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**Kyoto University**
A SUBRECURSIVE INACCESSIBLE ORDINAL

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INTRODUCTION

The purpose of the article is to prove the minimal subrecursive inaccessibility of the ordinal $\tau$ introduced by Wainer[6]. We call an ordinal $\alpha$ subrecursive inaccessible (or s-inaccessible) if the slow-growing hierarchy $\{G_\gamma | \gamma \leq \alpha\}$ of number-theoretic functions catches up with the fast-growing hierarchy $\{F_\gamma | \gamma \leq \alpha\}$, i.e., there exists $p < \omega$ such that for all $x > p$,

$$G_\alpha(x) < F_\alpha(x) \leq G_\alpha(x+1).$$

In the article, we will complete the proof of the result of [6] that $\tau$ is a minimal s-inaccessible, by showing

(I) Collapsing theorem (Section 2), and
(II) (3)-built-upness of $\tau$ (Section 3).

We will use the result of [4] (the strong normalization theorem) when we will show (I) and (II).

It is known from the results of Girard[3] (cf.[6,Example 4]) that the set-theoretic ordinal height of $\tau$ is $\sup\{|\text{ID}_\nu| : \nu < \omega\}$ where $\text{ID}_\nu$ is the theory for $\nu$-times iterated inductive definitions and $|\text{ID}_\nu|$ is its proof-theoretic ordinal. Hence (II) above indicates that Wainer's fundamental sequences for $|\text{ID}_\nu|$ ($\nu < \omega$) is natural in the sense of subrecursive hierarchy theory.

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§1. SUBRECURSIVE INACCESSIBILITY

In this section we will define a tree-ordinal $\tau$ following [6] and show that $\tau$ is minimal $s$-inaccessible (Theorem 1.10 below) assuming the collapsing theorem and (3)-built-upness of $\tau$ which will be proved in Sections 2 and 3 respectively. In the following, the letters $k, m, n, p, x$ denote non-negative integers.

DEFINITION 1.1.1(cf.[1]) The set $\Omega$ of the tree-ordinals consists of the infinitary terms generated inductively by:

(i) $0 \in \Omega$.

(ii) If $\alpha \in \Omega$, then $\alpha+1 \in \Omega$.

(iii) If $\alpha_x \in \Omega$ for all $x < \omega$, then $(\alpha_x)_{x<\omega} \in \Omega.$ (In this case we call $(\alpha_x)_{x<\omega}$ limit and write $\alpha[x]$ instead of $\alpha_x$.)

(2) For a given $p < \omega$, the subset $\Omega^{(p)}_{bu} \subseteq \Omega$ of $(p)$-built-up tree-ordinals consists of those $\alpha \in \Omega$ satisfying that:

$$\lambda[x] \prec_p \lambda[x+1] \text{ for all limit } \lambda \trianglelefteq \alpha \text{ and } x < \omega,$$

where the relations $\prec$ ($\prec_p$) on $\Omega$ are the transitive closure of

(i) $\beta < \beta+1$ ($\beta \prec_p \beta+1$) (ii) $\beta[x] < \beta$ for all $x < \omega$ ($\beta[p] \prec \beta$ resp.) if $\beta$ is limit.

Then we define the fast-growing $\{F_x\}_{\alpha \in \Omega}$ and slow-growing $\{G_x\}_{\alpha \in \Omega}$ hierarchies as follows:

$$F_0(x) = x+1,$$
$$G_0(x) = 0,$$
$$F_{\alpha+1}(x) = F^x_{\alpha}(F_{\alpha}(x)),$$
$$G_{\alpha+1}(x) = G_{\alpha}(x)+1,$$
$$F_\lambda(x) = F_{\lambda[x]}(x),$$
$$G_\lambda(x) = G_{\lambda[x]}(x),$$

where $\lambda$ is limit and the superscript $x$ denotes iteration $x$-times.
of $F_\alpha$ (i.e., if $F: \omega \rightarrow \omega$ then $F^0(x) = x$, $F^{m+1}(x) = F(F^m(x)))$.

**PROPOSITION 1.2** ([5, Theorem 3.1]). For some $p < \omega$, we assume $\alpha \in \Omega^{(p)}$-bu. Then the following holds:

1. $F_\alpha(x) < F_\alpha(x+1)$ and $G_\alpha(x) \leq G_\alpha(x+1)$ for $p \leq x+1$.
2. If $\beta \prec_m \alpha$ for $p \leq m$, then $F_\beta(x) < F_\alpha(x)$ and $G_\beta(x) < G_\alpha(x)$ for $x > m$.

**Proof.** Induction on $\alpha \in \Omega$ similarly to [5, Theorem 3.1].

**LEMMA 1.3.** For $p < \omega$ and $\alpha \in \Omega^{(p)}$-bu, the following holds:

1. For all $x > p$, $G_\alpha(x) < F_\alpha(x)$.
2. If $\alpha$ is $s$-inaccessible (see Intro. for the definition), then $\alpha$ is limit and $G_\alpha$ eventually dominates every $F_\beta$ with $\beta < \alpha$ (i.e., for all but finitely many $x$, $F_\beta(x) < G_\alpha(x)$).

**Proof.** (1) Induction on $\alpha$. (2) Clearly $\alpha$ cannot be 0. For any $\beta+1 \in \Omega^{(p)}$-bu and $x > p$,

$$G_{\beta+1}(x) = G_\beta(x)+1 < F_\beta(x)+1 \leq F_\beta(x+1) \leq F^{x+1}_\beta(x) = F_{\beta+1}(x).$$

Hence $\alpha$ must be limit. Assume $\beta < \alpha$. Then $\beta+1 < \alpha$ since $\alpha$ is limit, and then we can see that for some $m > p$, $\beta+1 <_m \alpha$. Hence $F_\beta(x+1) < F^{x+1}_\beta(x) = F_{\beta+1}(x) < F_\alpha(x) \leq G_\alpha(x+1)$, by 1.2.

**PROPOSITION 1.4** ([7, p. 215]). Let $p < \omega$ and $\alpha \in \Omega^{(p)}$-bu satisfy that $G_{\alpha[n+1]} = F_{\alpha[n]}$ for all $n < \omega$. Then $\alpha$ is $s$-inaccessible and, if $\alpha[0]$ is finite (i.e., $\alpha[0] = 0+1+\cdots+1$), then no $\beta < \alpha$ is $s$-inaccessible.

**Proof.** If $G_{\alpha[n+1]} = F_{\alpha[n]}$ for each $n$, then $F_\alpha(x) = F_{\alpha(x)}(x) = \ldots$
\[ G_{\alpha[x+1]}(x) \leq G_{\alpha[x+1]}(x+1) = G_{\alpha}(x+1) \] and so \( \alpha \) is s-inaccessible. If \( \alpha[0] \) is finite and \( \beta < \alpha \) were s-inaccessible then \( \alpha[0] < \beta \) since \( \beta \) is limit. So \( \alpha[n] < \beta \leq \alpha[n+1] \) for some \( n \). By 1.3, for sufficient large \( x \), \( G_{\alpha[n+1]}(x) = F_{\alpha}[n](x) < G_{\beta}(x) \leq G_{\alpha[n+1]}(x) \) since \( \beta \leq \alpha[n+1] \).

\[
\]

DEFINITION 1.5([6]). The sets \( \Omega_n \) of higher level tree-ordinals are defined by induction similarly to the case of \( \Omega \):

(1) \( 0 \in \Omega_n \).

(2) If \( \alpha \in \Omega_n \), then \( \alpha+1 \in \Omega_n \).

(3) If \( \alpha_\gamma \in \Omega_n \) for all \( \gamma \in \Omega_k(\kappa<n) \), then \( (\alpha_\gamma)_{\gamma \in \Omega_k} \in \Omega_n \). (In this case, we call \( (\alpha_\gamma)_{\gamma \in \Omega_k} \) limit and write \( \alpha[\gamma] \) instead of \( \alpha_\gamma \).

In the following we identify \( \Omega_0 \) with \( \omega \), and \( \Omega_1 \) with \( \Omega \).

DEFINITION 1.6([6, Definition 5]). The level \( n \) fast-growing hierarchies of functions \( \varphi_n: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n \) is defined by:

(1) \( \varphi_n(0, \beta) = \beta+1 \).

(2) \( \varphi_n(\alpha+1, \beta) = \varphi_n^\beta(\alpha, \varphi_n(\alpha, \beta)) \).

(3) \( \varphi_n(\lambda, \beta) = (\varphi_n(\lambda[\gamma], \beta))_{\gamma \in \Omega_k(\kappa<n)} \) for \( \lambda = (\lambda[\gamma])_{\gamma \in \Omega_k(\kappa<n)} \).

(4) \( \varphi_n(\lambda, \beta) = \varphi_n(\lambda[\beta], \beta) \) for \( \lambda = (\lambda[\gamma])_{\gamma \in \Omega_n} \),

where \( \varphi_n^\beta \) denotes the iteration \( \beta \)-times of \( \varphi_n \) (i.e., if \( \psi: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n \), then \( \psi^0(\alpha, \beta) = \beta \), \( \psi^{\delta+1}(\alpha, \beta) = \psi(\alpha, \psi^\delta(\alpha, \beta)) \), \( \psi^\lambda(\alpha, \beta) = (\psi^\lambda(\gamma)(\alpha, \beta))_{\gamma \in \Omega_m(\kappa<n)} \) for \( \lambda = (\lambda[\gamma])_{\gamma \in \Omega_m} \).

Note that, in the case \( n = 0 \), \( \varphi_0(\alpha, \beta) = F_\alpha(\beta) \) for \( \alpha \in \Omega_1 \) and
$\beta \in \Omega_0 (= \omega)$. We define $\omega_k \in \Omega_n$ by $\omega_k = (\gamma)_{\gamma \in \Omega_k}$ (i.e., $\omega_k[\gamma] = \gamma$).

DEFINITION 1.7([6, Definition 7]). The sets $T_n$ ($\subseteq \Omega_n$) of named tree-ordinals are defined inductively by:

(i) $0, 1, \omega_0, \ldots, \omega_{n-1} \in T_n$.

(ii) $T_k \subseteq T_n$ for $k < n$.

(iii) If $\alpha \in T_{n+1}$ and $\beta, \gamma \in T_n$, then $\varphi_n^\gamma(\alpha, \beta) \in T_n$.

COLLAPSING THEOREM([6]). Let $x < \omega$, $\alpha \in T_2$ and $\beta \in T_1$. Then

$$G_{\varphi_1}(\alpha, \beta)(x) = F_{\varphi_0}(G_{\beta}(x)),$$

where the function $c (= c_x)$ which collapses each $T_{n+1}$ to $T_n$ is defined by: $c0 = 0$, $c1 = 1$, $c\omega_0 = x$, $c\omega_{k+1} = \omega_k$.

$c(\varphi_{k+1}^\gamma(\delta, \xi)) = \varphi_k^{c\gamma}(c\delta, c\xi)$, $c(\varphi_{0}^\gamma(\delta, \xi)) = \varphi_0^\gamma(\delta, \xi)$. Hence, in particular, if $\alpha$ is generated in $T_2$ without reference to $\omega_0$ then, as $G_{\omega_0}(x) = x$, we have $G_{\varphi_1}(\alpha, \omega_0) = F_{\varphi_0}$.

Proof. See Section 2.

DEFINITION 1.8([6, Example 4]). We define $\tau = (\tau[x])_{x < \omega}$ by setting $\tau[0] = 3$, $\tau[n+1] = \varphi_1(\ldots \varphi_n(\varphi_{n+1}(3, \omega_n), \omega_{n-1}), \ldots, \omega_0)$.

THEOREM 1.9. $\tau$ is a minimal s-inaccessible tree-ordinal.

Proof. From Section 3, $\tau$ is (3)-built-up. Hence 1.4 and the collapsing theorem complete the proof.
§2. THE COLLAPSING THEOREM

In this section we will prove the collapsing theorem used in Section 1 using the strong normalization theorem in [4]. First, we introduce term structures \( \langle \bar{T}_n, NT_n, \cdot \rangle \rightarrow \) by considering each element in \( T_n \) as a finitary term and each defining equation of \( \phi_n \) (Definition 1.6) as a rewrite (or reduction) rule of the terms. Let \( \bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \ldots; \bar{\phi}_0, \bar{\phi}_1, \ldots \) be formal symbols.

DEFINITION 2.1. The sets \( \bar{T}_n \) of terms are defined inductively by:

(1) \( \bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_{n-1} \in \bar{T}_n \).

(ii) \( \bar{T}_k \subseteq \bar{T}_n \) for \( k < n \).

(iii) If \( a \in \bar{T}_{n+1} \) and \( b, c \in \bar{T}_n \), then \( \bar{\phi}_n^c(a, b) \in \bar{T}_n \).

Naturally, terms in \( \bar{T}_n \) are interpreted as tree-ordinals by the function \( \text{ord}: \bar{T}_n \rightarrow T_n \) such that (i) \( \text{ord}(\bar{0}) = 0 \), \( \text{ord}(\bar{1}) = 1 \), \( \text{ord}(\bar{\omega}_k) = \omega_k \), (ii) \( \text{ord}(\bar{\phi}_n^c(a, b)) = \phi_n^{\text{ord}(c)(\text{ord}(a), \text{ord}(b))} \).

Abbreviations. \( \bar{\phi}_n(a, b) = \bar{\phi}_n^1(a, b), b + 1 = \bar{\phi}_n(\bar{0}, b). \)

DEFINITION 2.2. The sets \( NT_n \) of normal terms in \( \bar{T}_n \): \( \text{dom}(a) \in \{ \phi, \{\bar{0}\}, \bar{T}_0, \ldots, \bar{T}_{n-1} \} \) and \( a[z] \) for \( a \in NT_n \), \( z \in \text{dom}(a) \) are defined inductively by:

(N1) \( \bar{0} \in NT_n \); \( \text{dom}(\bar{0}) = \phi \).

(N2) \( \bar{1} \in NT_n \); \( \text{dom}(\bar{1}) = \{\bar{0}\}, \bar{1}[\bar{0}] = \bar{0} \).

(N3) \( \bar{\omega}_1 \in NT_n \) (\( i < n \)); \( \text{dom}(\bar{\omega}_1) = \bar{T}_i \), \( \bar{\omega}_1[z] = z \).

(N4) \( NT_k \subseteq NT_n \) for \( k < n \).
(N5) Let \( a \in \mathbb{N}_n^{+1}, b, c \in \mathbb{N}_n \) and \( A = \varphi_n^c(a, b) \). Then \( A \in \mathbb{N}_n \) if one of the following holds:

(i) \( c = 0 \) and \( a = \tilde{0}(i.e., A = b+1); \) \( \text{dom}(A) = \{0\}, A[z] = b \).

(ii) \( \text{dom}(c) = \tilde{T}_k(k<n); \) \( \text{dom}(A) := \text{dom}(c), A[z] = \varphi_n^c[z](a, b) \).

(iii) \( c = 1 \) and \( \text{dom}(a) = \tilde{T}_k(k<n); \) \( \text{dom}(A) = \text{dom}(a), A[z] = \varphi_n(a[z], b) \).

A term-rewriting system \((S)\) (see e.g. Dershowitz[2] as for the definition) is introduced so that, for every term in \( \tilde{T}_n \) which is not normal, some rewrite rule in \((S)\) is applied to it:

**Definition of the rewrite rules of \((S)\):** For normal \( a, b, c \):

- *(R1)* \( \varphi_n^0(a, b) \rightarrow b \), *(R2)* \( \varphi_n^1(\tilde{0}, b) \rightarrow \varphi_n^b(\tilde{0}, \varphi_n^0(\tilde{0}, b)) \).
- *(R3)* \( \varphi_n(a+1, b) \rightarrow \varphi_n^b(a, \varphi_n(a, b)) \).
- *(R4)* \( \varphi_n^{c+1}(a, b) \rightarrow \varphi_n(a, \varphi_n^c(a, b)) \).
- *(R5)* \( \varphi_n(a, b) \rightarrow \varphi_n(a[b], b) \) if \( \text{dom}(a) = \tilde{T}_n \).

**Proposition 2.3.** For every \( a \in \tilde{T}_n \), \( a \in \mathbb{N}_n \) if and only if there is no \( b \in \mathbb{T} \) such that \( a \rightarrow b \) (where \( a \rightarrow b \) means that \( b \) is obtained from \( a \) by a single application of some rule of \((S)\)).

**Proof.** Induction on the length of \( a \). \( \square \)

**Strong Normalization Theorem:** Every term \( a \) in \( \tilde{T}_n \) is strongly normalizable (i.e., there is no infinite sequence such that \( a \rightarrow a_1 \rightarrow a_2 \rightarrow \ldots \)).
Proof. See [4, Theorem 1]. □

Now we introduce a function \( \tilde{c} \) which represents the function \( c \) (in the collapsing theorem) on the terms as follows: (for each fixed \( x < \omega \))

(i) \( \tilde{c} \bar{0} = \bar{0}, \quad \tilde{c} \bar{1} = \bar{1}, \quad \tilde{c} \bar{w}_0 = \bar{x}, \quad \tilde{c} \bar{w}_{k+1} = \bar{w}_k \).

(ii) \( \tilde{c}(\bar{\phi}_{n+1} \gamma(\delta, \xi)) = \bar{\phi}_n \tilde{c} \gamma(\tilde{c} \delta, \tilde{c} \xi) \) and \( \tilde{c}(\bar{\phi}_0 \gamma(\delta, \xi)) = \bar{\phi}_0 \gamma(\delta, \xi) \),

where \( \bar{x} \) is the numeral of \( x \) (i.e., if \( x = 0 \) then \( \bar{x} = \bar{0} \); if \( x = y+1 \) then \( \bar{x} = \bar{\phi}_0(\bar{0}, \bar{y}) = (\bar{y}+1) \)).

**Lemma 2.4.** Let \( a \in \bar{T}_n \). Then the following hold.

1. If \( a = b+1 \) for some \( b \), then \( \tilde{c}(b) = \tilde{c}b+1 \).
2. If \( a \in NT_n \) and \( \text{dom}(a) = \bar{T}_0 \), then \( \tilde{c}(a[\bar{x}]) = \tilde{c}a \) and \( \text{ord}(a[\bar{x}]) = \text{ord}(a) \) for \( x < \omega \).
3. If \( a \in NT_n \) and \( \text{dom}(a) = \bar{T}_k \) for some \( k > 0 \), then \( \text{ord}(a[b]) = \text{ord}(a)[\text{ord}(b)] \) and \( \text{ord}(\tilde{c}(a[b])) = \text{ord}(\tilde{c}a)[\text{ord}(\tilde{c}b)] \) for \( b \in \text{dom}(a) \).
4. If \( a \xrightarrow{1} b \), then \( \text{ord}(a) = \text{ord}(b) \) and \( \text{ord}(\tilde{c}a) = \text{ord}(\tilde{c}b) \).

*Proof.* (1)-(4) Induction on the length of \( a \). □

**Lemma 2.5.** If \( x < \omega \) and \( a \in \bar{T}_1 \), then \( G_{\text{ord}(a)}(x) = \text{ord}(\tilde{c}a) \).

*Proof.* From the strong normalization theorem, the proof is proceeded by transfinite induction on \( a \) over the well-founded ordering \( \ll \) (where \( \ll \) on \( \bar{T}_n \) is defined as the transitive closure of (i) \( b[z] \ll b \) for normal \( b \) with \( z \in \text{dom}(b) \), (iii) \( d \ll b \) for non-normal \( b \) with \( b \xrightarrow{1} d \)).

Case 1. \( a = \bar{0} \). This case is trivial.
Case 2. $a \in NT_1$ and $\text{dom}(a) = \{\tilde{0}\}$. Then $a = \tilde{1}$ or $b+1$ for some $b \in \tilde{T}_1$. If $a = \tilde{1}$, the assertion is trivial. If $a = b+1$, then $G_{\text{ord}(a)}(x) = G_{\text{ord}(b)}(x) + 1 = \text{ord}(\tilde{\omega}b) + 1 = \text{ord}(\tilde{\omega}a)$ by I.H. (= induction hypothesis) and 2.4(1).

Case 3. $a \in NT_1$ and $\text{dom}(a) = \tilde{T}_0$. By 2.4(2) and I.H.,

$G_{\text{ord}(a)}(x) = G_{\text{ord}(a[\tilde{x}])(x) = \text{ord}(\tilde{\sigma}(a[\tilde{x}])) = \text{ord}(\tilde{\omega}a)$. 

Case 4. $a \xrightarrow{1} b$ for some $b$. By 2.4(4) and I.H.,

$G_{\text{ord}(a)}(x) = G_{\text{ord}(b)}(x) = \text{ord}(\tilde{\omega}b) = \text{ord}(\tilde{\omega}a)$. \hfill \Box$

Proof of the collapsing theorem (in Section 1). For $a \in \tilde{T}_2$ and $b \in \tilde{T}_1$, we have $\tilde{\sigma}(\tilde{\phi}_1(a,b)) = \tilde{\varphi}_0(\tilde{\omega}a,\tilde{\omega}b)$ and hence $\text{ord}(\tilde{\sigma}(\tilde{\phi}_1(a,b))) = \varphi_0(\text{ord}(\tilde{\omega}a),\text{ord}(\tilde{\omega}b))$. Thus,

$G_{\phi_1(\text{ord}(a),\text{ord}(b))}(x) = G_{\text{ord}(\tilde{\phi}_1(a,b))}(x)$

$= \text{ord}(\tilde{\sigma}(\tilde{\phi}_1(a,b)))$ by 2.5

$= \varphi_0(\text{ord}(\tilde{\omega}a),\text{ord}(\tilde{\omega}b))$

$= F_{\text{ord}(\tilde{\omega}a)}(\text{ord}(\tilde{\omega}b))$

$= F_{\text{ord}(\tilde{\omega}a)}(G_{\text{ord}(b)}(x))$ by 2.5.

For given $\alpha \in T_2$ and $\beta \in T_1$, we choose $a$ and $b$ above such that

(i) $\text{ord}(a) = \alpha$, $\text{ord}(\tilde{\omega}a) = \omega \alpha$ and (ii) $\text{ord}(b) = \beta$ (we can choose such $a$ and $b$ since the elements of $T_n$ are constructed by the same way as to the element in $\tilde{T}_n$). This completes the proof. \hfill \Box

§3. (3)-BUILT-UPNESS OF $\tau$

In this section we will prove that $\tau$ is (3)-built-up. This completes the proof of Theorem 1.9 ($\tau$ is minimal $s$-inaccessible). First, we remark that the following proposition holds:
PROPOSITION 3.1([4, Lemma 3.4]). Let $\alpha \in T_n$ and $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$. Then $\alpha[\gamma] \in T_n$ for every $\gamma \in T_m$.

Proof. For a given $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n$, there is a normal $a \in T_n$ such that ord$(a) = \alpha$ by 2.4(4) and the strong normalization theorem. We fix such an $a \in T_n$ with the minimal length. The proof of this proposition can be proceeded by induction on the length of this term $a$ for $\alpha$. □

It follows from this proposition that we can use transfinite induction on the terms in $T_n (n<\omega)$ over the ordering $\prec$ of $T_n$.

DEFINITION 3.2. The step-down relations $\prec_k (k<\omega)$ on $\bigcup_{n<\omega} T_n$ are defined inductively as follows:

$\alpha \prec_k \beta$ if $\beta \neq 0$ and one of the following holds:

(i) $\alpha \prec_k \gamma$ if $\beta = \gamma + 1$,

(ii) $\alpha \prec_k \beta[k]$ if $\beta = (\beta[x])_{x \in \Omega_0}$,

(iii) $\alpha \prec_k \beta[\gamma]$ for all $\gamma \in T_m \setminus \{0\}$ if $\beta = (\beta[\gamma])_{\gamma \in \Omega_m} (m>0)$.

where $\alpha \preceq_k \delta$ means that $\alpha \prec_k \delta$ or $\alpha = \delta$.

Note that if $\alpha, \beta \in T_1$ then the relations $\prec_k$ defined above are the same as ones defined in Definition 1.1(2).

LEMMA 3.3. For $\alpha \in T_{n+1}$, $\beta \in T_n$ and $\gamma \in T_n \setminus \{0\}$, $\beta \prec_k \varphi_n \gamma(\alpha, \beta)$.

Proof. The lemma follows immediately from the two claims: □
CLAIM 1. Let $\alpha \in T_{n+1}$ and $\beta \in T_n$. If $\delta \prec_k \varphi_n(\alpha, \beta)$ for all $\delta \in T_n$, then $\beta \prec_k \varphi_n^\gamma(\alpha, \beta)$ for $\gamma \in T_n \setminus \{0\}$.

Proof of Claim 1. Transfinite induction on $\gamma \in T_n$.
Case 1. $\gamma = n+1$. Then $\beta \prec_k \varphi_n^\eta(\alpha, \beta) \prec_k \varphi_n(\alpha, \varphi_n^\eta(\alpha, \beta)) = \varphi_n^\gamma(\alpha, \beta)$ by I.H.
Case 2. $\gamma = (\gamma[x])_{x \in \Omega_0}$. Then $\beta \prec_k \varphi_n^{\gamma[k]}(\alpha, \beta) = \varphi_n^\gamma(\alpha, \beta)[k]$ by I.H. Hence $\beta \prec_k \varphi_n^\gamma(\alpha, \beta)$.
Case 3. $\gamma = (\gamma[\delta])_{\delta \in \Omega_m}$ for $0 < m < n$. We can prove that $\gamma[\delta] \in T_m \setminus \{0\}$ similarly to 3.1. Hence $\beta \prec_k \varphi_n^{\gamma[\delta]}(\alpha, \beta) = \varphi_n^\gamma(\alpha, \beta)[\delta]$ for $\delta \in T_m \setminus \{0\}$ by I.H. Therefore $\beta \prec_k \varphi_n^\gamma(\alpha, \beta)$.

CLAIM 2. Let $\alpha \in T_{n+1}$. Then $\beta \prec_k \varphi_n(\alpha, \beta)$ for all $\beta \in T_n$.

Proof of Claim 2. Transfinite induction on $\alpha \in T_{n+1}$.
Case 1. $\alpha = 0$. Then $\beta \prec_k \beta + 1 = \varphi_n(\alpha, \beta)$.
Case 2. $\alpha = \gamma + 1$. Then $\delta \prec_k \varphi_n(\gamma, \delta)$ for all $\delta \in T_n$ by I.H. Hence, by Claim 1, $\beta \prec_k \varphi_n(\gamma, \beta) \prec_k \varphi_n^\beta(\gamma, \varphi_n(\gamma, \beta)) = \varphi_n(\alpha, \beta)$.
Case 3. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$ for $m < n$. By I.H., $\beta \prec_k \varphi_n(\alpha[\gamma], \beta) = \varphi_n(\alpha, \beta)[\gamma]$ for $\gamma \in T_m$. Hence $\beta \prec_k \varphi_n(\alpha, \beta)$.
Case 4. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_n}$. By I.H., $\beta \prec_k \varphi_n(\alpha[\gamma], \beta) = \varphi_n(\alpha, \beta)$.

LEMMA 3.4. Let $\alpha \in T_{n+1}$ and $\beta, \delta, \gamma \in T_n$. If $\gamma \prec_k \delta$, then $\varphi_n^\gamma(\alpha, \beta) \prec_k \varphi_n^\delta(\alpha, \beta)$.

Proof. Transfinite induction on $\delta \in T_n$.
Case 1. $\delta = 0$. This case is trivial.
Case 2. $\delta = n+1$. By I.H. and 3.3, $\varphi_n^\gamma(\alpha, \beta) \prec_k \varphi_n^\eta(\alpha, \beta) \prec_k$
\( \varphi_n(\alpha, \varphi_n^\eta(\alpha, \beta)) = \varphi_n^\delta(\alpha, \beta) \).

Case 3. \( \delta = (\delta[x])_{x \in \Omega_0} \). By I.H., \( \varphi_n^\gamma(\alpha, \beta) \preceq_k \varphi_n^\delta[k](\alpha, \beta) = \varphi_n^\delta(\alpha, \beta)[k] \). Hence \( \varphi_n^\gamma(\alpha, \beta) \preceq_k \varphi_n^\delta(\alpha, \beta) \).

Case 4. \( \delta = (\delta[\xi])_{\xi \in \Omega_m} (0 < m < n) \). Then \( \varphi_n^\gamma(\alpha, \beta) \preceq_k \varphi_n^\delta[\xi](\alpha, \beta) = \varphi_n^\delta(\alpha, \beta)[\xi] \) for \( \xi \in T_m \setminus \{0\} \) by I.H. Hence \( \varphi_n^\gamma(\alpha, \beta) \preceq_k \varphi_n^\delta(\alpha, \beta) \). \( \Box \)

**Lemma 3.5.** Let \( \alpha, \gamma \in T_{n+1}, \beta \in T_n \setminus \{0\} \) and \( n > 0 \). If \( \gamma \preceq_k \alpha \), then \( \varphi_n(\gamma, \beta) \preceq_k \varphi_n(\alpha, \beta) \).

**Proof.** Transfinite induction on \( \alpha \in T_n \).

Case 1. \( \alpha = 0 \). This case is trivial.

Case 2. \( \alpha = \eta + 1 \). By I.H. and 3.3, \( \varphi_n(\gamma, \beta) \preceq_k \varphi_n(\eta, \beta) \preceq_k \varphi_n^\beta(\eta, \varphi_n(\eta, \beta)) = \varphi_n(\alpha, \beta) \) since \( \beta \neq 0 \).

Case 3. \( \alpha = (\alpha[x])_{x \in \Omega_0} \). By I.H., \( \varphi_n(\gamma, \beta) \preceq_k \varphi_n(\alpha[k], \beta) = \varphi_n(\alpha, \beta)[k] \). Hence \( \varphi_n(\gamma, \beta) \preceq_k \varphi_n(\alpha, \beta) \).

Case 4. \( \alpha = (\alpha[\xi])_{\xi \in \Omega_m} (0 < m < n) \). By I.H., \( \varphi_n(\gamma, \beta) \preceq_k \varphi_n(\alpha[\xi], \beta) = \varphi_n^\alpha(\beta)[\xi] \) for \( \xi \in T_m \setminus \{0\} \). Hence \( \varphi_n(\gamma, \beta) \preceq_k \varphi_n(\alpha, \beta) \).

Case 5. \( \alpha = (\alpha[\xi])_{\xi \in \Omega_n} \). By I.H., \( \varphi_n(\gamma, \beta) \preceq_k \varphi_n(\alpha[\beta], \beta) = \varphi_n(\alpha, \beta) \) for \( \beta \in T_n \setminus \{0\} \). \( \Box \)

**Theorem 3.6 (4, Theorem 3).** (1) Let \( \alpha \in T_n^+ \) and \( \alpha = (\alpha[\xi])_{\xi \in \Omega_m} \).

If \( \gamma, \delta \in T_m \) and \( \gamma \preceq_k \delta \), then \( \alpha[\gamma] \preceq_k \alpha[\delta] \) (where the sets \( T_n^+ \) (\( \leq T_n \)) are defined inductively by:

1. \( 0, 1, \omega_0, \ldots, \omega_{n-1} \in T_n^+ \)  \( \gamma \in T_n^+ \) for \( k < n \),
2. \( \gamma \in T_n^+ \) if \( \alpha \in T_{n+1}^+ \) and \( \beta \in T_n \setminus \{0\} \), then \( \varphi_n(\gamma, \beta) \in T_n^+ \).
3. Each \( \alpha \in T_1^+ \) is \( (k) \)-built-up for all \( k < \omega \).
Proof. (1) Similarly to the proof of 3.1, for a given \( \alpha \in T_n^+ \), we can take a normal term \( a \in \tilde{T}_n^+ \) with the minimal length such that \( \text{ord}(a) = \alpha \) (where the sets \( \tilde{T}_n^+ (\leq \tilde{T}_n) \) are defined inductively by:

(i) \( \tilde{0}, \tilde{1}, \tilde{\omega}_0, \ldots, \tilde{\omega}_{n-1} \in \tilde{T}_n^+ \), (ii) \( \tilde{T}_k^+ \subseteq \tilde{T}_n^+ \) for \( k < n \),

(iii) if \( a \in \tilde{T}_{n+1}^+ \), \( c \in \tilde{T}_n^+ \) and \( b \in \tilde{T}_n^+ \setminus \{0\} \), then \( \tilde{\varphi}_n^c(a, b) \in T_n^+ \).

Hence we fix such an \( a \in \tilde{T}_n^+ \). The proof of this theorem will be proceeded by the induction on the length of the term \( a \). We have the following cases:

Case 1. \( a = \tilde{\omega}_m \). Then \( \alpha = \omega_m \). We have \( \alpha[\gamma] = \gamma \prec_k \delta = \alpha[\delta] \).

Case 2. \( a = \tilde{\varphi}_n^t(d, b) \) and \( \text{dom}(d) = \tilde{T}_m \). Then \( \alpha = \varphi_n(\lambda, \beta) \) so that \( \lambda = (\lambda[\xi]) \xi \in \Omega_m \) \( \text{ord}(d) \) and \( \beta = \text{ord}(b) \in T_n^+ \setminus \{0\} \) from the definition of \( T_n^+ \) above and \( a \in \tilde{T}_n^+ \). Hence, by I.H. \( \lambda[\gamma] \prec_k \lambda[\delta] \) and 3.5. \( \varphi_n(\lambda, \beta)[\gamma] = \varphi_n(\lambda[\gamma], \beta) \prec_k \varphi_n(\lambda[\delta], \beta) = \varphi_n(\lambda, \beta)[\delta] \).

Case 3. \( a = \tilde{\varphi}_n^e(d, b) \) and \( \text{dom}(e) = \tilde{T}_m \). This case is treated similarly to Case 2, using 3.4. This completes the proof of (1).

(2) We can show that for each \( \alpha = (\alpha[\gamma]) \gamma \in \Omega_m \in T_n^+ \) and \( \gamma \in T_m^+ \), \( \alpha[\gamma] \in T_n^+ \) similarly to 3.1. Hence for each \( \alpha \in T_1^+ \) and limit \( \lambda \preceq \alpha \), we have \( \lambda \in T_1^+ \). Thus by (1), \( \lambda[x] \prec_k \lambda[x+1] \) for all \( k, x < \omega \) and limit \( \lambda \preceq \alpha \in T_1^+ \).

We remark that (k)-built-upness does not hold for some element in \( T_1 \) since, if we put \( \alpha = \varphi_1(\omega_0, 0) \), then \( \alpha[x] = \varphi_1(x, 0) = 1 \) for all \( x < \omega \).

THEOREM 3.7([4, Corollary 3.1]). \( \tau \) is (3)-built-up.

Proof. From the definition of \( \tau \) (Definition 1.8), \( \tau[x] \in T_1^+ \).
for every $x < \omega$. By 3.6(2), $\tau[x]$ is $(3)$-built-up. Hence it is sufficient to prove that $\tau[x] \lesssim_3 \tau[x+1]$. For this, we have

$$\tau[x] = \varphi_1(\ldots \varphi_x(3, \omega_{x-1}), \ldots, \omega_0) \lesssim_3 \varphi_1(\ldots \varphi_x(\omega_0, \omega_{x-1}), \ldots, \omega_0)$$

$$= \varphi_1(\ldots \varphi_x(\omega_x, \omega_{x-1}), \ldots, \omega_0) \lesssim_3 \varphi_1(\ldots \varphi_x(\varphi_{x+1}(3, \omega_x), \omega_{x-1}), \ldots, \omega_0)$$

$$= \tau[x+1]$$

from 3 $\lesssim_3 \omega_0$ and 3.5, 3.3. This completes the proof. □

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