilde{V}_{[t_0/h]+1}=0. \tag{4.21}$$

Taking

$$\gamma_1=2\gamma(4m)^23^m/\lambda,$$
$$\gamma_2=4\mu^2|B_2|/\lambda|B_1|=4\mu^2|B_{2r}|/\lambda|B_r|$$

in (4.20) and noting that (4.20) remains unchanged if v_l is replaced by v_l+ const, (4.20) are rewritten as follows:

$$\begin{cases}\dfrac{\widetilde{V}_n-\widetilde{V}_{n-1}}{h/r^2}+\gamma_1^{-1}\dfrac{1}{|B_r|}\displaystyle\int_{B_{2r}(x_0)}\left(\widetilde{v}_n-\widetilde{V}_n\right)^2dx\le\gamma_2 & (n;\ [(\bar{t}-R^2)/h]+1\le n\le[\bar{t}/h]+1)\\ \widetilde{V}_{[t_0/h]+1}=0\end{cases} \tag{4.22}$$

Then (4.22) is exchanged by

$$\begin{cases}\dfrac{W_n-W_{n-1}}{h/r^2}+{\gamma_1}^{-1}\dfrac{1}{|B_r|}\displaystyle\int_{B_{2r}(x_0)}(w_n-W_n)^2dx\le0 & (n;\ [(\bar{t}-R^2)/h]+1\le n\le[\bar{t}/h]+1)\\ W_{[t_0/h]+1}=0\end{cases} \tag{4.23}$$

Here we notice that, for $\sigma>0$ and $n=[t_0/h]+1,\cdots,[(t_0+\tau)/h]+1$,

$$w_n-W_n\ge\sigma-W_n>\sigma>0\quad\text{in}\quad B_\sigma^n.$$

Because from difference inequalities (4.23), it follows that

$$\begin{cases} \dfrac{W_n - W_{n-1}}{h/r^2} \le 0 \\ W_{[t_0/h]+1} = 0 \end{cases}$$

so that we obtain, for $n; [t_0/h] + 1 \le n \le [(t_0 + \tau)/h] + 1$

$$W_n \le W_{n-1} \le \cdots \le W_0 = 0. \tag{4.24}$$

Thus, again by difference inequalities (4.23) we have, for any $\sigma > 0$ and all n; $[t_0/h] + 1 \le n \le [(t_0 + \tau)/h] + 1$

$$\frac{W_n - W_{n-1}}{h/r^2} + \gamma_1{}^{-1}\frac{|B_\sigma^n|}{|B_r|}(\sigma - W_n)^2 \le \frac{W_n - W_{n-1}}{h/r^2} + \gamma_1{}^{-1}\frac{1}{|B_r|}\int_{B_{2r}(x_0)}(w_n - W_n)^2 dx \le 0,$$

so that

$$(\sigma - W_n)^{-2}\frac{\sigma - W_n - (\sigma - W_{n-1})}{h/r^2} \ge \gamma_1{}^{-1}\frac{|B_\sigma^l|}{|B_r|}.$$

Here noticing that for $a, b \ge 0$

$$-(a^{-1} - b^{-1}) \ge a^{-2}(a - b),$$

we have, for $n; [t_0/h] + 1 \le n \le [(t_0 + \tau)/h] + 1$

$$\frac{-\Big((\sigma - W_n)^{-1} - (\sigma - W_{n-1})^{-1}\Big)}{h/r^2} \ge \gamma_1{}^{-1}\frac{|B_\sigma^l|}{|B_r|}. \tag{4.25}$$

Multiplying (4.25) by h/r^2 and summing the resultant inequality from $n_0 := [t_0/h] + 2$ to $n_1 := [(t_0 + \tau)/h] + 1$, we obtain

$$\begin{aligned} &\frac{h}{r^2}\sum_{n=n_0}^{n_1}\gamma_1^{-1}\frac{|B_\sigma^n|}{|B_r|} \ge -\sum_{n=n_0+1}^{n_1}\{(\sigma - W_n)^{-1} - (\sigma - W_{n-1})^{-1}\} \\ =&(\sigma - W_{n_1})^{-1} - (\sigma - W_{n_1-1})^{-1} + \cdots + (\sigma - W_{n_0})^{-1} - (\sigma - W_{n_0-1})^{-1} \\ =&(\sigma - W_{n_1})^{-1} - (\sigma - W_{n_0-1})^{-1}, \end{aligned}$$

namely,

$$-(\sigma - W_{n_1})^{-1} + (\sigma - W_{n_0-1})^{-1} \ge \frac{h}{r^2}\sum_{n=n_0}^{n_1}\gamma_1^{-1}\frac{|B_\sigma^n|}{|B_r|}. \tag{4.26}$$

Noting (4.24), from (4.26) it follows that, for any $\sigma > 0$

$$\sigma^{-1} \ge \frac{h}{r^2}\sum_{n=n_0}^{n_1}\gamma_1^{-1}\frac{|B_\sigma^n|}{|B_r|} = \gamma_1^{-1}\frac{1}{r^2|B_r|}\int_{t_{[t_0/h]+1}}^{t_{[(t_0+\tau)/h]}}|B_\sigma(t)|dt$$

Namely we have

$$\int_{t_{[t_0/h]+1}}^{t_{[(t_0/h+\tau)/h]+1}}|B_\sigma(t)|dt \le \gamma_1 r^2|B_r|\sigma^{-1}. \tag{4.27}$$

Just now we are in a position to prove (4.9). Since

$$\begin{aligned}
&\frac{1}{|C_r^+||C_r^-|}\iint_{C_r^+(t_0,x_0)}\iint_{C_r^-(t_0,x_0)}\varphi(v(t,x)-v(t',x'))dtdxdt'dx' \\
\leq&\frac{1}{|C_r^+|}\iint_{C_r^+(t_0,x_0)}\varphi(v(t,x)-\int_{B_{2r}}v_{[t_0/h]+1}\eta^2/\int_{B_{2r}}\eta^2)dtdx \\
+&\frac{1}{|C_r^-|}\iint_{C_r^-(t_0,x_0)}\varphi(-v(t',x')+\int_{B_{2r}}v_{[t_0/h]+1}\eta^2/\int_{B_{2r}}\eta^2)dt'dx'.
\end{aligned}\tag{4.28}$$

To show (4.9) we need to estimate each term of (4.28). From now on we put

$$\begin{aligned}
I_1 =& \frac{1}{|C_r^+|}\iint_{C_r^+(t_0,x_0)}\varphi(v(t,x)-\int_{B_{2r}}v_{[t_0/h]+1}\eta^2/\int_{B_{2r}}\eta^2)dtdx, \\
I_2 =& \frac{1}{|C_r^-|}\iint_{C_r^-(t_0,x_0)}\varphi(-v(t',x')+\int_{B_{2r}(x_0)}v_{[t_0/h]+1}\eta^2dy/\int_{B_{2r}(x_0)}\eta^2dy)dt'dx'
\end{aligned}$$

To estimate I_1 and I_2 respectively, we shall classify the proof into two cases:

Case 1.(parabolic case) $2r^2 \geq h$,

Case 2.(elliptic case) $2r^2 < h$.

Now we consider the case of Case1 : $2r^2 \geq h$. Noting a definition of functions;$\tilde{v}, w, \varphi(\cdot)$ and setting, for $n = [t_0/h]+1,\cdots,[(t_0+\tau)/h]+1$

$$g(t,\cdot) = \gamma_2(n-([t_0/h]+1))\frac{h}{r^2}, \quad \text{for} \quad (n-1)h < t \leq nh \quad \text{in} \quad B_r$$

we have the calculations:

$$\begin{aligned}
I_1 =& \iint_{C_r^-(t_0,x_0)}\varphi(\tilde{v}(t,x))dtdx = \iint_{C_r^-(t_0,x_0)}\varphi(w(t,x)+g(t,x))dtdx \\
\leq& \iint_{C_r^-(t_0,x_0)}\varphi(w(t,x))dtxd + \iint_{C_r^-(t_0,x_0)}\varphi(g(t,x))dtdx \\
\leq& \iint_{C_r^-(t_0,x_0)}\varphi(w(t,x))dtdx + |C_r^-|\gamma_2\left(1+\frac{h}{r^2}\right)
\end{aligned}$$

where we have used that

$$\begin{aligned}
&g(t_1+t_2,\cdot) \leq g(t_1,\cdot)+g(t_2,\cdot) \quad \text{for} \quad t_1,t_2 : [t_0/h]h < t_1,\ t_2 < ([(t_0+\tau)/h]+1)h, \\
&g(t) \leq \gamma_2\left(1+\frac{h}{r^2}\right) \quad \text{for} \quad t \in [[t_0/h]h,\ ([(t_0+\tau)/h]+1)h].
\end{aligned}$$

Moreover, noticing that $h \leq 2r^2$, I_1 is estimated above by

$$\iint_{C_r^-(t_0.x_0)}\varphi(w(t,x))dtdx + 3\gamma_2|C_r^-|. \tag{4.29}$$

Thus we need to make a estimate for $\iint_{C_r^-(t_0,x_0)}\varphi(w(t,x))dtdx$. To do it, we distinguish the calculation into two ones:

$$\iint_{C_r^-(t_0,x_0)}\varphi(w(t,x))dtdx = \iint_{C_r^{--}(t_0,x_0)}\varphi(w(t,x))dtdx + \iint_{C_r^{-+}(t_0,x_0)}\varphi(w(t,x))dtdx \tag{4.30}$$

where we define

$$C_r^{--}(t_0,x_0)=(t_{[t_0/h]+1},\ t_{[(t_0+r^2)/h]+1})\times B_r(x_0),\quad C_r^{-+}(t_0,x_0)=(t_0,\ t_{[t_0/h]+1})\times B_r(x_0).$$

Let's put

$$I_1^1=\iint_{C_r^{--}(t_0,x_0)}\varphi(w(t,x))dtdx,\quad I_1^2=\iint_{C_r^{-+}(t_0,x_0)}\varphi(w(t,x))dtdx.$$

(Estimate for I_1^1) Frist we notice that

$$\begin{aligned}I_1^1=&\iint_{C_r^{--}(t_0,x_0)\cap\{w>1\}}\varphi(w(t,x))dtdx+\iint_{C_r^{--}(t_0,x_0)\cap\{0<w\le1\}}\varphi(w(t,x))dtdx\\ \le&\iint_{C_r^{--}(t_0,x_0)\cap\{w>1\}}\varphi(w(t,x))dtdx+|C_r^{--}|.\end{aligned}\tag{4.31}$$

For the first term we obtain, from (4.31)

$$\begin{aligned}&\iint_{C_r^{--}(t_0,x_0)\cap\{w>1\}}\varphi(w(t,x))dtdx\\ =&\int_1^\infty\sqrt{\sigma}(-dm(\sigma))=\int_1^\infty m(\sigma)d\sqrt{\sigma}\le\int_1^\infty\gamma_1\sigma^{-1}r^2|B_r|d\sqrt{\sigma}\\ =&\gamma_1r^2|B_r|.\end{aligned}\tag{4.32}$$

where $m(\sigma)=\int_{t_{[t_0/h]+1}}^{t_{[(t_0+r)/h]+1}}|B_\sigma(t)|dt$. Substuting (4.32) into (4.31), we have

$$I_1^1\le\gamma_1r^2|B_r|+|C_r^{--}|\le(\gamma_1+1)|C_r^{-}|.\tag{4.33}$$

(Estimation for I_1^2) Next we shall deal with I_1^2. Let's notice that

$$\begin{aligned}I_1^2=&\iint_{C_r^{-+}(t_0,x_0)\cap\{w>1\}}\varphi(w(t,x))dtdx=\int_{t_0}^{t_{[t_0/h]+1}}\int_{B_r(x_0)}\varphi(w_{[t_0/h]+1})dtdx\\ =&|t_{[t_0/h]+1}-t_0|\int_{B_r(x_0)}\varphi(w_{[t_0/h]+1})dx.\end{aligned}\tag{4.34}$$

Thus we have to make a estimate the quantity: $\int_{B_r(x_0)}\varphi(w_{[t_0/h]+1})dx$. Recalling the difinition of w_n, we have

$$w_{[t_0/h]+1}=\widetilde{v}_{[t_0/h]+1}=v_{[t_0/h]+1}-\int_{B_{2r}(x_0)}v_{[t_0/h]+1}\eta^2dx/\int_{B_{2r}(x_0)}\eta^2dx,$$

so that

$$\begin{aligned}&\int_{B_r(x_0)}\varphi(w_{[t_0/h]+1})dx=\int_{B_r(x_0)}\varphi(v_{[t_0/h]+1}-\int_{B_{2r}(x_0)}v_{[t_0/h]+1}\eta^2dy/\int_{B_{2r}(x_0)}\eta^2dy)dx\\ \le&\int_{B_{2r}(x_0)}\sqrt{\left|v_{[t_0/h]+1}(x)-\frac{1}{|B_{2\theta}|}\int_{B_{2r}}v_{[t_0/h]+1}\eta^2dy\right|}\,dx\\ &+|B_r|^{-\frac12}\sqrt{\int_{B_{2r}}\left|v_{[t_0/h]+1}(x)-\frac{1}{|B_{2r}|}\int_{B_{2r}(x_0)}v_{[t_0/h]+1}\eta^2dy\right|dx\ |B_r|}\\ \le&2|B_{2r}|^{\frac12}\left(\int_{B_{2r}(x_0)}\left|v_{[t_0/h]+1}(x)-\frac{1}{|B_{2r}|}\int_{B_{2r}(x_0)}v_{[t_0/h]+1}\eta^2dy\right|dx\right)^{\frac12}.\end{aligned}\tag{4.35}$$

Applying (4.3) in Lemma4.3 to (4.35) gives that

$$\int_{B_r} \varphi(w_{[t_0/h]+1})dx \le 2\gamma^{\frac{1}{2}}|B_{2r}|\left(\frac{16\mu^2}{\lambda^2}+\frac{2r^2}{\lambda h}\right)^{\frac{1}{4}}. \tag{4.36}$$

Substituting (4.36) into (4.34) and noting that $r^2 \ge h/2$ and that $|t_{[t_0/h]+1}-t_0| \le h$, we have

$$\begin{aligned} I_1^2 &= |t_{[t_0/h]+1}-t_0| \times \int_{B_r(x_0)} \varphi(w_{[t_0/h]+1})dx \le 2\gamma^{\frac{1}{2}}h|B_{2r}|\left(\frac{r^2}{h}\right)^{\frac{1}{4}}\left(\frac{32\mu^2}{\lambda^2}+\frac{2}{\lambda}\right)^{\frac{1}{4}} \\ &\le 2\gamma^{\frac{1}{2}}|B_{2r}|h^{\frac{3}{4}}r^{\frac{1}{2}}\left(\frac{32\mu^2}{\lambda^2}+\frac{2}{\lambda}\right)^{\frac{1}{4}} \\ &\le 2^2\gamma^{\frac{1}{2}}\left(\frac{16\mu^2}{\lambda^2}+\frac{1}{\lambda}\right)^{\frac{1}{4}} r^2|B_{2r}| \le 2^2\gamma^{\frac{1}{2}}\left(\frac{16\mu^2}{\lambda^2}+\frac{1}{\lambda}\right)^{\frac{1}{4}} 2^m|C_r^-|. \end{aligned} \tag{4.37}$$

Combining the estimates (4.33) and (4.37) for I_1^1 and I_1^2 with (4.29) gives that

$$I_1 = \iint_{C^-(t_0,x_0)} \varphi(\tilde{v})dtdx \le (\gamma_1+1)|C^-| + \gamma|C_r^-| + 3\gamma_2|C_r^-|. \tag{4.38}$$

where

$$\gamma = 2^{m+2}\gamma^{\frac{1}{2}}\left(\frac{16\mu^2}{\lambda^2}+\frac{1}{\lambda}\right)^{\frac{1}{4}}$$

Thus we have obtained the estmations for I_1.

(The estimation of I_2) Now we shall estimate the term

$$I_2 = \iint_{C_r^+(t_0,x_0)} \varphi(-v(t,x) + \int_{B_{2r}(x_0)} v_{[t_0/h]+1}\eta^2 dy / \int_{B_{2r}(x_0)} \eta^2 dy)dtdx.$$

By a transformation

$$\begin{cases} t-t_0 = -(t'-t_0) \\ x = x', \end{cases}$$

I_2 is becoming

$$I_2 = \int_{t_0}^{t_0+\tau} \int_{B_r(x_0)} \varphi(-v(-t'+2t_0,\, x') + \int_{B_{2r}(x_0)} v_{[t_0/h]+1}\eta^2 dy / \int_{B_{2r}(x_0)} \eta^2 dy)dt'dx'.$$

Thus, estimating $-v(-t'+2t_0,\, x')$ in $C_r^{--}(t_0,x_0)$ similarly as (4.33) and (4.37), we have

$$I_2 \le \gamma_1|C_r^+| + 3\gamma_2|C_r^+|. \tag{4.39}$$

As a result, substituting (4.38) and (4.39) into (4.28) gives that

$$\begin{aligned} &\frac{1}{|C_r^+||C_r^-|}\iint_{C_r^+(t_0,x_0)}\iint_{C_r^-(t_0,x_0)} \varphi(v(t,x)-v(t',x'))dtdxdt'dx' \\ \le&\frac{1}{|C_r^-|}I_1 + \frac{1}{|C_r^+|}I_2 \\ \le&2\gamma_1 + \gamma + 6\gamma_2 \end{aligned} \tag{4.40}$$

Next we shall deal with Case2, $2r^2 < h$. We need to distinguish our calculations into subcases:

$$\text{Case2-1,}\quad C_r^+(t_0,x_0)\cup C_r^-(t_0,x_0)\subset (t_{[t_0/h]},\quad t_{[t_0/h]+2})\times B_r(x_0),$$
$$\text{Case2-2,}\quad C_r^+(t_0,x_0)\cup C_r^-(t_0,x_0)\subset (t_{[t_0/h]-1},\quad t_{[t_0/h]+1})\times B_r(x_0).$$

Firstly we consider Case2-1. By using (4.27) with $\tau = h$, we have, for $\sigma > 0$

$$\int_{t_{[t_0/h]+1}}^{t_{[t_0/h]+2}} |B_\sigma(t)|dt \le \gamma_1 r^2|B_r|\sigma^{-1}.$$

Namely

$$\left|\{(t,x)\in C_{r,h}^-(t_{[t_0/h]+1},x_0) : w(t,x)>\sigma\}\right| \le \gamma_1 r^2|B_r|\sigma^{-1}. \tag{4.41}$$

Estimating similarly as (4.27) we also have

$$\left|\{(t,x)\in C_{r,h}^+(t_{[t_0/h]+1},x_0) : -w(t,x)>\sigma\}\right| \le \gamma_1 r^2|B_r|\sigma^{-1}. \tag{4.42}$$

Noticing that

$$\begin{aligned}\varphi(v(t,x)-v(t',x')) \le&\varphi(v(t,x)-\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\eta^2dy/\int_{B_{2r}(x_0)}\eta^2dy-\gamma_2 h/r^2)\\ &+\varphi(-v(t',x')+\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\eta^2dy/\int_{B_{2r}(x_0)}\eta^2dy+\gamma_2 h/r^2),\end{aligned}$$

we obtain from (4.41) and (4.42) that

$$\begin{aligned}&\frac{1}{|C_{r,h}^+|}\frac{1}{|C_{r,h}^-|}\iint_{C_{r,h}^+(t_{[t_0/h]+1},x_0)}\iint_{C_{r,h}^-(t_{[t_0/h]+1},x_0)}\varphi(v(t,x)-v(t',x'))dt'dx'dtdx\\ \le&\frac{1}{|C_{r,h}^-|}\iint_{C_{r,h}^-(t_{[t_0/h]+1},x_0)}\varphi(v(t,x)-\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\eta^2dy/\int_{B_{2r}(x_0)}\eta^2dy-\gamma_2h/r^2)dtdx\\ &+\frac{1}{|C_{r,h}^+|}\iint_{C_{r,h}^+(t_{[t_0/h]+1},x_0)}\varphi(-v(t',x')+\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\eta^2dy/\int_{B_{2r}(x_0)}\eta^2dy+\gamma_2h/r^2)dt'dx'\\ =&\frac{1}{|C_{r,h}^-|}\gamma_1r^2|B_r|+\frac{1}{|C_{r,h}^+|}\gamma_1r^2|B_r| = 2\gamma_1r^2|B_r|\frac{1}{|B_r|h} = 2\gamma_1\frac{r^2}{h}\le\gamma_1\end{aligned} \tag{4.43}$$

where we used the fact

$$|C_{r,h}^-| = |C_{r,h}^+| = h|B_r|$$

and that $2r^2 < h$. Noticing that

$$v(t,x) = v_{[t_0/h]+2}(x)\quad\text{in}\quad C_{r,h}^-(t_{[t_0/h]+1},x_0),$$
$$v(t',x') = v_{[t_0/h]+1}(x')\quad\text{in}\quad C_{r,h}^+(t_{[t_0/h]+1},x_0),$$

we deduce from (4.43) that

$$\begin{aligned}&\frac{1}{|B_r|}\frac{1}{|B_r|}\int_{B_r(x_0)}\int_{B_r(x_0)}\varphi(v_{[t_0/h]+2}(x)-v_{[t_0/h]+1}(x'))dx'dx\\ =&\frac{1}{|C_{r,h}^+|}\frac{1}{|C_{r,h}^-|}\iint_{C_{r,h}^+(t_{[t_0/h]+1},x_0)}\iint_{C_{r,h}^-(t_{[t_0/h]+1},x_0)}\varphi(v(t,x)-v(t',x'))dt'dx'dtdx\le\gamma_1.\end{aligned} \tag{4.44}$$

Now let's remark the following calculations;

$$\begin{aligned}
&\frac{1}{|C_r^+|}\frac{1}{|C_r^-|}\iint_{C_r^+(t_0,x_0)}\iint_{C_r^-(t_0,x_0)}\varphi(v(t,x)-v(t',x'))dt'dx'dtdx\\
=&\frac{1}{|C_r^+|}\iint_{C_r^+(t_0,x_0)}\frac{1}{|C_r^-|}\Big\{\iint_{(t_0,t_{[t_0/h]+1})\times B_r(x_0)}\varphi(v(t,x)-v(t',x'))dtdx\\
+&\iint_{(t_{[t_0/h]+1},t_0+r^2)\times B_r(x_0)}\varphi(v(t,x)-v(t',x'))dtdx\Big\}dt'dx'\\
=&\frac{1}{|B_r|}\int_{B_r(x_0)}\left(\frac{1}{|C_r^-|}\times|t_{[t_0/h]+1}-t_0|\times\int_{B_r(x_0)}\varphi(v_{[t_0/h]+1}(x)-v_{[t_0/h]+1}(x'))dx\right)dx'\\
+&\frac{1}{|B_r|}\int_{B_r(x_0)}\left(\frac{1}{|C_r^-|}\times|t_0+r^2-t_{[t_0/h]+1}|\times\int_{B_r(x_0)}\varphi(v_{[t_0/h]+2}(x)-v_{[t_0/h]+1}(x'))dx\right)dx'\\
\le&\frac{1}{|B_r|}\frac{1}{|B_r|}\int_{B_r(x_0)}\int_{B_r(x_0)}\varphi(v_{[t_0/h]+1}(x)-v_{[t_0/h]+1}(x'))dxdx'\\
+&\frac{1}{|B_r|}\frac{1}{|B_r|}\int_{B_r(x_0)}\int_{B_r(x_0)}\varphi(v_{[t_0/h]+2}(x)-v_{[t_0/h]+1}(x'))dxdx'. \qquad (4.45)
\end{aligned}$$

In the last inequality we used

$$|t_{[t_0/h]+1}-t_0|,\quad |t_0+r^2-t_{[t_0/h]+1}|\le r^2.$$

For the second term, we have (4.44). For first term, by noting the fact that

$$\varphi(t_1+t_2)\le\varphi(t_1)+\varphi(t_2)\quad\text{for}\quad t_1,t_2\in R$$

and exploiting (4.3) in Lemma 4.3, we obtain

$$\begin{aligned}
&(\text{First term of } (4.45))\\
\le&\frac{1}{|B_r|}\int_{B_r(x_0)}\varphi(v_{[t_0/h]+1}(x)-\frac{1}{|B_r|}\int_{B_r(x_0)}v_{[t_0/h]+1}dy)dx\\
+&\frac{1}{|B_r|}\int_{B_r(x_0)}\varphi(-v_{[t_0/h]+1}(x)+\frac{1}{|B_r|}\int_{B_r(x_0)}v_{[t_0/h]+1}dy)dx\\
\le&2\gamma^{\frac{1}{2}}\left(\frac{16\mu^2}{\lambda^2}+\frac{2r^2}{\lambda h}\right)^{\frac{1}{4}}\le 2\gamma^{\frac{1}{2}}\left(\frac{16\mu^2}{\lambda^2}+\frac{1}{\lambda}\right)^{\frac{1}{4}}. \qquad (4.46)
\end{aligned}$$

In the last inequality we used that $2r^2<h$. Therefore, substituting (4.44) and (4.46) into (4.45), we have

$$\frac{1}{|C_r^+|}\frac{1}{|C_r^-|}\iint_{C_r^+(t_0,x_0)}\iint_{C_r^-(t_0,x_0)}\varphi(v(t,x)-v(t',x'))dt'dx'dtdx\le\gamma_1+2\gamma^{\frac{1}{2}}\left(\frac{16\mu^2}{\lambda^2}+\frac{1}{\lambda}\right)^{\frac{1}{4}}. \qquad (4.47)$$

Next we deal with Case2-2. Then we obtain from (4.27) with $\tau=h$ that

$$\int_{t_{[t_0/h]}}^{t_{[t_0/h]+1}}|B_\sigma(t)|dt\le\gamma_1 r^2|B_r|\sigma^{-1}.$$

Namely we have

$$\left|\{(t,x)\in C_{r,h}^-(t_{[t_0/h]},\ x_0); w(t,x)>\sigma\}\right|\le\gamma_1 r^2|B_r|\sigma^{-1}.$$

Estimating similarly as (4.42)-(4.45), we have (4.47).

As a result we obtain from (4.40)and (4.47) the assertion of Lemma4.5.

5. Proof of Theorems

In this section we present the proof of our theorems. Firstly we consider Theorem1.1.

Proof of Theorem1.1. Since u_h is nonnegative in $C_r^+(t_{n_0}, x_0)$, we find that

$$u_h \geq 0 \quad \text{in} \quad C^+_{\sqrt{\tilde{n}_r h}}(t_{n-0}, x_0)$$

and that

$$u_{n_0-[\tilde{n}_r/4]-1} \geq 0 \quad \text{in} \quad B_{\sqrt{\tilde{n}_r h}}(x_0).$$

Thus, we can apply Lemma4.5 to u_h in $C^+_{\sqrt{\tilde{n}_r h}}(t_{n_0}, x_0)$, so that

$$\left(\frac{1}{|D^-|}\iint_{D^-}(u_h)^{\xi}dxdt\right)^{\frac{1}{\xi}} \leq \gamma^{\frac{1}{\xi}}\left(\frac{1}{|D^+|}\iint_{D^+}(u_h)^{-\xi}dxdt\right)^{-\frac{1}{\xi}} \tag{5.1}$$

where we put

$$\begin{aligned} D^+ &= (t_{n_0} - \frac{1}{4}\tilde{n}_r h,\ t_{n_0}) \times B_{\sqrt{\tilde{n}_r h}}(x_0), \\ D^- &= (t_{n_0-\tilde{n}_r},\ t_{n_0-\tilde{n}_r} + \frac{1}{4}\tilde{n}_r h) \times B_{\sqrt{\tilde{n}_r h}}(x_0) \end{aligned} \tag{5.2}$$

and γ, ξ are positive constants determined in Lemma4.5. Also we can adopt (2.40) in Lemma2.3 for u_h in D^+. Namely we obtain that

$$\left(\frac{1}{|D^+|}\iint_{D^+}(u_h)^{p}dxdt\right)^{-\frac{1}{p}} \leq \gamma \inf_{D^+_{1/2}} u_h \quad \text{for} \quad P > 1 \tag{5.3}$$

where

$$D^+_{1/2} = (t_{n_0} - \frac{1}{8}\tilde{n}_r h,\ t - n_0) \times B_{\frac{1}{2}\sqrt{\tilde{n}_r h}}(x_0).$$

Investigating similarly as above, we find it justified to exproit (2.41) in Lemma2.3 for u_h in D^-, so that

$$\left(\frac{1}{|D^-_{1/2}|}\iint_{D^-_{1/2}}(u_h)^{\xi}dxdt\right)^{\frac{1}{\xi}} \leq \gamma\left(\frac{1}{|D^+|}\iint_{D^-}(u_h)^{\tilde{q}}dxdt\right)^{-\frac{1}{\tilde{q}}} \quad \text{for} \quad q, \tilde{q}; 0 < q, \tilde{q} < 1 + \frac{2}{m} \tag{5.4}$$

where

$$D^-_{1/2} = (t_{n_0-\tilde{n}_r}, t_{n_0-\tilde{n}_r} + \frac{1}{8}\tilde{n}_r h) \times B_{\sqrt{\tilde{n}_r h}}(x_0).$$

Combining (5.3), in which $p = \xi$, and (5.4), in which $\tilde{q} = \xi$, $q \in (0, 1 + \frac{2}{m})$ with (5.1), and recalling the definition of $D^+, D^+_{1/2}, D^-$, and $D^-_{1/2}$, we conclude the assertion (1.10) of Theorem1.1.

Now, by using Theorem1.1, we can have Hölder estimate for a weak solution u_h of (1.1), which was derived in the paper [4], but the proof is entirely different from theirs.

Lemma5.1. *Let u_h be a weak solution of* (1.1) *and* $(\bar{t}, \bar{x})$ *be taken arbitrarily in* $\widetilde{Q_{h_0}}$ *with*

$$d = \frac{1}{4} min\Big(|\bar{t} - N_0 h_0|^{\frac{1}{2}}, dist(\bar{x}, \partial\Omega) \Big).$$

Then there exist positive constants γ *and* $\alpha; 0 < \alpha < 1$ *depending only on* λ, μ, m *and* d, *such that*

$$\Big| u_h(t_{n'}, x') - u_h(t_n, x) \Big| \leq \gamma \Big(|x - x'|^{\alpha}, |t_{n'} - t_n|^{\frac{\alpha}{2}} \Big) \tag{5.5}$$

for any $(t_{n'}, x'), (t_n, x) \in C_d^+(\bar{t}, \bar{x})$ *with* $\delta((t_{n'}, x'), (t_n, x)) \geq \sqrt{h}$.

Proof.First of all we need to notice the uniform boundedness of a weak solution u_h in $C_d^+(\bar{t}, \bar{x})$. From adopting lemma3.2 with $p = 2$, for u_h in $C_d^+(\bar{t}, \bar{x})$, it follows that there exists a positive constant U depending only on λ, μ, d and a bounds of

$$\iint_Q u_h^2 dx dt,$$

such that

$$|u_h| \leq U \quad \text{in} \quad C_d^+(\bar{t}, \bar{x}). \tag{5.6}$$

Let's take $(t_{n'}, x'), (t_n, x) \in C_d^+(\bar{t}, \bar{x})$ satisfying $\delta((t_{n'}, x'), (t_n, x)) \geq \sqrt{h}$, arbitrarily. Now, let's set notations:

$$\begin{aligned} M &= \sup_{C_d^+(t_n, x)} u_h, \\ m &= \inf_{C_d^+(t_n, x)} u_h \\ \widetilde{n}_d &= \text{the greatest number satisfying} \quad n < d^2/h. \end{aligned} \tag{5.7}$$

Since $M - u_h, u_h - m$ are weak solution of (1.1) and nonnegative in $C_d^+(t_n, x)$, we can apply Theorem1.1 for $M - u_h, u_h - m$ in $D_1^+(t_n, x)$, $D_1^-(t_{n-\widetilde{n}_d}, x) \subset C_d^+(t_n, x)$. Namely, we obtain

$$\frac{1}{|D_{1/2}^-|} \iint_{D_{1/2}^-(t_{n-\widetilde{n}_d}, x)} (M - u_h) dx dt \leq \gamma \inf_{D_{1/2}^+(t_n, x)} (M - u_h) \tag{5.8}$$

$$\frac{1}{|D_{1/2}^-|} \iint_{D_{1/2}^-(t_{n-\widetilde{n}_d}, x)} (u_h - m) dx dt \leq \gamma \inf_{D_{1/2}^+(t_n, x)} (u_h - m) \tag{5.9}$$

where

$$D^- = (t_{n_0 - \widetilde{n}_r}, \ t_{n_0 - \widetilde{n}_r} + \frac{1}{8}\widetilde{n}h) \times B_{\frac{1}{2}\sqrt{\widetilde{n}_r h}}(x_0),$$

$$D^+ = (t_{n_0} - \frac{1}{8}\widetilde{n}_r h, \ t_{n_0}) \times B_{\frac{1}{2}\sqrt{\widetilde{n}_r h}}(x_0).$$

Adding (5.8) by (5.9) and replacing $\max(\gamma, 1)$ by γ, we have

$$\operatorname*{osc}_{C_{d/4}^+(t_n, x)} u_h \leq (1 - \gamma^{-1}) \operatorname*{osc}_{C_d^+(t_n, x)} u_h \tag{5.10}$$

Repeating the above arguments which give (5.6), we have, for any positive integer ν

$$\omega_{d_\nu} \leq \vartheta^{\nu} \omega_{d_{\nu-1}}, \tag{5.11}$$

where

$$\vartheta = 1-\gamma^{-1}, \quad \omega_{d_\nu} = \operatorname{osc}_{C^+_{d_\nu}}, \quad d_\nu = \frac{d}{2^\nu} - \sum\nolimits_{k=1}^{\nu} \frac{\delta_{\nu+1-k}}{2^k}.$$

If

$$\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \sqrt{h} < \frac{d}{2},$$

then there exists positive integer ν such that

$$\frac{d}{2^{\nu+1}} < \max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \sqrt{h} \le \frac{d}{2^\nu}.$$

Now, adopting (5.11) successively gives that

$$\begin{aligned}
&|u_n(x) - u_{n'}| \le \omega_{d_\nu} \theta \omega_{d_{\nu-1}} \\
&\le \theta^\nu \omega_d = \left(\frac{1}{2}\right)^{\alpha\nu} \omega_d \\
&= \left(\frac{2}{d}\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \frac{2}{d}\sqrt{h}\right)^\alpha \omega_d \\
&\le \left(\frac{2}{d}\delta((t_{n'},x'),(t_n,x)) + \frac{2}{d}\sqrt{h}\right)^\alpha \omega_d.
\end{aligned} \tag{5.12}$$

where

$$\theta = \left(\frac{1}{2}\right)^{-\frac{log\theta}{log2}}, \quad \theta = 1-\gamma^{-1}, \quad \alpha = -\frac{log\theta}{log2}.$$

Since $\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} \ge \sqrt{h}$, from (5.12), it follows that

$$\begin{aligned}
&|u_n(x) - u_{n'}(x')| \le \omega_{d_\nu} \\
&\le \left(\frac{4}{d}\right)^\alpha \Big(\delta((t_{n'},x'),(t_n,x)\Big)\omega_d.
\end{aligned} \tag{5.13}$$

If

$$\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \sqrt{h} \ge \frac{d}{2},$$

from the boundedness of u_h in $C^+_d(\bar{t},\bar{x})$, we obtain

$$\begin{aligned}
&|u_n(x) - u_{n'}(x')| \le \omega_{d_\nu} \\
&\le 2U \\
&\le 2U\left(\frac{2}{d}\Big(\max|t_n - t_{n'}|^{1/2}, |x-x'| + \sqrt{h}\Big)\right)^\alpha \\
&= 2U\left(\frac{4}{d}\right)^\alpha \Big(\delta((t_{n'},x'),(t_n,x))\Big)^\alpha.
\end{aligned} \tag{5.14}$$

Proof of Theorem1.2. Now let's take $B_{2r} = B_{2r}(x_0) \subset \Omega$ with $r^2 \le h$ and fix it. Suppose that $u_n (1 \le n \le N)$ is nonegative in B_{2r}. By a scaling transformation: $x = x_0 + ry$ and setting on B_1

$$\widetilde{a}^{\alpha\beta}_n(y) = a^{\alpha\beta}_n(x_0+ry), \quad \widetilde{u}_n(y) = u_n(x_0+ry), \quad \widetilde{u}_{n-1}(y) = u_{n-1}(x_0+ry)$$

from (1.3), it follows that

$$\int_{B_1} \widetilde{a}_n^{\alpha\beta} D_\beta \widetilde{u}_n D_\alpha \varphi dy + \int_{B_1} \frac{\widetilde{u}_n - \widetilde{u}_{n-1}}{h/r^2} = 0 \quad \text{for any } \varphi = (\varphi^i) \in \overset{\circ}{W}{}_2^1(B_1). \tag{5.15}$$

Applying Harnack theorem on elliptic equations (see [3],Th.8.18., p194) to $\widetilde{u}_n$ in B_1, it follows from (5.15) that, for any p; $1 \le p < m/(m-2)$, and $q > m$, there exists a constant γ depending only on m, q, p and λ, μ such that

$$\left(\frac{1}{|B_1|}\int_{B_1} (\widetilde{u}_n)^p dy\right)^{\frac{1}{p}} \le \gamma \left\{ \inf_{B_1} \widetilde{u}_n + \left(\frac{1}{|B_1|}\int_{B_1} g^q dy\right)^{\frac{1}{q}} \right\}, \tag{5.16}$$

where we put

$$g = \frac{\widetilde{u}_n - \widetilde{u}_{n-1}}{h/r^2}.$$

Now, by adopting Hölder estimate Lemma5.1 for u_h in $C_d^+(x_0)$ with $d = \min\{\sqrt{h}, \operatorname{dist}(x_0, \partial\Omega)\}$, we have that

$$|g| \le \gamma h^{\alpha/2-1} r^2 \tag{5.17}$$

with positive constants γ, α; $0 < \alpha < 1$, independent of h, v_n, which were determined in Lemma5.1. Thus, noting that from $r^2 \le h$

$$h^{\alpha/2-1} r^2 \le r^{\alpha-2+2} = r^\alpha,$$

we obtain from (5.16) and (5.17) the assertion in Theorem1.2.

Proof of Theorem1.3. we take a cube $C^+_{\rho_0,\tau_0}(\bar{t},\bar{x}) \subset Q$ with $d = \frac{1}{4}\delta\{(\bar{t},\bar{x}), \partial Q\}$ and fix it. We use the notation: $u = u_h$, $v = u_h^{\pm}$. Now we shall improve Caccioppoli type inequality Lemma2.3. Firstly we consider the case p; $1 < p \le 2$. If $\sigma_2\tau > 3h$, we have (2.50) and (2.51). If $\sigma_2\tau \le 3h$, we remark that, adopting Hölder estimate Lemma5.1 for u_h in $C_d^+(\bar{t},\bar{x})$ yields the following calculations: For all $C^+_{\rho,\tau}(t_{n_0}, x_0) \subset C_d^+(\bar{t},\bar{x})$ and for any $\varepsilon > 0$

$$\begin{aligned}
&(\text{Quotient term})\\
=&\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} \frac{\pm u_h(t,\cdot) + \varepsilon - (\pm u(t-h,\cdot) + \varepsilon)}{h} (v(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot) dxdt\\
\ge& - h^{\alpha/2} 3(\sigma_2\tau)^{-1} \iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} |v(t,\cdot)+\varepsilon|^{p-1}\eta^2(\cdot) dxdt\\
\ge& - (\sigma_2\tau)^{-1} \iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} |v(t,\cdot)+\varepsilon|^p dyds - 2^p h^{p\alpha/2} |C^+_{\rho,\tau}|.
\end{aligned} \tag{5.18}$$

In the last inequality we used Young's inequality. Thus, calculating similarly as (2.52), we have, for all $C^+_{\rho,\tau}(t_{n_0} x_0) \subset C_d^+(\bar{t},\bar{x})$,

$$\begin{aligned}
&\iint_{C^+_{\rho,\tau}(t_{n_0} x_0)} |D(v+\varepsilon)^{p/2}|^2 \eta^2 \sigma dxdt\\
&+\frac{p^2}{\lambda(p-1)}\varepsilon^{p-1} \iint_{C^+_{\rho,\tau}(t_{n_0},x_0)\cap\{v\le 0\}} a^{\alpha\beta} D_\beta(v(t,\cdot))\eta D_\alpha \eta dxdt\\
&\le \frac{\mu^2 p^2}{\lambda^2(p-1)^2}\left((\sigma_2\tau)^{-1} + (\sigma_1\rho)^{-2}\right) \iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} (v+\varepsilon)^p dxdt + \frac{2^{p-1}3p}{\lambda(p-1)}(\sigma_2\tau)^{-1} h^{\frac{p\alpha}{2}} |C^+_{\rho,\tau}|.
\end{aligned} \tag{5.19}$$

We also remark that (2.20) holds for $v+\varepsilon = u_h^{\pm}+\varepsilon > 0$ in this case. Therefore we have, for all $C^+_{\rho,\tau}(t_{n_0},x_0) \subset C^+_d(\bar{t},\bar{x})$ and any $p; 1 < p \le 2$

$$\begin{aligned}
&\mathop{Sup}_{t_{n_0}-\tau(1-\sigma_2)\le t\le t_{n_0}} \int_{B_{\rho(1-\sigma_1)}(x_0)} (v+\varepsilon)^p(t,\cdot)dx + \iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)} \left|D(v+\varepsilon)^{p/2}\right|^2 dxdt \\
&+\frac{p^2}{\lambda(p-1)}\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)\cap\{v\le 0\}} a^{\alpha\beta}D_\beta(v(t,\cdot))\eta D_\alpha\eta dxdt \\
&\le\gamma\frac{p(2p-1)}{(p-1)^2}\left\{\left((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\right)\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}(v+\varepsilon)^p dxdt\right. \\
&\left.+\frac{2^{p-1}3p}{\lambda(p-1)}(\sigma_2\tau)^{-1}h^{\frac{p\alpha}{2}}|C^+_{\rho,\tau}|\right\}.
\end{aligned} \tag{5.20}$$

Since each term of the right hand of (5.20) is finite for $p;\ 1 < p \le 2$, we are able to pass ε to 0 in (5.20), so that we have

$$\begin{aligned}
&\mathop{Sup}_{t_{n_0}-\tau(1-\sigma_2)\le t\le t_{n_0}} \int_{B_{\rho(1-\sigma_1)}(x_0)} v^p(t,\cdot)dx + \iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)} \left|Dv^{p/2}\right|^2 dxdt \\
&\le\gamma\frac{p(2p-1)}{(p-1)^2}\left(1+\frac{p}{p-1}\right)\left\{\left((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}\right)\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}v^p dxdt\right. \\
&\left.+\frac{2^{p-1}3p}{\lambda(p-1)}(\sigma_2\tau)^{-1}h^{\frac{p\alpha}{2}}|C^+_{\rho,\tau}|\right\}.
\end{aligned} \tag{5.21}$$

Next we consider a case of $p > 2$. If $\sigma_2\tau > 3h$, then we have (2.60). If $\sigma_2\tau \le 3h$, using Hölder estimate Lemma5.1 and calculating similarly as (5.18) yields that, for any $C^+_{\rho,\tau}(t_{n_0},x_0) \subset C^+_d(\bar{t},\bar{x})$

$$\begin{aligned}
&(\text{Quotient term of (2.53)}) \\
&\ge\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}\frac{\pm u(t,\cdot)-\pm u(t-h,\cdot)}{h}(v^{(M)})^{p-1}(t,\cdot)\eta^2(\cdot)dxdt \\
&\ge-(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}(v^{(M)})^p(t,\cdot)dyds-2^p h^{p\alpha/2}|C^+_{\rho,\tau}|.
\end{aligned}$$

By calculating similarly as (2.62), we have, for all $C^+_{\rho,\tau}(t_{n_0},x_0) \subset C^+_d(\bar{t},\bar{x})$ and for any $t\in[t_{n_0}-\tau(1-\sigma_2),\ t_{n_0}]$

$$\begin{aligned}
0\ge&\int_{B_\rho(x_0)}v^{(M)}(t,\cdot)\eta^2dx-4(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho(x_0)}(v^{(M)})^p\eta^2dxdt-2^ph^{p\alpha/2}|C^+_{\rho,\tau}| \\
&+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}a^{\alpha\beta}(t,\cdot)D_\beta(v^{(M)})^{p/2}D_\alpha(v^{(M)})^{p/2}\eta^2(\cdot)\sigma(t)dxdt \\
&+2\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}a^{\alpha\beta}(t,\cdot)D_\beta v(v^{(M)})^{p-1}\eta D_\alpha\eta(\cdot)\sigma(t)dxdt.
\end{aligned} \tag{5.22}$$

Noting the boundedness of v in $C^+_d(\bar{t},\bar{x})$ from Lemma3.2, it's justified to pass M to the limit in

(5.22) for $p > 2$. Namely we have, for all $C^+_{\rho,\tau} \subset C^+_d(\bar{t},\bar{x})$ and any $t \in [t_{n_0} - \tau,\ t_{n_0}]$

$$\begin{aligned} 0 \geq &\int_{B_\rho} v(t,\cdot)^2_\eta dx - 4(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2 dxdt - 2^p h^{p\alpha/2}|C^+_{\rho,\tau}| \\ &+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v^{\frac{p}{2}} D_\alpha v^{\frac{p}{2}}\eta^2(\cdot)\sigma(t)dxdt \\ &+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)D_\beta v v^{p-1}\eta D_\alpha\eta(\cdot)\sigma(t)dxdt. \end{aligned} \tag{5.23}$$

By Young's inequality, we have (5.21) for all $C^+_{\rho,\tau} \subset C^+_d(\bar{t},\bar{x})$ and for any $p > 2$. As a result we have obtained (5.21) for all $C^+_{\rho,\tau} \subset C^+_d(\bar{t},\bar{x})$ and for any $p > 1$.

From now on by Moser's iterative procedure we shall estimate a bounds of a weak solution u_h of (1.1). Now we take $(t_{n_0}, x_0) \in C^+_{d/2}(\bar{t},\bar{x})$ arbitrarily and fix it. Also we take ρ_0, τ_0; $0 < \rho_0, \tau_0 < d/2$ arbitrarily. We proceed our inductive calculation similarly as the proof of Lemma2.2. By a scaling transform (2.41) and noting (2.42), from (5.18) we obtain that

$$\begin{aligned} &\operatorname*{Sup}_{0\geq t\geq\tilde{\tau}(1-\sigma_2)}\int_{B_{\tilde{\rho}(1-\sigma_1)}(0)} (v^{\frac{p}{2}})^2(t,\cdot)dy + \iint_{C^+_{\tilde{\rho}(1-\sigma_1),\tilde{\tau}(1-\sigma_2)}(0)} |Dv^{\frac{p}{2}}|^2 dyds \\ &\leq \gamma\frac{p}{p-1}\left(1+\frac{p}{p-1}\right)\left\{\left((\sigma_1\tilde{\rho})^{-2} + (\sigma_2\tilde{\tau})^{-1}\right)\iint_{C^+_{\tilde{\rho},\tilde{\tau}}(0)} (v^{\frac{p}{2}})^2 dyds + 2^p h^{p\alpha/2}(\sigma_2\tilde{\tau})^{-1}|C^+_{\tilde{\rho},\tilde{\tau}}|\right\} \end{aligned} \tag{5.24}$$

for $0 < \tilde{\rho} < 1$, $0 < \tilde{\tau} < \rho_0^{-2}\tau_0$, $\sigma_1, \sigma_2 \in (0.1)$ and any $p > 1$.

Let's take sequences p_ν, ρ_ν and τ_ν as follows: For $\nu = 0, 1, \cdots$,

$$p_\nu := p(1+\frac{2}{m})^\nu, \rho_\nu := \rho\left(\frac{1}{2}+(\frac{1}{2})^{\nu+1}\right) \quad \text{and} \quad \tau_\nu := \rho^2\left(\frac{1}{4}+\frac{3}{4}(\frac{3}{4})^\nu\right)^\nu.$$

Noticing that, since $a/(a-1) \geq b/(b-1)$ for $1 < a \leq b$,

$$\frac{p_\nu}{p_\nu - 1} < \frac{p}{p-1}$$

and exploiting Sobolev's type inequality(see [9],p76) and (5.24) successively we have that

$$\begin{aligned} &\iint_{C^+_{\rho_{\nu+1},\tau_{\nu+1}}} (v^{\frac{p_\nu}{2}})^{2(1+\frac{2}{m})} dyds \\ &\leq \left(\frac{p}{p-1}\right)^2 2^{2(1+\frac{2}{m})}\beta^{2(1+\frac{2}{m})}\left\{\gamma\left[2^{3(\nu+3)} + \theta^{-1}2^{\nu+2}\right]\right. \\ &\quad \left.\times \iint_{C^+_{\rho_\nu,\tau_\nu}} (v^{\frac{p_\nu}{2}})^2 dyds + \theta^{-1}2^{\nu+2}h^{\alpha p/2}|C^+_{\rho_\nu,\tau_\nu}| + 2^{\nu+4}\iint_{C^+_{\rho_\nu,\tau_\nu}} (v^{\frac{p_\nu}{2}})^2 dyds\right\}^{1+\frac{2}{m}} \\ &\leq \beta^{2(1+\frac{2}{m})}\left(\max(\gamma,1)\right)^{1+\frac{2}{m}} 2^{2(\nu+4)(1+\frac{2}{m})}\left\{\left[1+\theta^{-1}\right]\iint_{C^+_{\rho_\nu,\tau_\nu}} (v^{\frac{p_\nu}{2}})^2 dyds\right. \\ &\quad \left.+\theta^{-1}h^{p\alpha/2}\iint_{C^+_{\rho_\nu,\tau_\nu}} 1 dyds + \iint_{C^+_{\rho_\nu,\tau_\nu}} (v^{\frac{p_\nu}{2}})^2 dyds\right\}^{1+\frac{2}{m}} \\ &\leq \beta^{2(1+\frac{2}{m})}\left(\max(\gamma,1)\right)^{1+\frac{2}{m}} 2^{2(\nu+4)(1+\frac{2}{m})}\left(1+\theta^{-1}\right)^{1+\frac{2}{m}}\left(\iint_{C^+_{\rho_\nu,\tau_\nu}}\left((v^{\frac{p_\nu}{2}})^2 + h^{\alpha p/2}\right)dyds\right)^{1+\frac{2}{m}}. \end{aligned} \tag{5.25}$$

Since $h^{p\alpha/2}$ is constant, we obtain from (5.25) that, for any $\nu = 0, 1, \cdot,$

$$\iint_{C^+_{\rho_{\nu+1},\tau_{\nu+1}}} \left((v^{p_\nu})^{2(1+\frac{2}{m})} + h^{p\alpha/2}\right) dyds$$
$$\leq \beta^{2(1+\frac{2}{m})}\left(\max(\gamma,1)\right)^{1+\frac{2}{m}} 2^{2(\nu+4)(1+\frac{2}{m})}\left(1+\theta^{-1}\right)^{1+\frac{2}{m}} \left(\iint_{C_{\rho_{\nu+1},\tau_{\nu+1}}} \left(v^{p_\nu} + h^{p\alpha/2}\right) dyds\right)^{1+\frac{2}{m}}. \tag{5.26}$$

Divinding the both side of(5.26) by $\rho^m_{\nu+1}\tau_{\nu+1}$, and taking the power of order $1/p_{\nu+1}$ in the resultant inequality, we have, for any $\nu = 0, 1, \cdots,$

$$\left(\rho^{-m}_{\nu+1}\tau^{-1}_{\nu+1}\iint_{C^+_{\rho_{\nu+1},\tau_{\nu+1}}} \left(v^{p_{\nu+1}} + h^{p\alpha/2}\right) dyds\right)^{\frac{1}{p_{\nu+1}}}$$
$$\leq \left[\beta^2\left\{\max(\gamma,1)\right\}2^8(4)^{p^{-1}\nu(1+\frac{2}{m})^{-\nu}}(1+\theta)^{p^{-1}(1+\frac{2}{m})^{-\nu}}\right.$$
$$\times\, (\theta^{\frac{2}{m}})^{p^{-1}(1+\frac{2}{m})^{-\nu-1}}\left(\rho_\nu^{-m}\tau_\nu^{-1}\iint_{C^+_{\rho_\nu,\tau_\nu}} (v^{p_\nu} + h^{p\alpha/2})dyds\right)^{\frac{1}{p_\nu}}.$$

By iterating infinitely with starting p we have

$$\sup_{C^+_{1/2,\theta/2}} v \leq \left[\beta^2\left\{\max(\gamma,1)\right\}2^8\right]^{p^{-1}(1+\frac{m}{2})} 4^{p^{-1}\sum_{j=0}^\infty j(1+\frac{2}{m})^{-j}}(1+\theta)^{p^{-1}(1+\frac{m}{2})}\theta^p$$
$$\times \left(\rho_\nu^{-m}\tau_\nu^{-1}\iint_{C^+_{\rho_\nu,\tau_\nu}} (v^p + h^{p\alpha/2})dyds\right)^{\frac{1}{p}} \tag{5.27}$$

Now we are in a position to show (1.12). Let's classify our proof into two cases: Case1 : $r^2 > h$ and Case2 : $r^2 \leq h$.

Firstly we consider Case1 : $r^2 > h$. Then, by taking $\rho_0{}^2 = \tau_0 = r^2$ and using $r^2 > h$ in (5.27), we have the assertion of Theorem1.3.

Case2. $r^2 \leq h$.Then we deduce from applying Harnack theorem on elliptic equations(see [3], Th.8.17.,p 194) for $\widetilde{u}_n$ in B_1 that, for any $p > 1$ and $q > m$, there exists a positive constant γ depending only on $m, q, p$and λ, μ such that, setting

$$\widetilde{v}_n = \max\{\pm\widetilde{u}_n, 0\} \quad (n = 1, 2, \cdots, N),$$

$$\sup_{B_1}\widetilde{v}_n \leq \gamma\left\{\left(\frac{1}{|B_1|}\int_{B_1}(\widetilde{v}_n)^p\right)^{\frac{1}{p}} + \left(\frac{1}{|B_1|}\int_{B_1} g^q\right)^{\frac{1}{q}}\right\} \tag{5.28}$$

holds for n; $1 \leq n \leq N$ where

$$g = \frac{\widetilde{u}_n - \widetilde{u}_{n-1}}{h/r^2}.$$

Thus, noting (5.17) and that $r^2 \leq h$, the assertion of Theorem1.3 is obtained.

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