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THE FAST AND SLOW GROWING HIERARCHIES AND THE INDUCTIVE DEFINITIONS

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§0. INTRODUCTION

The aim of subrecursive hierarchy theory is to assign ordinal notations to computable functions in such a way as to reflect their computational complexity. We shall consider here, as the complexity measure, the termination proofs of some algorithms for computing them, in particular, the proofs given in $ID_{<\omega}$ (= $\bigcup_{\nu<\omega} ID_\nu$; the theory of finitely iterated inductive definitions). Then the function whose termination proof is given in $ID_{<\omega}$ is called ***provably computable*** in $ID_{<\omega}$.

On the relation between termination proofs and subrecursive hierarchies, Wainer[15],[16] introduced a subrecursive inaccessible ordinal τ, so that for x > 0,

$$G_\tau(x) < F_\tau(x) \leqq G_\tau(x+1)$$

where G_τ and F_τ are the slow and fast-growing functions at τ, respectively. This means that the slow-growing hierarchy catches up with the fast-growing one at stage τ. Then he stated that the ordinal height of τ is $\sup\{|ID_\nu|:\nu<\omega\}$ where $|ID_\nu|$ is the proof-theoretic ordinal of ID_ν, based on the results of Girard[8].

In this article, we shall demonstrate the following (I)~(III) on the relation between termination proofs in $ID_{<\omega}$ and the slow and fast-growing hierarchies:

(I) We introduce an ordinal τ' such that for $x > 3$,

$$G_{\tau'}(x) < F_{\tau'}(x) \leqq G_{\tau'}(G_{\tau'}(x)).$$

This means also that the slow-growing hierarchy catches up with the fast-growing hierarchy at τ'. The reason why we change the definition of τ is to show the collapsing lemma in Section 5.

(II) For each $\alpha \prec \tau'$, the function F_α is provably computable in $ID_{<\omega}$.

(III) If a computable function $f:\omega \longrightarrow \omega$ is provably computable in $ID_{<\omega}$, then f is ***dominated*** by F_α for some $\alpha \prec \tau'$ (i.e., there is an $m < \omega$ such that $f(x) < F_\alpha(x)$ for $x > m$).

Our demonstration here is based on the results of Buchholz[4] on the functions provably computable in $ID_\nu(\nu \leqq \omega)$. On the other hand, Arai[2] had already studied these functions by means of the slow-growing hierarchy. Here we shall first prove the relation of (I) which is a direct estimation of the fast-growing function at τ' by the slow- growing hierarchy at τ', following the idea of [16]; secondly, we shall prove (II) and (III) which imply that τ' corresponds to the proof-theoretic ordinal of $ID_{<\omega}$. Moreover, we shall consider only the case of ID_ν where $\nu < \omega$. The author does not know how to construct τ' which implies (I) and corresponds to the proof theoretic ordinal of ID_ω.

§1. A SUBRECURSIVE INACCESSIBLE ORDINAL τ'

1A. In this section, we shall introduce a (tree-) ordinal τ' and prove that

$$\text{(I)} \qquad G_{\tau'}(x) < F_{\tau'}(x) \leqq G_{\tau'}(G_{\tau'}(x)) \quad \text{for } x > 3.$$

The definition of τ' is slightly changed from that of τ in [15], [16]. The reason why we change the definition of τ is to apply directly Buchholz' method in [4] to our case; we need this change to prove the collapsing lemma of Section 5.

In the following, the letters k, m, n, p, x denote non-negative integers.

1B. TREE ORDINALS AND (p)-BUILT-UPNESS. The hierarchies of number-theoretic functions considered here are defined by recursions over the set of countable ordinals which has an assignment of *fundamental sequences* at limit stages. For a countable limit ordinal λ, we call $\langle\lambda[x]\rangle_{x<\omega}$ a fundamental sequence for λ when it satisfies:

(i) $\lambda[0] < \lambda[1] < \lambda[2] < \cdots < \lambda$,

(ii) $\sup\{\lambda[x] : x < \omega\} = \lambda$.

Following [6], here we shall define the set Ω of countable tree-ordinals which is constructed by assigning the arbitrary chosen fundamental sequences at limit stages as follows:

DEFINITION 1.1 (Countable tree-ordinals). The set Ω of the *countable tree-ordinals* consists of the infinitary terms generated inductively by:

(i) $0 \in \Omega$.

(ii) If $\alpha \in \Omega$, then $\alpha+1 \in \Omega$.

(iii) If $\alpha_x \in \Omega$ for all $x < \omega$, then $(\alpha_x)_{x<\omega} \in \Omega$.

For a given $\alpha \in \Omega$ such that $\alpha = (\alpha_x)_{x<\omega}$, we call α 'limit' and write $\alpha[x]$ for α_x. According to the inductive definition of Ω, proofs and definitions will usually be by induction over the well-founded 'sub-tree' partial ordering on Ω which is denoted $\prec$ and defined as the transitive closure of

(i) $0 \preceq \beta$,

(ii) $\beta \prec \beta+1$,

(iii) $\beta[x] \prec \beta$ for all $x < \omega$ if β is limit.

In order to ensure that $\prec$-predecessors of α are linearly and hence well-ordered, and to develop basic domination properties; we need to restrict attention to tree-ordinals α possessing additional structure.

DEFINITION 1.2 ((p)-built-up tree-ordinals). For a given $p < \omega$, the subset $\Omega^{(p)\text{-bu}} \subset \Omega$ of (p)-*built-up* tree-ordinals consists of those $\alpha \in \Omega$ satisfying that:

$$\lambda[x] \prec_p \lambda[x+1] \quad \text{for all limit } \lambda \preceq \alpha \text{ and } x < \omega,$$

where the relation $\prec_p$ on Ω is defined as the transitive closure of (i) $0 \preceq_p \beta$, (ii) $\beta \prec_p \beta+1$, (iii) $\beta[p] \prec \beta$ if β is limit.

Built-upness and the other related notions on fundamental sequences are studied in [1],[11],[12],[13]. In [16], Wainer used the notion of structuredness (or niceness in [6]) as bases to develop his theory of subrecursive inaccessible ordinals. From the author and Aoyama's study of [11], we can prove the same results to Wainer[16] when we use the notion of (p)-built-upness instead of the structuredness.

LEMMA 1.3. *Let* $p < \omega$ *and* $\alpha \in \Omega^{(p)\text{-bu}}$. *Then the following hold.*

(1) *If* $\beta \prec_m \alpha$ *and* $p \leqq m < n$, *then* $\beta \prec_n \alpha$.

(2) *If* $\beta \prec \alpha$, *then* $\beta \prec_m \alpha$ *for some* $m < \omega$.

(3) *If* $p \leqq m$ *and* $\beta \prec_m \alpha$, *then* $\beta+1 \preceq_{m+1} \alpha$.

Proof. By induction on α. See Lemma 2.3 and Cor.2.8 in [11].□

PROPOSITION 1.4 ([16]). *For each* $p < \omega$ *and* $\alpha \in \Omega^{(p)\text{-bu}}$, *the set* $\{\gamma: \gamma \prec \alpha\}$ *is linearly and hence well-ordered by* $\prec$. *Furthermore, if* $\gamma \prec \alpha$ *then* $\gamma+1 \preceq \alpha$.

Proof: If $\gamma \prec \alpha$ and $\delta \prec \alpha$, choose any m such that $\gamma \prec_m \alpha$ and $\delta \prec_m \alpha$. Then we have $\gamma = \delta$ or $\gamma \prec_m \delta$ or $\delta \prec_m \gamma$. Hence we have $\gamma = \delta$ or $\gamma \prec \delta$ or $\delta \prec \gamma$. Furthermore, if $\gamma \prec \alpha$, then $\gamma \prec_m \alpha$ for some $m < \omega$. Hence $\gamma+1 \preceq_{m+1} \alpha$ by 1.3. Therefore $\gamma+1 \preceq \alpha$. □

1C. HIERARCHIES $\{F_\alpha\}_{\alpha\in\Omega}$, $\{G_\alpha\}_{\alpha\in\Omega}$, $\{F'_\alpha\}_{\alpha\in\Omega}$. We define the fast-growing $\{F_\alpha\}_{\alpha\in\Omega}$ and slow-growing $\{G_\alpha\}_{\alpha\in\Omega}$ hierarchies as follows:

$$F_0(x) = x+1, \qquad G_0(x) = 0,$$

$$F_{\alpha+1}(x) = F_\alpha^x(F_\alpha(x)), \qquad G_{\alpha+1}(x) = G_\alpha(x)+1,$$

$$F_\lambda(x) = F_{\lambda[x]}(x), \qquad G_\lambda(x) = G_{\lambda[x]}(x),$$

where λ is limit and the superscript x denotes iteration x-times of F_α(i.e.,if $F:\omega \longrightarrow \omega$ then $F^0(x) = x$, $F^{m+1}(x) = F(F^m(x))$).

Moreover, we introduce an auxiliary fast-growing hierarchy $\{F'_\alpha\}_{\alpha\in\Omega}$ as follows:

$$F'_0(x) = x+1,$$

$$F'_{\alpha+1}(x) = F'^{x}_{\alpha}(F'_\alpha(x)),$$

$$F'_\lambda(x) = F'_{\lambda[z]}(x), \text{ where } z = F'_{\lambda[1]}(x).$$

Then we have the following proposition which state that these hierarchies indexed by (p)-built-up ordinals have elementary properties on increase and domination.

PROPOSITION 1.5. *For some* $p < \omega$, *we assume* $\alpha \in \Omega^{(p)\text{-bu}}$. *Then the following holds:*

(1) $F_\alpha(x) < F_\alpha(x+1)$, $G_\alpha(x) \leqq G_\alpha(x+1)$ *and* $F'_\alpha(x) < F'_\alpha(x+1)$ *for* $p \leqq x+1$.

(2) *If* $\beta <_m \alpha$ *for* $p \leqq m$, *then* $F_\beta(x) < F_\alpha(x)$, $G_\beta(x) < G_\alpha(x)$ *and* $F'_\beta(x) < F'_\alpha(x)$ *for* $x > m$.

Proof. By induction on α. See Theorem 3.1 of [11]. □

1D. SUBRECURSIVE INACCESSIBILITY. Now let us define the subrecursive inaccessibility on these hierarchies:

DEFINITION 1.6. Let $p < \omega$. We call $\alpha \in \Omega^{(p)\text{-bu}}$ *subrecursive inaccessible* (or *s-inaccessible* for short) if for all $x > p$,

$$G_\alpha(x) < F_\alpha(x) \leqq F'_\alpha(x) \leqq G_\alpha(G_\alpha(x)).$$

This definition slightly differs from the original subrecursive inaccessibility in [15],[16], but they have the same meaning which the slow-growing function at α catches up with the fast-growing one.

LEMMA 1.7. *Let* $p < \omega$ *and* $\alpha \in \Omega^{(p)\text{-bu}}$.

(1) *For all* $x > p$, $G_\alpha(x) < F_\alpha(x) \leqq F'_\alpha(x)$.

(2) *If* α *is s-inaccessible, then* α *is limit and* G_α^2 *dominates every* F'_β *with* $\beta \prec \alpha$ *(i.e., for all but finitely many* x, $F_\beta(x) < G_\alpha(G_\alpha(x))$).

Proof. (1) Induction on α. (2) Clearly α cannot be 0. For any $\beta+1 \in \Omega^{(p)\text{-bu}}$ and $x > p$,

$$G_{\beta+1}(x) = G_\beta(x)+1 < F'_\beta(x)+1 \leqq F'_\beta(x+1) \leqq F'_\beta(F'_\beta(x)) \leqq F'^{x+1}_\beta(x) = F'_{\beta+1}(x).$$

Hence α must be limit. On the other hand, we can prove that if $\beta \prec \alpha$, then there is an $m < \omega$ such that $\beta \prec_m \alpha$ since α is (p)-built-up. Hence, F_α dominates every F_β with $\beta \prec \alpha$. Therefore G_α^2 dominates every F_β with $\beta \prec \alpha$. □

PROPOSITION 1.8 ([16,p.215]). *Let* $p < \omega$ *and* $\alpha \in \Omega^{(p)\text{-bu}}$ *satisfy that*

$$G_{\alpha[n+1]} = F'_{\alpha[n]} \textit{ for all } n < \omega.$$

Then α *is s-inaccessible and, if* $\alpha[0]$ *is finite (i.e.,* $\alpha[0] = 0+1+\cdots+1$), *then no* $\beta \prec \alpha$ *is s-inaccessible.*

Proof. If $G_{\alpha[n+1]} = F'_{\alpha[n]}$ for each n and $z = F_{\alpha[1]}(x)$, then

$$z = F_{\alpha[1]}(x) = G_{\alpha[2]}(x) < G_{\alpha[x]}(x) = G_\alpha(x)$$

for $x > \max(2,p)$. Hence we have that

$$F'_\alpha(x) = F'_{\alpha[z]}(x) = G_{\alpha[z+1]}(x) \leqq G_{\alpha[G_\alpha(x)]}(x)$$
$$\leqq G_{\alpha[G_\alpha(x)]}(G_\alpha(x)) = G_\alpha(G_\alpha(x)) = G^2_\alpha(x).$$

(Since α is limit and (p)-built-up, we have $G_\alpha(x) \geqq x$ $(x > p)$.) So α is s-inaccessible. If also $\alpha[0]$ is finite and $\beta \prec \alpha$ were s-inaccessible then $\alpha[0] \prec \beta$ since β is limit. So $\alpha[n] \prec \beta \precsim \alpha[n+1]$ for some n. By 1.7, for sufficient large x,

$$G^2_{\alpha[n+1]}(x) = F'^2_{\alpha[n]}(x) \leqq F'^{x+1}_{\alpha[n]}(x) = F_{\alpha[n]+1}(x) < G^2_\beta(x)$$

since $\alpha[n]+1 \prec \beta$. This is a contradiction, since $\beta \precsim \alpha[n+1]$ and therefore $G^2_\beta(x) \leqq G^2_{\alpha[n+1]}(x)$ for sufficiently large x. □

This proposition suggests a method for constructing a minimal s-inaccessible which we shall denote

$$\tau' = (\tau'[x])_{x<\omega}.$$

First choose $\tau' = 3$ for which F_3 dominates all functions elementary in $\{F_\beta : \beta \prec 3\}$. Then if $\tau'[0],\dots,\tau'[n]$ have already defined, choose $\tau'[n+1]$ so that $G_{\tau'[n+1]} = F_{\tau'[n]}$.

1E. A MINIMAL S-INACCESSIBLE τ'. We introduce a minimal s-inaccessible τ' as in [15],[16]. Just as the fast-growing hierarchy uses countable tree-ordinals α to name big number-theoretic functions F'_α, we can use uncountable tree ordinals α to name big ordinal-functions $\varphi(\alpha):\Omega \longrightarrow \Omega$. These can be used to name bigger number-theoretic functions $F'_{\varphi(\alpha)(\beta)}$ etc. This idea leads to a collection of higher level fast-growing hierarchies $\varphi_n(\alpha):\Omega_n \longrightarrow \Omega_n$ where α ranges over the next higher tree class Ω_{n+1}.

DEFINITION 1.9 ([15]). The sets Ω_n of *higher level tree-*

ordinals are defined by induction similarly to the case of Ω:

(i) $0 \in \Omega_n$.

(ii) If $\alpha \in \Omega_n$, then $\alpha+1 \in \Omega_n$.

(iii) If $\alpha_\gamma \in \Omega_n$ for all $\gamma \in \Omega_k (k<n)$, then $(\alpha_\gamma)_{\gamma\in\Omega_k} \in \Omega_n$.

As in the case of Ω, we call $(\alpha_\gamma)_{\gamma\in\Omega_k}$ limit and write $\alpha[\gamma]$ instead of α_γ. We shall identify Ω_0 with ω, and Ω_1 with Ω, in the following.

DEFINITION 1.10 ([15,Definition 5]). The level n fast-growing hierarchies of functions $\varphi_n:\Omega_{n+1}\times\Omega_n \longrightarrow \Omega_n$ is defined by:

(i) $\varphi_n(0,\beta) = \beta+1$,

(ii) $\varphi_n(\alpha+1,\beta) = \varphi_n{}^{\beta}(\alpha,\varphi_n(\alpha,\beta))$,

(iii) $\varphi_n(\lambda,\beta) = (\varphi_n(\lambda[\gamma],\beta))_{\gamma\in\Omega_k}$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_k}$ $(k<n)$,

(iv) $\varphi_n(\lambda,\beta) = \varphi_n(\lambda[z],\beta)$, $z = \varphi_n(\lambda[1],\beta)$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_n}$

where $\varphi_n{}^{\beta}$ denotes the iteration β-times of φ_n(i.e.,if $\psi:\Omega_{n+1}\times\Omega_n \longrightarrow \Omega_n$, then $\psi^0(\alpha,\beta) = \beta$, $\psi^{\delta+1}(\alpha,\beta) = \psi(\alpha,\psi^{\delta}(\alpha,\beta))$, $\psi^{\lambda}(\alpha,\beta) = (\psi^{\lambda[\gamma]}(\alpha,\beta))_{\gamma\in\Omega_m}$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_m}$.

Note that, in the case $n = 0$, $\varphi_0(\alpha,\beta) = F_\alpha(\beta)$ for $\alpha \in \Omega_1$ and $\beta \in \Omega_0 (= \omega)$. We define $\omega_k \in \Omega_n$ by $\omega_k = (\gamma)_{\gamma\in\Omega_k}$ (i.e., $\omega_k[\gamma] = \gamma$).

DEFINION 1.11 ([15,Definition 7]). The sets T_n $(\subset \Omega_n)$ of *named tree-ordinals* are defined inductively by:

(i) $0, 1, \omega_0,\cdots,\omega_{n-1} \in T_n$.

(ii) $T_k \subset T_n$ for $k < n$.

(iii) If $\alpha \in T_{n+1}$ and β, $\gamma \in T_n$, then $\varphi_n{}^{\gamma}(\alpha,\beta) \in T_n$.

THEOREM 1.12 (Collapsing theorem [15]). *Let* $x < \omega$, $\alpha \in T_2$ *and* $\beta \in T_1$. *Then*

$$G_{\varphi_1(\alpha,\beta)}(x) = F'_{c\alpha}(G_\beta(x)),$$

where the function c *(*$= c_x$*) which collapses each* T_{n+1} *to* T_n *is defined by:* $c0 = 0$, $c1 = 1$, $c\omega_0 = x$, $c\omega_{k+1} = \omega_k$, $c(\varphi_{k+1}{}^{\gamma}(\delta,\xi)) = \varphi_k{}^{c\gamma}(c\delta,c\xi)$, $c(\varphi_0{}^{\gamma}(\delta,\xi)) = \varphi_0{}^{\gamma}(\delta,\xi)$. *Hence, in particular, if* α *is generated in* T_2 *without reference to* ω_0 *then, as* $G_{\omega_0}(x) = x$, *we have* $G_{\varphi_1(\alpha,\omega_0)} = F'_{c\alpha}$.

We shall prove this theorem in Section 3 by using the strong normalization theorem in Section 2. Together with Proposition 1.8, we can construct a minimal s-inaccessible ordinal as follows:

DEFINITION 1.13 ([15,Example 4]). We define $\tau' = (\tau'[x])_{x<\omega}$ by setting $\tau'[0] = 3$,

$$\tau'[n+1] = \varphi_1(\ldots\varphi_n(\varphi_{n+1}(3,\omega_n),\omega_{n-1}),\ldots,\omega_0).$$

THEOREM 1.14. τ' *is a minimal s-inaccessible tree-ordinal.*

Proof. From Section 4, τ' is (3)-built-up. Hence 1.8 and the collapsing theorem(1.12) complete the proof. □

§2. PROVABLE COMPUTABILITY OF F_α ($\alpha \prec \tau'$)

2A. In this section, we shall prove Theorem 2.10;

(II) for $\alpha \prec \tau'$, F_α and F'_α are provably computable in $ID_{<\omega}$.

(For the definition of $ID_{<\omega}$, see Section 5.) As the corollaries of this proof, we shall prove also that the collapsing theorem in Section 3, and (3)-built-upness of τ' in Section 4 which we had used to prove (I) in Section 1.

To prove (II) above, we shall introduce the term structures for the sets T_n ($n<\omega$). Then we shall prove the strong normalization theorem for the structures. Our method here is the same as that of [4,Section 2] and our results of this section (and of Sections 3,4 below) comes from those of [10].

2B. THE TERM STRUCTURES. We introduce term structures $<\bar{T}_n, NT_n, \cdot[\cdot], \longrightarrow>$ ($n<\omega$) by considering each element in T_n as a finitary term and each defining equation of φ_n (Definition 1.10) as a rewrite (or reduction) rule of the terms. Let $\bar{0}$, $\bar{1}$, $\bar{\omega}_0$, $\bar{\omega}_1$, ...; $\bar{\varphi}_0$, $\bar{\varphi}_1$, ... be formal symbols.

DEFINITION 2.1. The sets $\bar{T}_n$ of *terms* are defined inductively by:

(i) $\bar{0}$, $\bar{1}$, $\bar{\omega}_0$, $\bar{\omega}_1, \ldots, \bar{\omega}_{n-1} \in \bar{T}_n$.

(ii) $\bar{T}_k \subset \bar{T}_n$ for $k < n$.

(iii) If $a \in \bar{T}_{n+1}$ and b, $c \in \bar{T}_n$, then $\bar{\varphi}_n{}^c(a,b) \in \bar{T}_n$.

Naturally, terms in $\bar{T}_n$ are interpreted as tree-ordinals by the function ord: $\bar{T}_n \longrightarrow T_n$ such that (i) $\mathrm{ord}(\bar{0}) = 0$, $\mathrm{ord}(\bar{1}) = 1$, $\mathrm{ord}(\bar{\omega}_k) = \omega_k$, (ii) $\mathrm{ord}(\bar{\varphi}_n{}^c(a,b)) = \varphi_n{}^{\mathrm{ord}(c)}(\mathrm{ord}(a),\mathrm{ord}(b))$.

Abbreviations. $\bar{\varphi}_n(a,b) = \bar{\varphi}_n{}^{\bar{1}}(a,b)$, $b+1 = \bar{\varphi}_n(\bar{0},b)$.

DEFINITION 2.2 (Normal terms). The sets NT_n of *normal terms* in $\bar{T}_n$; $\mathrm{dom}(a) \in \{\phi,\{\bar{0}\},\bar{T}_0,\ldots,\bar{T}_{n-1}\}$ and $a[s]$ for $a \in NT_n$, $s \in \mathrm{dom}(a)$ are defined inductively by:

(N1) $\bar{0} \in NT_n$; $\mathrm{dom}(\bar{0}) = \phi$.

(N2) $\bar{1} \in NT_n$; $\mathrm{dom}(\bar{1}) = \{\bar{0}\}$, $\bar{1}[\bar{0}] = \bar{0}$.

(N3) $\bar{\omega}_i \in NT_n$ $(i<n)$; $\mathrm{dom}(\omega_i) = \bar{T}_i$, $\omega_i[s] = s$.

(N4) $NT_k \subset NT_n$ for $k < n$.

(N5) Let $a \in NT_{n+1}$, $b,c \in NT_n$ and $A = \bar{\varphi}_n{}^c(a,b)$. Then $A \in NT_n$ if one of the following holds:

(i) $c = \bar{1}$ and $a = \bar{0}$(i.e.,$A = b+1$); $\mathrm{dom}(A) = \{\bar{0}\}$, $A[s] = b$.

(ii) $\mathrm{dom}(c) = \bar{T}_k(k<n)$; $\mathrm{dom}(A) := \mathrm{dom}(c)$, $A[s] = \bar{\varphi}_n{}^{c[s]}(a,b)$.

(iii) $c = \bar{1}$ and $\mathrm{dom}(a) = \bar{T}_k(k<n)$; $\mathrm{dom}(A) = \mathrm{dom}(a)$, $A[s] = \bar{\varphi}_n(a[s],b)$.

Next we introduce a *term-rewriting system*(S) (see e.g., Dershowitz[7] as for the definition) so that, for every term in $\bar{T}_n$ which is not normal, some rewrite rule in (S) is applicable to it.

Definition of the rewrite rules of (S): For normal a,b,c;

(R1) $\bar{\varphi}_n{}^{\bar{0}}(a,b) \longrightarrow b$, (R2) $\bar{\varphi}_n(\bar{1},b) \longrightarrow \bar{\varphi}_n{}^b(\bar{0},\bar{\varphi}_n(\bar{0},b))$,

(R3) $\bar{\varphi}_n(a+1,b) \longrightarrow \bar{\varphi}_n{}^b(a,\bar{\varphi}_n(a,b))$, (R4) $\bar{\varphi}_n^{\,c+1}(a,b) \longrightarrow \bar{\varphi}_n(a,\bar{\varphi}_n^{\,c}(a,b))$,

(R5) $\bar{\varphi}_n(a,b) \longrightarrow \bar{\varphi}_n(a[z],b)$ with $z = \varphi_n(a[1],b)$
if $\mathrm{dom}(a) = \bar{T}_n$.

Every rule in (R1)-(R5) may be applied to a term $A \in \bar{T}_n$ if A contains a subterm of the left-hand side of the rule. Then the rule is used by replacing the subterm to the right-hand side of the rule. We write $A \xrightarrow{1} B$ to indicate that the term B is obtained from the term A by a single application of some rule. We have the following fundamental proposition.

PROPOSITION 2.3. (1) *For every* $a \in \bar{T}_n$, $a \in NT_n$ *if and only if there is no* $b \in \bar{T}_n$ *such that* $a \xrightarrow{1} b$.

(2) (i) *If* $a \in NT_n$ *and* $a = b+1$ *for some* b, *then* $\mathrm{ord}(a) = \mathrm{ord}(b)+1$.

(ii) *If* $a \in NT_n$ *and* $\mathrm{dom}(a) = \bar{T}_k$ $(k<n)$, *then* $\mathrm{ord}(a) = (\mathrm{ord}(a)[\gamma])_{\gamma\in\Omega_k}$ *and* $\mathrm{ord}(a[b]) = \mathrm{ord}(a)[\mathrm{ord}(b)]$ *for* $a \in \bar{T}_k$.

(iii) *If* $a \in \bar{T}_n$ *and* $a \xrightarrow{1} b$, *then* $\mathrm{ord}(a) = \mathrm{ord}(b)$.

Proof. Induction on the length of a. □

2C. THE STRONG NORMALIZATION THEOREM. Now we say that a term $A \in \bar{T}_n$ is *strongly normalizable* if every derivation sequence starting at A(i.e.,$A \xrightarrow{1} A' \xrightarrow{1} A'' \xrightarrow{1} \cdots$) is finite (cf.[9]). Then we prove the following theorem:

THEOREM 2.4 (Strong normalization theorem[10,Theorem 1]). *Every term* a *in* $\bar{T}_n$ *is strongly normalizable.*

We can also show that the term rewriting system (S) has the Church-Rosser property (i.e., if $A \Rightarrow B$ and $A \Rightarrow C$, then there is a D such that $B \Rightarrow D$ and $C \Rightarrow D$, where $a \Rightarrow b$ indicates that b is obtained from a by a finite (perhaps empty) series of reduction "$\xrightarrow{1}$".). This can be shown by induction on the length of terms A. However, we do not need this property in the article.

We devote the rest of this section to prove the strong normalization theorem and, as a corollary, prove Theorem 2.10. First, we introduce the subsets W_n of $\bar{T}_n$ which express all strongly normalizable terms of $\bar{T}_n$. Then we prove that $\bar{T}_n = W_n$ in $ID_{<\omega}$.

DEFINITION 2.5. For $n < \omega$, the sets W_n $(\subset \bar{T}_n)$ are defined inductively by:

(W1) $\bar{0} \in W_n$.

(W2) If $a \in \bar{T}_n$ is normal and $a[s] \in W_n$ for all $s \in \mathrm{dom}(a)$, then $a \in W_n$.

(W3) If $a \in \bar{T}_n$ is not normal and $b \in W_n$ for all b such that $a \xrightarrow{1} b$, then $a \in W_n$.

We can easily show that every term in W_n is strongly normalizable as follows. From the inductive definition of W_n, the following partial ordering $<<$ on $\bigcup_{n<\omega} W_n$ is well-founded: $<<$ is defined as the transitive closure of

(i) $\bar{0} \underline{\underline{<<}} a$,

(ii) $a[s] << a$ where a is normal and $s \in \mathrm{dom}(a)$,

(iii) $b << a$ where a is not normal and $a \xrightarrow{1} b$.

Hence, if $A \in W_n$, there is no infinite sequence $<A_i : i<\omega>$ such that $A = A_0$, $A_{i+1} << A_i$. Thus, in particular, every term in W_n is strongly normalizable.

We remark here that, as usual, we can extend $<<$ to the lexicographic orderings $<<$ on $W_{n+1}\times W_n$ and $W_{n+1}\times W_n\times W_n$ which are also well-founded. To prove the strong normalization theorem, we show the following theorem.

THEOREM 2.6. *For each* $a \in \bar{T}_n$, "$a \in W_n$" *is provable in* $ID_{<\omega}$.

LEMMA 2.7. ($ID_{<\omega}$) *Let* $a \in W_{n+1}$ *and* $b, c \in W_n$. *If* $\bar{\varphi}_n(a,d) \in W_n$ *for all* $d \in W_n$, *then* $\bar{\varphi}_n{}^c(a,b) \in W_n$.

Proof. By induction on $(a,b,c) \in W_{n+1}\times W_n\times W_n$ over $<<$. Let $A = \bar{\varphi}_n{}^c(a,b)$. We have the following cases:

Case 1. $A \in NT_n$ and $dom(A) = \{\bar{0}\}$: Then $A = \bar{\varphi}_n(\bar{0},b)$. By the assumption, $A \in W_n$.

Case 2. $A \in NT_n$ and $dom(A) = \bar{T}_k (k<n)$: Let $s \in \bar{T}_k$.

(i) $dom(c) = \bar{T}_k$: Then $c[s] << c$. By I.H.(= induction hypotheses), $A[s] = \bar{\varphi}_n{}^{c[s]}(a,b) \in W_n$.

(ii) $c = \bar{1}$ and $dom(a) = \bar{T}_k$: By the assumption, $A \in W_n$. Hence $A[s] \in W_n$ by (W2). Hence, $A \in W_n$ by (W2).

Case 3. $A \in \bar{T}_n \setminus NT_n$: Let $A \xrightarrow{1} B$. We will show $B \in W_n$.

(i) $A = \bar{\varphi}_n{}^{\bar{0}}(a,b)$ and $B = b$: Then $B \in W_n$.

(ii) $A = \bar{\varphi}_n(a,b)$: Since $A \in W_n$ by the assumption, $B \in W_n$ by (W3).

(iii) $A = \bar{\varphi}_n{}^{e+1}(a,b)$ and $B = \bar{\varphi}_n(a,\bar{\varphi}_n{}^e(a,b))$: From $e << e+1$ and I.H., $\bar{\varphi}_n{}^e(a,b) \in W_n$. Hence $B \in W_n$ by the assumption.

(iv) In all other cases(e.g.,$A = \bar{\varphi}_n{}^c(a,b)$, $B = \bar{\varphi}_n{}^c(a',b)$ and

$a \xrightarrow{1} a'$), $B \in W_n$ follows immediately from I.H. Hence $A \in W_n$ by (W3). □

LEMMA 2.8. ($ID_{<\omega}$) *For* $a \in W_{n+1}$ *and* $b \in W_n$, $\bar{\varphi}_n(a,b) \in W_n$.

Proof. By induction on $(a,b) \in W_{n+1} \times W_n$ over $<<$. Let $A = \bar{\varphi}_n(a,b)$.

Case 1. $A \in NT_n$ and $dom(A) = \{\bar{0}\}$: Then $A = \bar{\varphi}_n(\bar{0},b)$ and $A[\bar{0}] = b \in W_n$. Hence $A \in W_n$ by (W2).

Case 2. $A \in NT_n$ and $dom(A) = \bar{T}_k (k<n)$: Then $dom(a) = \bar{T}_n$ and $A[s] = \bar{\varphi}_n(a[s],b)$ for $s \in dom(A)$. From $a[s] << a$ and I.H., $A[s] \in W_n$. Hence $A \in W_n$ by (W2).

Case 3. $A \in \bar{T}_n \setminus NT_n$: Let $A \xrightarrow{1} B$. We will show $B \in W_n$.

(i) $A = \bar{\varphi}_n(a'+1,b)$ and $B = \bar{\varphi}_n{}^b(a', \bar{\varphi}_n(a',b))$: By I.H., $\bar{\varphi}_n(a',d) \in W_n$ for all $d \in W_n$. Hence, $\bar{\varphi}_n(a',b) \in W_n$ and $B \in W_n$ by 2.7.

(ii) $A = \bar{\varphi}_n(a,b)$ and $B = \bar{\varphi}_n(a[z],b)$ where $a \in NT_{n+1}$, $dom(a) = \bar{T}_n$, $z = \bar{\varphi}_n(a[1],b)$: Since $a \in W_{n+1}$, $b \in W_n$ and $1 \in W_n$, we have $a[1] \in W_{n+1}$ by (W2). So $z \in W_n$ from I.H. and $a[1] << a$. Hence $a[z] \in W_{n+1}$ by (W2). Therefore $B \in W_n$ from $a[z] << a$ and I.H.

(iii) In all other cases (e.g., $A = \bar{\varphi}_n(a,b)$, $B = \bar{\varphi}_n(a',b)$ and $a \xrightarrow{1} a'$), $B \in W_n$ follows immediately from I.H. Hence $A \in W_n$ by (W3). □

LEMMA 2.9. ($ID_{<\omega}$) *For* $a \in W_{n+1}$ *and* $b,c \in W_n$, $\bar{\varphi}_n{}^c(a,b) \in W_n$.

Proof. Immediate from 2.7 and 2.8.

Proof of Theorem 2.6. By induction on the length of $a \in T_n$.

Clearly, $\bar{0}$, $\bar{1}$, $\bar{\omega}_0, \ldots, \bar{\omega}_{n-1} \in W_n$ and $W_k \subset W_n$ for $k < n$. By 2.9, $\bar{\varphi}_n{}^c(d,b) \in W_n$ for $d \in W_{n+1}$ and b, $c \in W_n$. This completes the proof. □

Proof of Theorem 2.4(Strong normalization theorem). From 2.6, we have $T_n = W_n$. Hence, if we consider the well-founded ordering $<<$ on W_n defined above, it is also the well-founded ordering on T_n. If there were an infinite sequence $\{a_i\}_{i<\omega}$ such that $a_0 \xrightarrow{1} a_1 \xrightarrow{1} a_2 \xrightarrow{1} \cdots$, then it is an infinite descending sequence on $<<$ such that $\cdots << a_2 << a_1 << a_0$. This contradicts the well-foundedness of $<<$ on T_n. Hence the proof of the strong normalization theorem is completed. □

THEOREM 2.10. *For each* $\alpha \prec \tau'$, F_α *and* F'_α *are provably computable in* $ID_{<\omega}$.

Proof. Let $\alpha \prec \tau'$. Then $\alpha \prec \tau'[m] \in T_1$ for some $m < \omega$. Hence $\alpha \in T_1$ and there is an $a \in \bar{T}_1$ such that $\alpha = ord(a)$. From 2.9, $\forall x(\bar{\varphi}_0(a,\bar{x}) \in W_0)$ is provable in $ID_{<\omega}$ where $\bar{x}$ is the numeral of x (i.e., if $x = 0$, then $\bar{x}$ is the numeral of x (i.e., if $x = 0$ then $\bar{x} = \bar{0}$; if $x = 1$ then $\bar{x} = \bar{1}$; if $x > 1$ then $\bar{x} = \bar{\varphi}_0(0,\overline{x-1})$). Hence $\forall x \exists y(\bar{\varphi}_0(a,\bar{x}) \xrightarrow{1} \cdots \xrightarrow{1} \bar{y})$ is provable in $ID_{<\omega}$. On the other hand, $ord(\bar{\varphi}_0(a,\bar{x})) = F'_\alpha(x)$ and $ord(\bar{y}) = y$. And we have that if $b \xrightarrow{1} d$ then $ord(b) = ord(d)$. Hence $\forall x \exists y(\bar{\varphi}_0(a,\bar{x}) \xrightarrow{1} \cdots \xrightarrow{1} \bar{y})$ equals to $\forall x \exists y(F'_\alpha(x)=y)$. Therefore F'_α is provably computable in $ID_{<\omega}$. Moreover, we have $F_\alpha(x) \leqq F'_\alpha(x)$. Hence $\forall x \exists y(F'_\alpha(x)=y)$ implies $\forall x \exists y(F_\alpha(x)=y)$ in $ID_{<\omega}$. Therefore F_α is also provably computable in $ID_{<\omega}$. □

§3. COLLAPSING THEOREM

3A. As a corollary of the strong normalization theorem proved above, we shall prove the collapsing theorem(Theorem 1.12) used in Section 1:

THEOREM 1.12 (Collapsing Theorem [15]). *Let* $x < \omega$, $\alpha \in T_2$ *and* $\beta \in T_1$. *Then*

$$G_{\varphi_1(\alpha,\beta)}(x) = F'_{c\alpha}(G_\beta(x)),$$

where the function c *(=* c_x*) which collapses each* T_{n+1} *to* T_n *is defined by:* $c0 = 0$, $c1 = 1$, $c\omega_0 = x$, $c\omega_{k+1} = \omega_k$, $c(\varphi_{k+1}{}^{\gamma}(\delta,\xi)) = \varphi_k{}^{c\gamma}(c\delta,c\xi)$, $c(\varphi_0{}^{\gamma}(\delta,\xi)) = \varphi_0{}^{\gamma}(\delta,\xi)$. *Hence, in particular, if* α *is generated in* T_2 *without reference to* ω_0 *then, as* $G_{\omega_0}(x) = x$, *we have* $G_{\varphi_1(\alpha,\omega_0)} = F'_{c\alpha}$.

We introduce a function $\bar{c}$ which represents the function c on the terms as follows: (for each fixed $x < \omega$) (i) $\bar{c}\bar{0} = \bar{0}$, $\bar{c}\bar{1} = \bar{1}$, $\bar{c}\bar{\omega}_0 = \bar{x}$, $\bar{c}\bar{\omega}_{k+1} = \bar{\omega}_k$,
(ii) $\bar{c}(\bar{\varphi}_{n+1}{}^{\gamma}(\delta,\xi)) = \bar{\varphi}_n{}^{\bar{c}\gamma}(\bar{c}\delta,\bar{c}\xi)$ and $\bar{c}(\bar{\varphi}_0{}^{\gamma}(\delta,\xi)) = \bar{\varphi}_0{}^{\gamma}(\delta,\xi)$,
where $\bar{x}$ is the ***numeral*** of x(i.e.,if $x = 0$ then $\bar{x} = \bar{0}$; if $x = 1$ then $\bar{x} = \bar{1}$; if $x > 1$ then $\bar{x} = \bar{\varphi}_0(\bar{0},\overline{x-1})$ $(= \overline{x-1}+1)$).

LEMMA 3.1. *Let* $a \in \bar{T}_n$ *and* $x < \omega$. *Then the following hold.*

(1) *If* $a = b+1$ *for some* b, *then* $\bar{c}(b) = \bar{c}b+1$.

(2) *If* $a \in NT_n$ *and* $\mathrm{dom}(a) = \bar{T}_0$, *then* $\bar{c}(a[\bar{x}]) = \bar{c}a$ *and* $\mathrm{ord}(a[\bar{x}]) = \mathrm{ord}(a)$.

(3) *If* $a \in NT_n$ *and* $\mathrm{dom}(a) = \bar{T}_k$ *for some* $k > 0$, *then*

$\mathrm{ord}(a[b]) = \mathrm{ord}(a)[\mathrm{ord}(b)]$ *and*

$\mathrm{ord}(\bar{c}(a[b])) = \mathrm{ord}(\bar{c}a)[\mathrm{ord}(\bar{c}b)]$ *for* $b \in \mathrm{dom}(a)$.

(4) *If* $a \xrightarrow{1} b$, *then* $\mathrm{ord}(a) = \mathrm{ord}(b)$ *and* $\mathrm{ord}(\bar{c}a) = \mathrm{ord}(\bar{c}b)$.

Proof. (1)-(4) Induction on the length of a. □

LEMMA 3.2. *If* $x < \omega$ *and* $a \in \bar{T}_1$, *then* $G_{\mathrm{ord}(a)}(x) = \mathrm{ord}(\bar{c}a)$.

Proof. From the strong normalization theorem(Theorem 2.4), the proof is proceeded by transfinite induction on a over the well-founded ordering $\ll$ (where $\ll$ on $\bar{T}_n$ is defined as the transitive closure of (i) $\bar{0} \underline{\underline{\ll}}\, b$, (ii) $b[z] \ll b$ for normal b with $z \in \mathrm{dom}(b)$, (iii) $d \ll b$ for non-normal b with $b \xrightarrow{1} d$).

Case 1. $a = \bar{0}$. This case is trivial.

Case 2. $a \in NT_1$ and $\mathrm{dom}(a) = \{\bar{0}\}$. Then $a = \bar{1}$ or $b+1$ for some $b \in \bar{T}_1$. If $a = \bar{1}$, the assertion is trivial. If $a = b+1$, then

$$G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(b)}(x)+1 = \mathrm{ord}(\bar{c}b)+1 = \mathrm{ord}(\bar{c}a)$$

by I.H. and 3.1(1).

Case 3. $a \in NT_1$ and $\mathrm{dom}(a) = \bar{T}_0$. By 3.1(2) and I.H.,

$$G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(a[\bar{x}])}(x) = \mathrm{ord}(\bar{c}(a[\bar{x}])) = \mathrm{ord}(\bar{c}a).$$

Case 4. $a \xrightarrow{1} b$ for some b. By 3.1(4) and I.H.,

$$G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(b)}(x) = \mathrm{ord}(\bar{c}b) = \mathrm{ord}(\bar{c}a).$$ □

Proof of the collapsing theorem(Theorem 1.12). For $a' \in \bar{T}_2$ and $b \in \bar{T}_1$, we have $\bar{c}(\bar{\varphi}_1(a,b)) = \bar{\varphi}_0(\bar{c}a,\bar{c}b)$ and hence $\mathrm{ord}(\bar{c}(\bar{\varphi}_1(a,b)) = \varphi_0(\mathrm{ord}(\bar{c}a),\mathrm{ord}(\bar{c}b))$. Thus we have

$$\begin{aligned} G_{\varphi_1(\mathrm{ord}(a),\mathrm{ord}(b))}(x) &= G_{\mathrm{ord}(\bar{\varphi}_1(a,b))}(x) \\ &= \mathrm{ord}(\bar{c}(\bar{\varphi}_1(a,b)) \quad \text{by } 3.2 \end{aligned}$$

$$= \varphi_0(\mathrm{ord}(\bar{c}a), \mathrm{ord}(\bar{c}b))$$
$$= F_{\mathrm{ord}(\bar{c}a)}(\mathrm{ord}(\bar{c}b))$$
$$= F_{\mathrm{ord}(\bar{c}a)}(G_{\mathrm{ord}(b)}(x)) \text{ by 3.2.}$$

For given $\alpha \in T_2$ and $\beta \in T_1$, we choose a and b above such that (i) $\mathrm{ord}(a) = \alpha$, $\mathrm{ord}(\bar{c}a) = c\alpha$ and (ii) $\mathrm{ord}(b) = \beta$ (we can choose such a and b since the elements of T_n are constructed by the same way as to the element in $\bar{T}_n$). This completes the proof. □

We recall that (Definition 1.13);

$\tau'[0] = 3$, $\tau'[n+1] = \varphi_1(\ldots\varphi_n(\varphi_{n+1}(3,\omega_n),\omega_{n-1}),\ldots,\omega_0)$.

We have the following figure from the collapsing theorem and Proposition 1.8:

$$\begin{array}{llll}
\tau'[0] = 3 & \cdots\cdots\cdots & G_{\tau'[0]}(x) & = 3 \\
\tau'[1] = \varphi_1(3,\omega_0) & \cdots\cdots\cdots & G_{\tau'[1]}(x) & = F'_{\tau'[0]}(x) \\
\tau'[2] = \varphi_1(\varphi_2(3,\omega_1),\omega_0) & \cdots & G_{\tau'[2]}(x) & = F'_{\tau'[1]}(x) \\
\tau'[3] & \cdots\cdots\cdots & G_{\tau'[3]}(x) & = F'_{\tau'[2]}(x) \\
\vdots & & & \vdots \\
\tau' = (\tau'[x])_{x<\omega} & \cdots & \multicolumn{2}{l}{G_{\tau'}(x) < F_{\tau'}(x) \leqq F'_{\tau'}(x) \leqq G^2_{\tau'}(x)} \\
(\tau' \text{ is minimal s-inaccessible}) & & & (x > p).
\end{array}$$

Figure 3.1.

§4. (3)-BUILT-UPNESS OF τ'

4A. In this section we shall prove the following theorems:

THEOREM 4.9. *Every element in* T_1^+ *is* (k)*-built-up for all* $k < \omega$.

THEOREM 4.10. τ' *is* (3)*-built-up.*

This corollary completes the proof of Theorem 1.14 that τ' is minimal s-inaccessible.

The notion of built-upness of fundamental sequences is first introduced by Schmidt[12] and the author and Aoyama[11] studied some other related notions of built-upness including (p)-built-upness.

In this section, we shall also introduce the sets of T_n^* ($\subseteq T_n^+ \subseteq T_n$) for the use of the next section. To begin with, we prove the following proposition which is needed to prove our theorems below.

PROPOSITION 4.1 ([10,Lemma 3.4]). *Let* $\alpha \in T_n$ *and* $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m}$. *Then* $\alpha[\gamma] \in T_n$ *for every* $\gamma \in T_m$. *Moreover, if* $\gamma \in T_m\setminus\{0\}$, *then* $\alpha[\gamma] \in T_n\setminus\{0\}$.

Proof. For a given $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m} \in T_n$, there is a normal $a \in \bar{T}_n$ such that $\mathrm{ord}(a) = \alpha$ by 2.3(2)(iii) and the strong normalization theorem. We fix such an $a \in T_n$ with the minimal length. The proof of this proposition can be proceeded by induction on the length of this term a for α. □

It follows from this proposition that we can use transfinite induction on the terms in T_n $(n<\omega)$ over the ordering $\prec$ of T_n which is defined in the same way as $\prec$ in Ω; i.e., $\prec$ is the transitive closure of (i) $0 \preceq \alpha$, (ii) $\alpha \prec \alpha+1$, (iii) $\alpha[\gamma] \prec \alpha$ for all $\gamma \in T_n$ if $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_n}$. Next we extend the relation $\prec_k$ $(k<\omega)$ on Ω to higher level tree-ordinals from this proposition.

DEFINITION 4.2. The *step-down relations* $\prec_k$ $(k<\omega)$ on $\cup_{n<\omega}T_n$ are defined inductively as follows: For α, $\beta \in T_n$,

$\alpha \prec_k \beta$ if $\beta \neq 0$ and one of the following holds;

(i) $\alpha \preceq_k \gamma$ if $\beta = \gamma + 1$,

(ii) $\alpha \preceq_k \beta[k]$ if $\beta = (\beta[x])_{x\in\Omega_0}$,

(iii) $\alpha \prec_k \beta[\gamma]$ for all $\gamma \in T_m\setminus\{0\}$ if $\beta = (\beta[\gamma])_{\gamma\in\Omega_m}$ $(m>0)$.

where $\alpha \preceq_k \delta$ means that $\alpha \prec_k \delta$ or $\alpha = \delta$.

Note that if α, $\beta \in T_1$ then the relations $\prec_k$ defined above are the same as ones defined in Definition 1.2.

LEMMA 4.3. *For* $\alpha \in T_{n+1}$, $\beta \in T_n$ *and* $\gamma \in T_n\setminus\{0\}$, $\beta \prec_k \varphi_n^{\gamma}(\alpha,\beta)$.

Proof. The lemma immediately follows from the two claims. □

CLAIM 1. *Let* $\alpha \in T_{n+1}$ *and* $\beta \in T_n$. *If* $\delta \prec_k \varphi_n(\alpha,\delta)$ *for all* $\delta \in T_n$, *then* $\beta \prec_k \varphi_n^{\gamma}(\alpha,\beta)$ *for* $\gamma \in T_n\setminus\{0\}$.

Proof of Claim 1. Transfinite induction on $\gamma \in T_n$.

Case 1. $\gamma = \eta+1$. Then $\beta \preceq_k \varphi_n^{\eta}(\alpha,\beta) \prec_k \varphi_n(\alpha,\varphi_n^{\eta}(\alpha,\beta)) = \varphi_n^{\gamma}(\alpha,\beta)$ by I.H.

Case 2. $\gamma = (\gamma[x])_{x\in\Omega_0}$. Then $\beta \preceq_k \varphi_n^{\gamma[k]}(\alpha,\beta) = \varphi_n^{\gamma}(\alpha,\beta)[k]$ by I.H. Hence $\beta \prec_k \varphi_n^{\gamma}(\alpha,\beta)$.

Case 3. $\gamma = (\gamma[\delta])_{\delta\in\Omega_m}(0<m<n)$. From 4.1, $\gamma[\delta] \in T_n\setminus\{0\}$ for $\delta \in T_m\setminus\{0\}$. Hence $\beta \prec_k \varphi_n^{\gamma[\delta]}(\alpha,\beta) = \varphi_n^{\gamma}(\alpha,\beta)[\delta]$ for $\delta \in T_m\setminus\{0\}$ by I.H. Therefore $\beta \prec_k \varphi_n^{\gamma}(\alpha,\beta)$. □

CLAIM 2. *Let* $\alpha \in T_{n+1}$. *Then* $\beta \prec_k \varphi_n(\alpha,\beta)$ *for all* $\beta \in T_n$.

Proof of Claim 2. Transfinite induction on $\alpha \in T_{n+1}$.

Case 1. $\alpha = 0$. Then $\beta \prec_k \beta + 1 = \varphi_n(\alpha,\beta)$.

Case 2. $\alpha = \gamma+1$. Then $\delta \prec_k \varphi_n(\gamma,\delta)$ for all $\delta \in T_n$ by I.H. Hence, by Claim 1, $\beta \prec_k \varphi_n(\gamma,\beta) \preceq_k \varphi_n^{\beta}(\gamma,\varphi_n(\gamma,\beta)) = \varphi_n(\alpha,\beta)$.

Case 3. $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m}(m<n)$. By I.H., $\beta \prec_k \varphi_n(\alpha[\gamma],\beta) = \varphi_n(\alpha,\beta)[\gamma]$ for $\gamma \in T_m$. Hence $\beta \prec_k \varphi_n(\alpha,\beta)$.

Case 4. $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_n}$. By I.H., $\beta \prec_k \varphi_n(\alpha[z],\beta) = \varphi_n(\alpha,\beta)$ where $z = \varphi_n(\alpha[1],\beta)$. □

LEMMA 4.4. *Let* $\alpha \in T_{n+1}$ *and* $\beta, \delta, \gamma \in T_n$. *If* $\gamma \prec_k \delta$, *then* $\varphi_n^{\gamma}(\alpha,\beta) \prec_k \varphi_n^{\delta}(\alpha,\beta)$.

Proof. Transfinite induction on $\delta \in T_n$.

Case 1. $\delta = 0$. This case is trivial.

Case 2. $\delta = \eta+1$. By I.H. and 4.3, $\varphi_n^{\gamma}(\alpha,\beta) \preceq_k \varphi_n^{\eta}(\alpha,\beta) \prec_k \varphi_n(\alpha,\varphi_n^{\eta}(\alpha,\beta)) = \varphi_n^{\delta}(\alpha,\beta)$.

Case 3. $\delta = (\delta[x])_{x\in\Omega_0}$. By I.H., $\varphi_n^{\gamma}(\alpha,\beta) \preceq_k \varphi_n^{\delta[k]}(\alpha,\beta) = \varphi_\alpha^{\delta}(\alpha,\beta)[k]$. Hence $\varphi_n^{\gamma}(\alpha,\beta) \prec_k \varphi_n^{\delta}(\alpha,\beta)$.

Case 4. $\delta = (\delta[\xi])_{\xi\in\Omega_m}(0<m<n)$. Then $\varphi_n^{\gamma}(\alpha,\beta) \prec_k \varphi_n^{\delta[\xi]}(\alpha,\beta) =$

$\varphi_n{}^{\delta}(\alpha,\beta)[\xi]$ for $\xi \in T_m\setminus\{0\}$ by I.H. Hence $\varphi_n{}^{\gamma}(\alpha,\beta) \prec_k \varphi_n{}^{\delta}(\alpha,\beta)$. □

LEMMA 4.5. *Let* α, $\gamma \in T_{n+1}$, $\beta \in T_n\setminus\{0\}$ *and* $n > 0$. *If* $\gamma \prec_k \alpha$, *then* $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha,\beta)$.

Proof. Transfinite induction on $\alpha \in T_n$.

Case 1. $\alpha = 0$. This case is trivial.

Case 2. $\alpha = \eta+1$. By I.H. and 4.3, $\varphi_n(\gamma,\beta) \preceq_k \varphi_n(\eta,\beta) \prec_k \varphi_n{}^{\beta}(\eta,\varphi_n(\eta,\beta)) = \varphi_n(\alpha,\beta)$ since $\beta \neq 0$.

Case 3. $\alpha = (\alpha[x])_{x\in\Omega_0}$. By I.H., $\varphi_n(\gamma,\beta) \preceq_k \varphi_n(\alpha[k],\beta) = \varphi_n(\alpha,\beta)[k]$. Hence $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha,\beta)$.

Case 4. $\alpha = (\alpha[\xi])_{\xi\in\Omega_m}(0<m<n)$. By I.H., $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha[\xi],\beta) = \varphi_\alpha(\beta)[\xi]$ for $\xi \in T_m\setminus\{0\}$. Hence $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha,\beta)$.

Case 5. $\alpha = (\alpha[\xi])_{\xi\in\Omega_n}$. By I.H., $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha[z],\beta) = \varphi_n(\alpha,\beta)$ for $\beta \in T_n\setminus\{0\}$ where $z = \varphi_n(\alpha[1],\beta)$. □

4B. THE SUBSETS T_n^+ OF T_n $(n<\omega)$.

We shall define the subset T_n^+ for each $n < \omega$, and prove that every element of T_1^+ is built-up.

DEFINITION 4.6. The subset $T_n^+ \subseteq T_n$ are defined inductively as follows:

(i) $0, 1, \omega_0, \omega_1,\ldots,\omega_{n-1} \in T_n^+$.

(ii) $T_k^+ \subseteq T_n^+$ for $k < n$.

(iii) If $\alpha \in T_{n+1}^+$, $\gamma \in T_n^+$ and $\beta \in T_n^+\setminus\{0\}$, then $\varphi_n^{\gamma}(\alpha,\beta) \in T_n^+$.

Note that the definition of T_n^+ above differs from that of T_n only in the restriction on β in (iii).

PROPOSITION 4.7 (cf.4.1). *Let* $\alpha \in T_n^+$ *and* $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m}$. *Then* $\alpha[\gamma] \in T_n^+$ *for every* $\gamma \in T_m^+$. *Moreover, if* $\gamma \in T_m^+\setminus\{0\}$, *then* $\alpha[\gamma] \in T_n^+\setminus\{0\}$.

Proof. It is proceeded in the same way as 4.1. First, we introduce the subsets $\bar{T}_n^+ \subseteqq \bar{T}_n$ of terms of T_n^+ as 2.1:

(i) $\bar{0}$, $\bar{1}$, $\bar{\omega}_0$, $\bar{\omega}_1, \ldots, \bar{\omega}_{n-1} \in \bar{T}_n^+$.

(ii) $\bar{T}_k^+ \subseteqq \bar{T}_n^+$ for $k < n$.

(iii) If $\alpha \in \bar{T}_{n+1}^+$, $\gamma \in \bar{T}_n^+$ and $\beta \in \bar{T}_n^+\setminus\{\bar{0}\}$, then $\varphi_n^\gamma(\alpha,\beta) \in \bar{T}_n^+$.

Then we can prove that for each $\alpha \in T_n^+$, there is a term $a \in \bar{T}_n^+$ such that $\mathrm{ord}(a) = \alpha$. And the strong normalization theorem on $\bar{T}_n^+$ holds since if $a \xrightarrow{1} a'$ and $a \in \bar{T}_n^+$, then $a' \in \bar{T}_n^+$. Hence we can prove this proposition in the same way as 4.1. □

THEOREM 4.8 ([10,Theorem 3]). *Let* $\alpha \in T_n^+$ *and* $\alpha = (\alpha[\xi])_{\xi\in\Omega_m}$. *If* γ, $\delta \in T_m$ *and* $\gamma \prec_k \delta$, *then* $\alpha[\gamma] \prec_k \alpha[\delta]$.

Proof. From the proof of 4.7., for a given $\alpha \in T_n^+$, we can take a normal term $a \in \bar{T}_n^+$ with the minimal length such that $\mathrm{ord}(a) = \alpha$ The proof of this theorem is proceeded by induction on the length of this term a. We have the following cases:

Case 1. $a = \bar{\omega}_m$. Then $\alpha = \omega_m$. We have $\alpha[\gamma] = \gamma \prec_k \delta = \alpha[\delta]$.

Case 2. $a = \bar{\varphi}_n(d,b)$ and $\mathrm{dom}(d) = \bar{T}_m$. Then $\alpha = \varphi_n(\lambda,\beta)$ so that $\lambda = (\lambda[\xi])_{\xi\in\Omega_m} = \mathrm{ord}(d)$ and $\beta = \mathrm{ord}(b) \in T_n^+\setminus\{0\}$ from the definition of T_n^+ above and $a \in \bar{T}_n^+$. Hence, by I.H. $\lambda[\gamma] \prec_k \lambda[\delta]$ and 4.5, $\varphi_n(\lambda,\beta)[\gamma] = \varphi_n(\lambda[\gamma],\beta) \prec_k \varphi_n(\lambda[\delta],\beta) = \varphi_n(\lambda,\beta)[\delta]$.

Case 3. $a = \bar{\varphi}_n{}^e(d,b)$ and $\mathrm{dom}(e) = \bar{T}_m$. This case is treated similarly to Case 2, using 4.4. □

THEOREM 4.9. *Each* $\alpha \in T_1^+$ *is* (k)*-built-up for all* $k < \omega$.

Proof. For each $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m} \in T_n^+$ and $\gamma \in T_m^+$, $\alpha[\gamma] \in T_n^+$ from 4.7. Hence for each $\alpha \in T_1^+$ and limit $\lambda \leqq \alpha$, we have $\lambda \in T_1^+$. Thus by 4.8, $\lambda[x] \prec_k \lambda[x+1]$ for all k, $x < \omega$ and limit $\lambda \leqq \alpha \in T_1^+$. □

The reason why we introduce the set T_n^+ is that (k)-built-upness does not hold for some element in T_1 since, if we put $\alpha = \varphi_1(\omega_0, 0)$, then $\alpha[x] = \varphi_1(x,0) = 1$ for all $x < \omega$.

THEOREM 4.10 ([10,Corollary 3.1]). τ' *is* (3)*-built-up.*

Proof. Let $x < \omega$. From the definition of τ'(1.13), $\tau'[x] \in T_1^+$. By 4.9, $\tau'[x]$ is (3)-built-up. Hence it is sufficient to prove that $\tau'[x] \prec_3 \tau'[x+1]$. For this, we have

$$
\begin{aligned}
\tau'[x] &= \varphi_1(\ldots\varphi_x(3,\omega_{x-1})\ldots,\omega_0) \\
&\prec_3 \varphi_1(\ldots\varphi_x(\omega_0,\omega_{x-1})\ldots,\omega_0) \\
&\prec_0 \varphi_1(\ldots\varphi_x(\varphi_1(z,\omega_0),\omega_{x-1})\ldots,\omega_0) \\
&\qquad\qquad \text{where } z = \varphi_2(\ldots\varphi_x(1,\omega_{x-1})\ldots,\omega_1) \\
&= \varphi_1(\ldots\varphi_x(\omega_1,\omega_{x-1})\ldots,\omega_0) \\
&\prec_0 \varphi_1(\ldots\varphi_x(\varphi_2(z',\omega_1),\omega_{x-1})\ldots,\omega_0) \\
&\qquad\qquad \text{where } z' = \varphi_2(\ldots\varphi_x(1,\omega_{x-1})\ldots,\omega_1) \\
&= \varphi_1(\ldots\varphi_x(\omega_2,\omega_{x-1})\ldots,\omega_0) \\
&\prec_0 \qquad \vdots \\
&= \varphi_1(\ldots\varphi_x(\omega_x,\omega_{x-1})\ldots,\omega_0) \\
&\prec_3 \varphi_1(\ldots\varphi_x(\varphi_{x+1}(3,\omega_x),\omega_{x-1})\ldots,\omega_0)
\end{aligned}
$$

$= \quad \tau'[x+1]$

from $3 \prec_3 \omega_0$ and from 3.5, 3.3. □

4C. THE SUBSETS T_n^* OF $T_n^+ (n<\omega)$. Here we shall introduce the sets T_n^* of terms which are used in the next section.

DEFINITION 4.11. The subset $T_n^* \subseteqq T_n^+$ are defined inductively as follows:

(i) $0, 1, \omega_0, \omega_1, \ldots, \omega_{n-1} \in T_n^*$.

(ii) $T_k^* \subseteqq T_n^*$ for $k < n$.

(iii) If $\alpha \in T_{n+1}^*$, $\gamma \in T_n^*$ and $\beta \in T_n^* \setminus T_{n-1}^*$, then $\varphi_n^{\gamma}(\alpha, \beta) \in T_n^*$ where $T_{-1}^* = \{0\}$.

Note that similarly to the case of the sets T_n^+, the definition of T_n^* above differs from that of T_n^+ only in the restriction on β in (iii). We can prove the same propositions as the case of T_n^+ in the same way as the corresponding proofs.

PROPOSITION 4.12 (cf.4.1). *Let* $\alpha \in T_n$ *and* $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$. *Then* $\alpha[\gamma] \in T_n^*$ *for every* $\gamma \in T_m^*$. *Moreover, if* $\gamma \in T_m^* \setminus T_{m-1}^*$, *then* $\alpha[\gamma] \in T_n^* \setminus T_{m-1}^*$.

DEFINITION 4.13 (cf.4.2). The *step-down relations* $\prec_k^*$ $(k<\omega)$ on $\cup_{n<\omega} T_n^*$ are defined inductively as follows: For α, $\beta \in T_n^*$,

$\alpha \prec_k^* \beta$ if $\beta \neq 0$ and one of the following holds;

(i) $\alpha \preceq_k^* \gamma$ if $\beta = \gamma + 1$,

(ii) $\alpha \preceq_k^* \beta[k]$ if $\beta = (\beta[x])_{x \in \Omega_0}$,

(iii) $\alpha \prec_k^* \beta[\gamma]$ for all $\gamma \in T_m^* \setminus (T_{m-1}^* \cup \{\omega_{m-1}\})$

if $\beta = (\beta[\gamma])_{\gamma\in\Omega_m}$ $(m>0)$.

LEMMA 4.14 (cf.4.3). *For* $\alpha \in T^*_{n+1}$, $\beta \in T^*_n \setminus T^*_{n-1}$ *and* $\gamma \in T^*_n \setminus \{0\}$, we have $\beta <^*_k \varphi_n{}^{\gamma}(\alpha,\beta)$.

LEMMA 4.15 (cf.4.4). *Let* $\alpha \in T^*_{n+1}$, δ, $\gamma \in T^*_n$ *and* $\beta \in T^*_n \setminus T^*_{n-1}$. *If* $\gamma <^*_k \delta$, *then* $\varphi_n{}^{\gamma}(\alpha,\beta) <^*_k \varphi_n{}^{\delta}(\alpha,\beta)$.

LEMMA 4.16 (cf.4.5). Let α, $\gamma \in T^*_{n+1}$, $\beta \in T^*_n \setminus T^*_{n-1}$ *and* $n > 0$. *If* $\gamma <^*_k \alpha$, *then* $\varphi_n(\gamma,\beta) <^*_k \varphi_n(\alpha,\beta)$.

THEOREM 4.17 (cf.4.8). *Let* $\alpha \in T^*_n$ *and* $\alpha = (\alpha[\xi])_{\xi\in\Omega_m}$. *If* γ, $\delta \in T^*_m$ *and* $\gamma <^*_k \delta$, *then* $\alpha[\gamma] <^*_k \alpha[\delta]$.

The next lemma is used in the next section.

LEMMA 4.18. (1) *If* $\alpha \in T^*_{m+1} \setminus (T^*_m \cup \{\omega_m\})$, *then* $\omega_m <^*_k \alpha$ *for all* $k < \omega$.

(2) $k <^*_k \omega_0$ *for* $k < \omega$.

(3) $\omega_i <^*_0 \omega_n$ *for* $i < n$.

Proof. (1) If $\alpha \in T^*_{m+1} - (T^*_m \cup \{\omega_m\})$, then α is of the form: $\varphi_{m+1}{}^{\gamma}(\ \ldots,\omega_m)\ldots)$. Hence 4.14 completes the proof. (2) Trivial for the definition of $<^*_k$. (3) It is sufficient to prove that $\omega_i <^*_0 \omega_{i+1}$. By (1) we have $\omega_i <^*_0 \alpha$ for all $\alpha \in T_{i+1} \setminus (T_i \cup \{\omega_i\})$. From the definition of $<^*_0$ and $\omega_i[\alpha] = \alpha$, this completes the proof. □

§5. PROVABLY COMPUTABLE FUNCTIONS IN $ID_{<\omega}$

5A. In this section we shall prove the following theorem:

THEOREM 5.1. *If a* Π^0_2 *-sentence* $\forall x\exists yA(x,y)$ $(A \in \Sigma^0_1)$ *is provable in* $ID_\nu(\nu<\omega)$, *then there is an* $\alpha \prec \tau'[\nu+1]$ *such that for all* $n > 1$, *there is an* $k < F_\alpha(n)$ $A(n,k)$.

Clearly, this theorem implies (III) in Introduction. Here we shall prove this theorem in the same way as Buchholz[4].

5B. THE SYSTEM $ID_\nu(\nu<\omega)$. We introduce the system ID_ν for $\nu < \omega$ following [4,Section 4].

Preliminaries. Let L denote the first-order language consisting of the following symbols:

(i) the logical constants $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$,

(ii) number variables(indicated by x, y),

(iii) a constant 0(zero) and a unary function symbol '(successor),

(iv) constants for primitive recursive predicates (among them the symbol < for the arithmetic 'less' relation).

By $s,t,t_0,\ldots$ we denote arbitrary L-terms. The constant terms 0,0',0'',...are called numerals; we identify numerals and natural numbers and denote them by i,j,k,m,n,u,v,w. A formula of the shape $Rt_1\cdots t_n$ or $\neg Rt_1\cdots t_n$, where R is a n-ary predicate symbol of L, is called an ***arithmetic prime formula*** (abbreviated by a.p.f.).

Let X be a unary and Y a binary predicate variable. A *positive operator form* is a formula $\mathfrak{A}_y(X,Y,y,x)$ of L(X,Y) in which only X,Y,y,x occur free and all occurrences of X are positive. The language L_{ID} is obtained from L by adding a binary predicate constant $P^{\mathfrak{A}}$ and a 3-ary predicate constant $P^{\mathfrak{A}}_{<}$ for each positive operator form $\mathfrak{A}$.

Abbreviations.

$$t \in P^{\mathfrak{A}}_{s} := P^{\mathfrak{A}}_{s}t := P^{\mathfrak{A}}st, \qquad t \notin P^{\mathfrak{A}}_{s} := \neg(t \in P^{\mathfrak{A}}_{s}),$$

$$P^{\mathfrak{A}}_{<s}t_0t_1 := P^{\mathfrak{A}}_{<}st_0t_1, \qquad \mathfrak{A}_s(X,x) := \mathfrak{A}(X,P^{\mathfrak{A}}_{<s},s,x).$$

The formal theory ID_ν with $\nu < \omega$ is an extension of Peano Arithmetic, formulated in the language L_{ID}, by the following axioms:

$(P^{\mathfrak{A}}.1)$ $\quad \forall y\forall x(\mathfrak{A}_y(P^{\mathfrak{A}}_y,x) \longrightarrow x \in P^{\mathfrak{A}}_y)$.

$(P^{\mathfrak{A}}.2)_{<\nu}$ $\quad \forall x(\mathfrak{A}_u(F,x) \longrightarrow F(x)) \longrightarrow \forall x(P^{\mathfrak{A}}_u x \longrightarrow F(x))$, for each L_{ID}-formula $F(x)$ and each $u < \nu$.

$(P^{\mathfrak{A}}.3)$ $\quad \forall y\forall x_0\forall x_1(P^{\mathfrak{A}}_{<y}x_0x_1 \longleftrightarrow x_0 < y \wedge x_1 \in P^{\mathfrak{A}}_{x_0})$.

4C. THE INFINITARY SYSTEM $\varphi ID^{\infty}_{<\omega}$. As in [4,Section 4], the infinitary system $\varphi ID^{\infty}_{<\omega}$ shall be formulated in the language $L_{ID}(N)$ which arises from L_{ID} by adding a new unary predicate symbol N. This is a technical tool which shall help us to keep control over the numerals n occuring in $\exists$-inferences $A(n) \vdash \exists xA(x)$ of $\varphi ID^{\infty}_{<\omega}$-derivations. Following Tait[14] we assume all formulas to be in negation normal form, i.e., the formulas are built up from atomic and negated atomic formulas by means of $\wedge,\vee,\forall,\exists$. If A is a complex formula we consider $\neg A$ as a notation for the corresponding negation normal form.

Definition of the length $|A|$ *of a* $L_{ID}(N)$*-formula* A

1. $|Nt| := |\neg Nt| := 0$.

2. $|A| := 1$, if A is an a.p.f. or a formula $(\neg)P^{\mathfrak{A}}_{s}t$.

3. $|P^{\mathfrak{A}}_{<s}t_0t_1| := |\neg P^{\mathfrak{A}}_{<s}t_0t_1| := 2$.

4. $|A\wedge B| := |A\vee B| := \max\{|A|,|B|\}+1$.

5. $|\forall xA| := |\exists xA| := |A|+1$.

PROPOSITION 5.2. $|\neg A| = |A|$, *for each* $L_{ID}(N)$*-formula* A.

As before we use the letters u,v to denote numbers $< \omega$.

Inductive definition of formula sets $Pos_v (v<\omega)$

1. All L(N)-formulas belong to Pos_v.

2. All formulas $P^{\mathfrak{A}}_{u}t$, $(\neg)P^{\mathfrak{A}}_{<u}t_0t_1$ with $u \leq v$ belong to Pos_v.

3. All formulas $\neg P^{\mathfrak{A}}_{u}t$ with $u < v$ belong to Pos_v.

4. If A and B belong to Pos_v, then the formulas $A\wedge B$, $A\vee B$, $\forall xA$, $\exists xA$ also belong to Pos_v.

REMARK 5.3. *If* $P^{\mathfrak{A}}_{u}t \in Pos_v$, *then also* $\mathfrak{A}_u(P^{\mathfrak{A}}_{u},t) \in Pos_v$.

Notations

- In the following A,B,C always denote closed $L_{ID}(N)$-formulas.
- Γ,Γ',Δ denote finite sets of closed $L_{ID}(N)$-formulas; we write, e.g., Γ,Δ,A for $\Gamma\cup\Delta\cup\{A\}$.
- A^N denotes the result of restricting all quantifiers in A to N.
- $t\in N := Nt$, $t\notin N := \neg Nt$.
- As before we use the letters $\alpha,\beta,\gamma,\delta$ to denote elements of T^*_n.

DEFINITION 5.4. $\gamma \prec_\Gamma \alpha :\Longleftrightarrow \gamma \prec_k \alpha$,

where $k := \max(\{3\}\cup\{3n : \neg Nn \in \Gamma\})$.

PROPOSITION 5.5. (1) $\gamma \prec_{\Gamma} \alpha$ *and* $\Gamma \subset \Delta \Rightarrow \gamma \prec_{\Delta} \alpha$. *((0)-built-upness of all elements of* T_n^**: Theorem 4.17.)*

(2) $\gamma \prec_{\Gamma\cup\{0\notin N\}} \alpha \Rightarrow \gamma \prec_{\Gamma} \alpha$.

Basic inference rules

($\wedge$) $A_0, A_1 \vdash A_0 \wedge A_1$.

($\vee$) $A \vdash A\vee B$; $B \vdash A \vee B$.

($\forall^{\infty}$) $(A(n))_{n\in\omega} \vdash \forall x A(x)$.

($\exists$) $A(n) \vdash \exists x A(x)$.

(N) $n \in N \vdash n' \in N$.

($P^{\mathfrak{A}}_{<u}$) $P^{\mathfrak{A}}_{j}n \vdash P^{\mathfrak{A}}_{<u}jn$, if $j < u < \omega$.

($\neg P^{\mathfrak{A}}_{<u}$) $\neg P^{\mathfrak{A}}_{j}n \vdash \neg P^{\mathfrak{A}}_{<u}jn$, if $j < u < \omega$.

Every instance $(A_i)_{i\in I} \vdash A$ of these rules is called a basic inference. If $(A_i)_{i\in I} \vdash A$ is a basic inference with $A \in Pos_V$, then $A_i \in Pos_V$ for all $i \in I$. This property will be used in the proof of 5.10.

The system $\varphi ID^{\infty}_{<\omega}$ consists of the language $L_{ID}(N)$ and a certain derivability relation $\vdash^{\alpha}_{m} \Gamma$ ("Γ is derivable with order $\alpha \in T_n^*$ and cut degree $m \in \omega$") which we introduce below by an iterated inductive definition.

Inductive definition of $\vdash^{\alpha}_{m} \Gamma$ $(\alpha\in T_n^*,\ m\in\omega)$

(Ax1) $\vdash^{\alpha}_{m} \Gamma, A$, if A is a true a.p.f. or $A \equiv 0 \in N$ or $A \equiv \neg P^{\mathfrak{A}}_{<u}jn$ with $u \leqq j$.

(Ax2) $\vdash^{\alpha}_{m} \Gamma, \neg A, A$, if $A \equiv n \in N$ or $A \equiv P^{\mathfrak{A}}_{u}n$.

(Bas) If $(A_i)_{i\in I} \vdash A$ is a basic inference with $A \in \Gamma$ and $\forall i\in I(\vdash^{\alpha}_{m} \Gamma, A_i)$, then $\vdash^{\alpha+1}_{m} \Gamma$.

$(P^{\mathfrak{A}}_u)$ $\quad \vdash^{\alpha}_m \Gamma$, $n \in N \wedge \mathfrak{A}^N_u(P^{\mathfrak{A}}_u, n)$ and $P^{\mathfrak{A}}_u n \in \Gamma \Rightarrow \vdash^{\alpha+3}_m \Gamma$.

(Cut) $\quad \vdash^{\alpha}_m \Gamma, \neg C$ and $\vdash^{\alpha}_m \Gamma, C$ and $|C| < m \Rightarrow \vdash^{\alpha+1}_m \Gamma$

$$(\Omega_{u+1}) \quad \left.\begin{array}{l} \mathrm{dom}(\alpha) = \Omega_{u+1} \text{ and } \vdash^{\alpha[1]}_m \Gamma, P^{\mathfrak{A}}_u n \text{ and} \\ \forall z \in \Omega_{u+1} \forall \Delta \subset Pos_u (\vdash^{z}_1 \Delta, P^{\mathfrak{A}}_u n \Rightarrow \vdash^{\alpha[z]}_m \Delta, \Gamma) \end{array}\right\} \Rightarrow \vdash^{\alpha+1}_m \Gamma.$$

$(\prec)$ $\quad \vdash^{\beta}_m \Gamma$ and $\beta \prec_{\Gamma} \alpha \Rightarrow \vdash^{\alpha}_m \Gamma$.

LEMMA 5.6. (1) $\vdash^{\alpha}_m \Gamma$ *and* $m \leqq k$, $\Gamma \subset \Delta \Rightarrow \vdash^{\alpha}_k \Delta$.

(2) $\vdash^{\alpha}_m \Gamma \Rightarrow \vdash^{\gamma+\alpha}_m \Gamma$. $(\gamma+\alpha = \varphi^{\alpha}_n(0,\gamma).)$

(3) $\vdash^{\alpha}_m \Gamma$, $0 \notin N \Rightarrow \vdash^{\alpha}_m \Gamma$.

Proof.(cf.[4,Lemma 4.2].) Induction on α using 5.5 and the relation that $(\gamma+\alpha)[\delta] = \gamma+\alpha[\delta]$ for all $\delta \in \Omega_k$ with $\alpha = (\alpha[\delta])_{\delta\in\Omega_k}$. □

LEMMA 5.7 (Inversion). *Let* $(A_i)_{i\in I} \vdash A$ *be a basic inference* $(\wedge)$, $(\forall^{\infty})$, $(P^{\mathfrak{A}}_{<u})$, $(\neg P^{\mathfrak{A}}_{<u})$. *Then* $\vdash^{\alpha}_m \Gamma, A$ *implies* $\forall i \in I(\vdash^{\alpha}_m \Gamma, A_i)$.

Proof. Similar to [4.Lemma 4.3] by induction on α. □

LEMMA 5.8 (Reduction). *Suppose* $\vdash^{\alpha}_m \Gamma_0, \neg C$ *and* $|C| \leqq m$, *where* C *is formula of the shape* $A \vee B$ *or* $\exists x A(x)$ *or* $P^{\mathfrak{A}}_{<u} jn$ *or* $\neg P^{\mathfrak{A}}_u n$ *or a false a.p.f. Then* $\vdash^{\beta}_m \Gamma, C$ *implies* $\vdash^{\alpha+\beta}_m \Gamma_0, \Gamma$.

Proof. Similar to [4,Lemma 4.4] from induction on β and the relation that $\alpha+(\beta+1) = (\alpha+\beta)+1$. □

THEOREM 5.9 (Cutelimination). $\vdash^{\alpha}_{m+1} \Gamma$ *and* $\alpha \in T^*_{\nu+1}$, $\nu < \omega$, $m > 0 \Rightarrow \vdash^{z}_m \Gamma$ *where* $z = \varphi^{\alpha}_{\nu+1}(1, \varphi_{\nu+1}(1, \varphi^{k}_{\nu+1}(2, \omega_{\nu})))$ *for all* $k < \omega$.

Proof.(cf.[4,Theorem 4.5].) Induction on α. Let $z = \varphi^{\alpha}_{\nu+1}(1,\varphi^{k}_{\nu+1}(1,\varphi^{k}_{\nu+1}(2,\omega_{\nu})))$.

1. Suppose $\alpha = \gamma+1$, $A \in \Gamma$ and $\forall i\in I(\vdash^{\gamma}_{m+1}\Gamma,A_i)$, where $(A_i)_{i\in I}\vdash A$ is a basic inference $(\mathcal{I})$. Then by I.H. we have $\forall i\in I(\vdash^{\beta}_{m}\Gamma,A_i)$ where $\beta = \varphi^{\gamma}_{\nu+1}(1,\varphi^{k}_{\nu+1}(1,\varphi^{k}_{\nu+1}(2,\omega_{\nu})))$. By $(\mathcal{I})$ we have $\vdash^{\beta+1}_{m}\Gamma$ and then $\vdash^{z}_{m}\Gamma$ since $\beta+1 = \varphi_{\nu+1}(0,\beta) \prec_0 \varphi_{\nu+1}(1,\beta) = z$ by 4.5.

2. Suppose $\alpha = \gamma+1$, $\vdash^{\gamma}_{m+1}\Gamma,\neg C$, $\vdash^{\gamma}_{m+1}\Gamma,C$ and $|C| = m$. Then by I.H. we have $\vdash^{\beta}_{m}\Gamma,\neg C$ and $\vdash^{\beta}_{m}\Gamma,C$ where β is as 1. We may assume that C fulfills the condition of 5.8. By $(\prec)$ and 5.8, we have $\vdash^{(\beta+1)+\beta}_{m}\Gamma$. Hence $\vdash^{z}_{m}\Gamma$ since $(\beta+1)+\beta = \varphi^{\beta}_{\nu+1}(0,\varphi_{\nu+1}(0,\beta)) = \varphi_{\nu+1}(1,\beta) = z$.

3. Suppose $\alpha = \gamma+3$, $P^{\mathfrak{A}}_{u}n \in \Gamma$ and $\vdash^{\gamma}_{m+1}\Gamma,B$ with $B = n\in N\wedge\mathfrak{A}^{N}_{u}(P^{\mathfrak{A}}_{u},n)$. Then by I.H. and $(\prec)$ we have $\vdash^{\beta}_{m}\Gamma,B$ where $\beta = \varphi^{\gamma+2}_{\nu+1}(1,\varphi^{k}_{\nu+1}(1,\varphi^{k}_{\nu+1}(2,\omega_{\nu})))$. By $(P^{\mathfrak{A}}_{u})$ we get $\vdash^{\beta+3}_{m}\Gamma$ and hence $\vdash^{z}_{m}\Gamma$ since $\beta+3 = \varphi^{3}_{\nu+1}(0,\beta) \prec_2 \varphi^{\beta}_{\nu+1}(1,\varphi_{\nu+1}(0,\beta)) = \varphi_{\nu+1}(1,\beta) = z$.

4. In all other cases the assertion follows from I.H. and the fact that $\beta+1 = \varphi_{\nu+1}(0,\beta) \prec_0 \varphi_{\nu+1}(1,\beta) = z$ as in 1 above. □

THEOREM 5.10 (Collapsing Lemma). $\vdash^{\alpha}_{1}\Gamma$ *and* $\Gamma \subset Pos_{v}$, $\alpha \in T^{*}_{v+2}$ $\Rightarrow \vdash^{z}_{1}\Gamma$ *where* $z = \varphi_{v+1}(\alpha,\omega_{v})$.

Proof.(cf.[4,Theorem 4.6].) Induction on α.

1. Suppose $\alpha = (\alpha[\delta])_{\delta\in\Omega_{u+1}}$, $\vdash^{\alpha[1]}_{1}\Gamma,P^{\mathfrak{A}}_{u}n$ and $\vdash^{\alpha[z]}_{1}\Delta,\Gamma$ for all $z \in T^{*}_{u+1}$, $\Delta \subset Pos_{u}$ with $\vdash^{z}_{1}\Delta,P^{\mathfrak{A}}_{u}n$. Then $u \leqq v$.

Case 1. $u < v$. We have $\varphi_{v+1}(\alpha[z],\omega_{v}) = \varphi_{v+1}(\alpha,\omega_{v})[z]$ for all $z \in T^{*}_{u+1}$. Hence the assertion follows by (Ω_{u+1}).

Case 2. $u = v$. Then $\Gamma\cup\{P^{\mathfrak{A}}_{u}n\} \subset Pos_{u}$ and by I.H. $\vdash^{\beta}_{1}\Gamma,P^{\mathfrak{A}}_{u}n$ where $\beta = \varphi_{v+1}(\alpha[1],\omega_{v})$. Since $\beta \in T^{*}_{u+1}$ we get $\vdash^{z}_{1}\Gamma$ where $z =$

$\varphi_{v+1}(\alpha[\beta],\omega_v)$. But $z = \varphi_{v+1}(\alpha[\beta],\omega_v) = \varphi_{v+1}(\alpha,\omega_v)$ from the definition of φ_{v+1} (see 1.10).

2. In all other cases the assertion follows from the I.H. $\square$

DEFINITION 5.11. $L(N)_+ :=$ {A:A is a sentence of L(N) in which N occurs only positively}. For $\Gamma = \{A_1,\dots,A_n\} \subset L(N)_+$ we define:

$$\models\Gamma(k) \;:\Longleftrightarrow\; \begin{cases} A_1\vee\cdots\vee A_n \text{ is true in the standard model} \\ \text{when } N \text{ is interpreted as } \{i<\omega\colon 3i<k\}. \end{cases}$$

LEMMA 5.12.
$$\left.\begin{array}{l} \vdash_1^{\alpha} i_1\notin N,\dots,i_m\notin N,\Gamma \textit{ and } \alpha \in T_1^*, \\ \Gamma \subset L(N)_+,\ n \geqq \max\{3,3i_1,\dots,3i_m\} \end{array}\right\} \Rightarrow \models\Gamma(F_\alpha(n)).$$

Proof. (cf. [4, Lemma 4.7].) Induction on α. Let $\Gamma_0 = \{i_1\notin N,\dots,i_m\notin N\}$ and $k = \max\{3,3i_1,\dots,3i_m\} \leqq n$.

1. (Ax1)$\vdash_1^{\alpha}\Gamma_0,\Gamma$. The assertion is trivial for $0 < F_\alpha(n)$.

2. (Ax2)$\vdash_1^{\alpha}\Gamma_0,\Gamma$. The assertion follows from $n < F_\alpha(n)$.

3. If $\vdash_1^{\alpha}\Gamma_0,\Gamma$ is the conclusion of a basic inference $\neq$ (N), then the assertion follows from the I.H. and the relation $F_\beta(n) < F_{\beta+1}(n)$.

4. Suppose $\alpha = \beta+1$, $N(j+1) \in \Gamma$, $\vdash_1^{\beta} \Gamma_0,\Gamma,Nj$. By I.H. we have $\models\Gamma\cup\{Nj\}(F_\alpha(n))$. Then we have $F_\beta(n) < F_\beta^2(n) < F_\beta^3(n) < F_\beta^4(n) \leqq F_\beta^{n+1}(n) = F_\alpha(n)$. So, $F_\beta(n)+3 \leqq F_\alpha(n)$. Hence $\models\Gamma(F_\beta(n))$.

5. Suppose $\vdash_1^{\beta}\Gamma_0,\Gamma$ with $\beta <_{\Gamma_0\cup\Gamma} \alpha$. Then we have $F_\beta(n) < F_\alpha(n)$ since $n \geqq k$. The assertion follows from the I.H.

6. Suppose $\alpha = \beta+1$, $\vdash_1^{\beta}\Gamma_0,\Gamma,i_0\in N$ and $\vdash_1^{\beta}i_0\notin N,\Gamma_0,\Gamma$. Let $\hat{n} = F_\alpha(n)$. Then we have $n < \hat{n} < F_\beta(\hat{n}) = F_\beta^2(n) < F_\alpha(n)$.

6.1. $\hat{n} < 3i_0$. From $\vdash_1^{\beta}\Gamma_0,\Gamma,i_0\in N$ we obtain by the I.H.

$\models\Gamma\cup\{i_0\in N\}(\hat{n})$ and then $\models\Gamma(\hat{n})$, since $\neg(3i_0<\hat{n})$. Using $\hat{n} < F_\alpha(n)$ we get the assertion.

6.2. $3i_0 \leqq \hat{n}$. From $\vdash_1^\beta i_0\notin N, \Gamma_0, \Gamma$ and $\max\{k, 3i_0\} \leqq \hat{n}$ we obtain by I.H. $\models\Gamma(F_\beta(\hat{n}))$ and then $\models\Gamma(F_\alpha(n))$. □

THEOREM 5.13 (Bounding). *If* $\vdash_1^\alpha \forall x\in N(\exists y\in N)A^N(x,y)$, *where* $0 \prec \alpha \in T_1^*$, $\nu \leqq \omega$, $m > 0$ *and* $A(x,y)$ *a* Σ_1^0*-formula of the language* L, *then* $\forall n>1$, $\exists k<F_{\alpha+1}(n)(A(n,k))$.

Proof.(cf.[4.Theorem 4.8].) From the premise we obtain $\vdash_1^\alpha n\notin N, \exists y\in N(A^N(n,y))$ for all $n < \omega$. Then by 5.12 we get $\models\{\exists y\in N(A^N(n,y))\}(F_\alpha(\hat{n}))$ for all $n < \omega$ and all $\hat{n} \geqq \max\{3,3n\}$. Hence $\forall n\exists k<F_\alpha(3n+3)A(n,k)$. From $3n+3 < 4n+2 = F_1^2(n)$ since $n > 1$. We have $F_\alpha(3n+3) < F_\alpha(F_1^2(n)) \leqq F_\alpha^3(n) \leqq F_\alpha^{n+1}(n) = F_{\alpha+1}(n)$ since $1 \leqq_1 \alpha$ by $0 \prec \alpha$ and 1.3(3). □

4C. EMBEDDING $ID_\nu(\nu<\omega)$ INTO $\varphi ID_{<\omega}^\infty$. In the remaining part of this section we show that ID_ν $(\nu<\omega)$ can be embedded into $\varphi ID_{<\omega}^\infty$ and finally we prove the theorem that if a Π_2^0-sentence $\forall x\exists yA(x,y)(A\in\Sigma_1^0)$ is provable in $ID_\nu(\nu<\omega)$ then there is an $\alpha \prec \tau'$ such that $\forall n>1$ $\exists k<F_\alpha(n)(A(n,k))$.

Abbreviations. $k^\sim = \varphi_{\nu+1}^{k+1}(2,\omega_\nu)$.

$\alpha \dashrightarrow_n \beta :\Longleftrightarrow \exists\alpha_0,\dots,\alpha_n(\alpha_0 = \alpha \wedge \alpha_n = \beta \wedge \forall i<n(\alpha_i+1 \leqq_2^* \alpha_{i+1})$.

LEMMA 5.14. (1) $k^\sim+1 \prec_0^* (k+1)^\sim$. (2) $k^\sim \dashrightarrow_9 (k+1)^\sim$.

Proof.(cf.[4,Lemma 4.9].) (1) From $\varphi_{\nu+1}(0,k^\sim) \prec_0^* \varphi_{\nu+1}(2,k^\sim) = (k+1)^\sim$ by 4.5. (2) From the relation that, since $2 \prec_2^* k^\sim$, $k^\sim+3$

$<_2^* \varphi_{\nu+1}(1,k^\sim) <_2^* \varphi_{\nu+1}(1,k^\sim)+3 <_2^* \varphi_{\nu+1}^2(1,k^\sim) <_2^* \varphi_{\nu+1}^2(1,k^\sim)+3 <_2^* \varphi_{\nu+1}^3(1,k^\sim) <_2^* \varphi_{\nu+1}(2,k^\sim) = (k+1)^\sim$ □

LEMMA 5.15. $\vdash_0^{k^\sim} \neg A, A$ *where* $k = |A|$.

Proof. Similar to [4,Lemma 1.10]. □

LEMMA 5.16. $\vdash^{(k^\sim+1)+\omega_\nu} \neg F(0), \neg\forall x\in N(F(x)\longrightarrow F(x')), n\notin N, F(n)$ *where* $k = |F|$.

Proof. Similar to [4,Lemma 4.11]. □

DEFINITION 5.17. For $A \in Pos_u$ let A^* denote the result of replacing all occurrences of $P_u^{\mathfrak{U}}$ in A by $F(\cdot)$. $\{A_1,\dots,A_m\}^* = \{A_1^*,\dots,A_m^*\}$.

PROPOSITION 5.18. $\Gamma_0\cup\Gamma \subset Pos_u$, $\alpha \in T_{u+1}^*(u+1\leqq\nu)$, $k = |F|$, $\vdash_1^\alpha \Gamma_0,\Gamma \Rightarrow \vdash_1^{(k^\sim+1)+\alpha} \Gamma_0, \neg(\forall x\in N(\mathfrak{U}_u^N(F,x)\longrightarrow F(x))), \Gamma^*$.

Proof. Similar to [4,P.151 Proposition]. □

LEMMA 5.19. $\alpha \in T_{u+1}^*$, $\Delta \subset Pos_u$, $k = |F|$, $\vdash_1^\alpha \Delta, P_u^{\mathfrak{U}}n \Longrightarrow \vdash_1^{(k^\sim+1)+\alpha} \Delta, \neg(\forall x\in N(\mathfrak{U}_u^N(F,x)\longrightarrow F(x)), F(n)$.

Proof. From 5.18. □

LEMMA 5.20. $\vdash_1^{(k^\sim+1)+\omega_{u+1}} \neg\forall x\in N(\mathfrak{U}_u^N(F,x)\longrightarrow F(x)), \neg P_u^{\mathfrak{U}}n, F(n)$ *with* $k = |F|$.

Proof. Similar to [4,Lemma 4.13] from 5.19. □

THEOREM 5.21. *If the sentence* A *is provable in* $ID_\nu (\nu<\omega)$, *then there is a* k $<\omega$ *such that* $\vdash_k^z A^N$ *where* $z = \varphi_{\nu+1}^k(2,\omega_\nu)$.

LEMMA 5.22. (1) $(k^\sim+1)+k^\sim \dashrightarrow_9 (k+2)^\sim$.

(2) $k^\sim+4 \prec_3^* (k+1)^\sim$.

Proof. (1) $(k^\sim+1)+k^\sim = \varphi_{\nu+1}(1,k^\sim) \prec_0 \varphi_{\nu+1}(2,k^\sim) = (k+1)^\sim \dashrightarrow_9 (k+2)^\sim$. (2) $k^\sim+4 = \varphi_{\nu+1}(0,k^\sim)+3 \prec_3^* \varphi_{\nu+1}^{k^\sim}(0,\varphi_{\nu+1}(0,k^\sim)) = \varphi_{\nu+1}(1,k^\sim) \prec_0^* (k+1)^\sim$. □

PROPOSITION 5.23. *For every mathematical axiom* $A(v_1,\ldots,v_m)$ *of* ID_ν, *there is a* k $<\omega$ *such that* $\vdash_1^{k^\sim} A(i_1,\ldots,i_m)^N$ *for all* $i_1,\ldots,i_m < \omega$. *(*$v_1,v_2,\ldots$ *denote variables of the language* L.*)*

Proof. Similar to [4,p.152 Proposition 1] from the relations $(k^\sim+1)+\omega_\nu \prec_0 (k+1)^\sim = \varphi_{\nu+1}(1,k^\sim) \prec_0 (k+1)^\sim \dashrightarrow_9 (k+2)^\sim$, $\omega_{u+1} \preceq_0 \omega_\nu$ and $k^\sim+4 \prec_3 (k+1)^\sim \dashrightarrow_{18} (k+3)$. □

PROPOSITION 5.24. *By* PL1 *we denote Tait's calculus for first-order predicate logic in the language* L_{ID} *(cf.*[14]*). If* $\Gamma(v_1,\ldots,v_m)$ *is derivable in* PL1, *then there is a* k $<\omega$ *such that* $\vdash_0^{k^\sim} i_1\notin N,\ldots,i_m\notin N,\Gamma(i_1,\ldots,i_m)$ *for all* $i_1,\ldots,i_m<\omega$.

Proof. Similar to [4,p.152 Proposition 2]. □

Proof of Theorem 5.21. Suppose $ID_\nu \vdash A$ (A closed). Then PL1 $\vdash$

$\neg(A_1\wedge\cdots\wedge A_n), A$ where every A_i is the universal closure of an axiom of ID_ν. By 5.23 and 5.24, there is an $m < \omega$ such that $\vdash_1^{m^\sim} (A_1\wedge\cdots\wedge A_n)^N$ and $\vdash_0^{m^\sim} \neg(A_1\wedge\cdots\wedge A_n)^N, A^N$. By a cut with cut formula $(A_1\wedge\cdots\wedge A_n)^N$ we obtain now $\vdash_k^{k^\sim} A^N$ with $k = \max\{|(A_1\wedge\cdots\wedge A_n)^N|, m\}+1$, since $\vdash_k^{k^\sim+1} B \Rightarrow \vdash_k^{(k+1)^\sim} B$. □

Proof of Theorem 5.1. Suppose $ID_\nu \vdash A$ (A closed). Then by 5.21, $\vdash_k^\alpha A^N$ where $\alpha = \varphi_{\nu+1}^k(2,\omega_\nu)$ for some $0 < k < \omega$. If $k > 1$, then by 5.9(Cutelimination) $\vdash_{k-1}^{\alpha'} A^N$ where

$$\alpha' = \varphi_{\nu+1}^\alpha(1,\varphi_{\nu+1}(1,\varphi_{\nu+1}^k(2,\omega_\nu))) = \varphi_{\nu+1}^\alpha(1,\varphi_{\nu+1}(1,\alpha))$$
$$= \varphi_{\nu+1}(2,\alpha) = \varphi_{\nu+1}^{k+1}(2,\omega_\nu).$$

By iterating this argument, we obtain $\vdash_1^\beta A^N$ where

$$\beta = \varphi_{\nu+1}^{k+m}(2,\omega_\nu) \text{ for some } m < \omega.$$

Then by iterating 5.10(Collapsing) we have $\vdash_1^\gamma A^N$ where

$$\gamma = \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}^{k+m}(2,\omega_\nu),\omega_{\nu-1})\ldots,\omega_0).$$

And we have $\gamma \prec \tau'[\nu+1]$ since

$$\gamma = \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}^{k+m-1}(2,\varphi_{\nu+1}(2,\omega_\nu)),\omega_{\nu-1})\ldots,\omega_0)$$
$$\prec \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}^{\omega_0}(2,\varphi_{\nu+1}(2,\omega_\nu)),\omega_{\nu-1})\ldots,\omega_0)$$
$$\prec \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}^{\varphi_1(z,\omega_0)}(2,\varphi_{\nu+1}(2,\omega_\nu)),\omega_{\nu-1})\ldots,\omega_0)$$
$$\text{where } z = \varphi_2(\ldots\varphi_\nu(\varphi_{\nu+1}^1(2,\varphi_{\nu+1}(2,\omega_\nu)),\omega_{\nu-1})\ldots,\omega_1)$$
$$= \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}^{\omega_1}(2,\varphi_{\nu+1}(2,\omega_\nu)),\omega_{\nu-1})\ldots,\omega_0)$$
$$\prec \; \vdots$$
$$\prec \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}^{\omega_\nu}(2,\varphi_{\nu+1}(2,\omega_\nu)),\omega_{\nu-1})\ldots,\omega_0)$$
$$= \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}(3,\omega_\nu),\omega_{\nu-1})\ldots,\omega_0)$$
$$= \tau'[\nu+1].$$

Hence $\gamma \prec \tau'[\nu+1] \prec \tau'$. Also $\gamma+1 \prec \tau'[\nu+1]$ since $\gamma \in T_1^*$ and $\tau'[\nu+1]$ is (0)-built-up. By 5.13(Bounding) we have $\forall n>1$, $\exists k<F_{\gamma+1}(n)(A(n,k))$. □

REFERENCES

1. AOYAMA,K. and N.KADOTA, A note on built-upness. Memoirs of the Fac.of Sci.,Kyushu Univ.,Ser.A,42(1988)159-165.
2. ARAI,T., A slow growing analogue to Buchholz' proof. To appear in Ann.Pure Appl.Logic
3. BUCHHOLZ,W., A new system of proof-theoretic ordinal functions. Ann.Pure Appl.Logic 32(1986)195-207.
4. BUCHHOLZ,W., An independence result for (Π^1_1-CA)+BI. Ann. Pure Appl.Logic 33(1987),131-155.
5. COQUAND,T. and C.PAULIN, Inductively defined types. COLOG-88 Springer Lecture Notes in Computer Science 417(1990)50-66.
6. DENNIS-JONES,E.C,and S.S.WAINER, Subrecursive hierarchies via direct limits. Springer Lect.Notes in Math.1104(1984), 117-128.
7. DERSHOWITZ,N., Orderings for term-rewriting systems. Theoret. Comput.Sci.17(1982),279-301.
8. GIRARD,J.-Y., Π^1_2 - logic, Part 1:Dilators. Ann.Math.Logic 21 (1981),75-219.
9. HINDLEY,J.R. and J.P.SELDIN, Introduction to combinators and lambda-calculus. Cambridge Univ.Press, 1986.
10. KADOTA,N., On Wainer's notation for a minimal subrecursive inaccessible ordinal. To appear in Zeit.Math.Logik.
11. KADOTA,N.,and K.AOYAMA, Some extensions of built-upness on systems of fundamental sequences. Zeit.Math.Logik 36(1990), 357-364.
12. SCHMIDT,D., Built-up systems of fundamental sequences and hierarchies of number theoretic functions. Arch.math.Logik 18(1976)47-53, postscript 18(1977)145-146.

13. SHIMODA,M., Elementary properties of a system of fundamental sequences for Γ_0. Springer Lecture Notes in Math. 1388(1989) 141-152.

14. TAIT,W.W., Normal derivability in classical logic. In: J.Barwise, ed., The Syntax and Semantics of Infinitary Languages, Springer Lecture Notes in Math. 72(1968)204-236.

15. WAINER,S.S., Slow growing versus fast growing. J.Symb.Logic 54(1989),608-614.

16. WAINER,S.S., Hierarchies of provably computable functions.In Mathematical Logic, edited by P.Petkov, Plenum Press, New York, 1990, 211-220.

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