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WEAKLY NORMAL CLOSURES OF FILTERS ON $P_\kappa\lambda$

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The study of filters on $P_\kappa\lambda$ started by Jech [5] as a natural generalization of that of filters on an uncountable regular cardinal κ. Several notions including weak normality have been generalized. However, there are two versions proposed as weak normality for filters on $P_\kappa\lambda$. One is due to Abe [1] as the generalization of weak normality for filters on κ due to Kanamori [6] and the other is due to Mignone [8]. While Mignone's version is weaker than normality, Abe and Kanamori's versions are not in general. In fact, Abe [2] has proved, generalizing Kanamori [5], that a filter is weakly normal in the sense of Abe iff it is weakly normal in the sense of Mignone and there exists no disjoint family of $cf\lambda$-many positive sets. Therefore Abe and Kanamori's versions are essentially large cardinal properties and Mignone's version seems to be the most natural formulation of weak normality.

In this paper, we study about weak normality in the sense of Mignone. In [8], Mignone studies weak normality of canonically defined filters. We complement his chart and try to find the weakly normal closures of these filters (i.e. the minimal weakly normal filter extending them). Therefore our result is a natural refinement of Carr [4].

It is now well-known that combinatorics on $P_\kappa\lambda$ is not a naive generalization of that on κ. For example, Menas [7] showed that stationarity on $P_\kappa\lambda$ can be characterized by 2-dimensional regressive functions, but not by 1-dimensional ones when $\lambda > \kappa$. We show in terms of weak normality that combinatorics on $P_\kappa\lambda$ varies drastically with respect to $cf\lambda$.

1. PRELIMINARIES

Throughout this paper, κ and λ denote an uncountable regular cardinal and a cardinal strictly larger than κ respectively. $P_\kappa\lambda$ denotes the set $\{x \subset \lambda : |x| < \kappa\}$. By "filter", we mean κ-complete fine (i.e. containing $\{x : \alpha \in x\}$ as an element for any $\alpha < \lambda$) filter on $P_\kappa\lambda$. Especially we are interested in the final segment filter $FSF_{\kappa\lambda}$, the strong club filter $SCF_{\kappa\lambda}$ defined by Carr [4] and the club filter $CF_{\kappa\lambda}$. In stead of their original definitions, the following characterizations due to Carr and Menas are used freely.

1.1 PROPOSITION ([4], [7]). *Let* $X \subset P_\kappa\lambda$.

(1) $X \in FSF_{\kappa\lambda}$ *iff* $\{x : y \subset x\} \subset X$ *for some* $y \in P_\kappa\lambda$.
(2) $X \in SCF_{\kappa\lambda}$ *iff* $\{x : f''x \subset Px\} \subset X$ *for some* $f : \lambda \to P_\kappa\lambda$.
(3) $X \in CF_{\kappa\lambda}$ *iff* $\{x : f''x^2 \subset Px\} \subset X$ *for some* $f : \lambda^2 \to P_\kappa\lambda$.

Throughout this paper, F and G stand for a filter and $\pi_n : \lambda^n \to \lambda$ denotes a bijection for each $2 \leq n < \omega$.

The following theorem is essentially due to Menas, and reformulated more clearly by Carr. We present a simplified proof, which is a prototype of our arguments.

1.2 THEOREM ([4], [7]). $SCF_{\kappa\lambda} \subsetneq CF_{\kappa\lambda}$.

PROOF: It is enough to show that $\{x : \pi_2''x^2 \subset x\} \notin SCF_{\kappa\lambda}$. Otherwise

$$(*) \qquad \{x : f''x \subset Px\} \subset \{x : \pi_2''x^2 \subset x\}$$

for some $f : \lambda \to P_\kappa\lambda$ such that $f(\gamma)$ is closed under π_2^{-1} for any $\gamma < \lambda$.

For each $\gamma < \lambda$, let $x_\gamma \in P_\kappa\lambda$ be the closure of $\{\gamma\}$ under f, i.e. $x_\gamma = \bigcup_{n<\omega} x_{\gamma,n}$, where $x_{\gamma,n}$ is defined inductively by $x_{\gamma,0} = \{\gamma\}$ and $x_{\gamma,n+1} = \bigcup f''x_{\gamma,n}$.

Now pick $\beta \in \lambda - \bigcup_{\alpha<\kappa} x_\alpha$. We claim that $\{\pi_2(\alpha,\beta) : \alpha < \kappa\} \subset x_\beta$, which immediately gives the required contradiction.

Fix $\alpha < \kappa$. Then $\pi_2(\alpha,\beta) \in x_\alpha \cup x_\beta$, since $x_\alpha \cup x_\beta$ is closed under f and since $(*)$. But $\pi_2(\alpha,\beta) \notin x_\alpha$. Otherwise $\beta \in x_\alpha$, since x_α is closed under π_2^{-1}. Contradiction. ∎

As an advantage of the simplification, the condition on the witnessing function can be weakened a great deal, i.e. we have similarly that $\{x : \rho''x^2 \subset Px\} \notin SCF_{\kappa\lambda}$ for any $\rho : \lambda^2 \to P_\kappa\lambda - \{0\}$ such that $\{(\alpha,\beta) : \gamma \in \rho(\alpha,\beta)\} \in P_\kappa\lambda^2$ for any $\gamma < \lambda$.

DEFINITION. *F is weakly normal iff $\Delta_{\alpha<\lambda}X_\alpha \in F$ for any $\{X_\alpha : \alpha < \lambda\} \subset F$ such that $X_\alpha \supset X_{\alpha'}$ for any $\alpha < \alpha' < \lambda$ ("descending" for short).*

We introduce weak normality analogs of the diagonal intersection and the diagonal operation Δ defined by Carr [4].

DEFINITION. *Let* $\{X_\alpha : \alpha < \lambda\} \subset PP_\kappa\lambda$.

(1) $\widetilde{\Delta}_{\alpha<\lambda}X_\alpha = \{x : \forall\alpha\in x\exists\alpha'\geq\alpha\ x \in X_{\alpha'}\}$.
(2) $X \in \widetilde{\Delta}F$ *iff* $X = \widetilde{\Delta}_{\alpha<\lambda}X_\alpha$ *for some* $\{X_\alpha : \alpha < \lambda\} \subset F$.

Let us summarize some basic properties of $\widetilde{\Delta}$, which are used without any mention of.

1.3 Proposition.

(1) *$X \in \tilde{\Delta} F$ iff $X = \Delta_{\alpha<\lambda} X_\alpha$ for some descending $\{X_\alpha : \alpha < \lambda\} \subset F$.*
(2) *$\tilde{\Delta} F$ is a (possibly improper) filter.*
(3) *$F \subset \tilde{\Delta} F \subset \Delta F$.*
(4) *F is weakly normal iff $F = \tilde{\Delta} F$.*
(5) *$F \subset G$ implies $\tilde{\Delta} F \subset \tilde{\Delta} G$.*

Therefore, when a weakly normal filter such as $CF_{\kappa\lambda}$ extends F, the weakly normal closure of F exists and extends $\tilde{\Delta} F$, $\tilde{\Delta}\tilde{\Delta} F$ and so on.

Definition. *$\tilde{\Delta}^\alpha F$ is defined inductively by $\tilde{\Delta}^0 F = F$, $\tilde{\Delta}^{\alpha+1} F = \tilde{\Delta}(\tilde{\Delta}^\alpha F)$ and $\tilde{\Delta}^\gamma F =$ the minimal filter extending $\bigcup_{\beta<\gamma} \tilde{\Delta}^\beta F$ when γ is limit.*

Thus the weakly normal closure of F (if it exists) can be written as $\tilde{\Delta}^\alpha F$ for some α.

2. Case $cf\lambda < \kappa$

Throughout this section, we assume that $cf\lambda < \kappa$. We have already known

2.1 Proposition ([8]). *$FSF_{\kappa\lambda}$ and $SCF_{\kappa\lambda}$ are weakly normal.*

But in fact,

2.2 Theorem. *Any filter is weakly normal.*

Proof: Let $\{X_\alpha : \alpha < \lambda\} \subset F$ be descending. Then $\bigcap_{\xi<cf\lambda} X_{\lambda_\xi} = \bigcap_{\alpha<\lambda} X_\alpha \subset \Delta_{\alpha<\lambda} X_\alpha$. Thus $\Delta_{\alpha<\lambda} X_\alpha \in F$. ∎

Let us summarize the situation below $CF_{\kappa\lambda}$.

normal		non-normal	
weakly normal			non-weakly normal
$CF_{\kappa\lambda}$	$\supsetneq$	$SCF_{\kappa\lambda}$ $\not\cup$ $FSF_{\kappa\lambda}$	—

3. Case $cf\lambda = \kappa$

Throughout this section, we assume that $cf\lambda = \kappa$ and fix a strictly increasing enumeration $\{\lambda_\xi : \xi < \kappa\}$ of a club subset of λ. Define $\xi : \lambda \to \kappa$ by $\xi_\alpha = \min\{\zeta : \alpha < \lambda_\zeta\}$.

3.1 THEOREM. *$\tilde{\Delta}F$ is the minimal weakly normal filter extending F.*

PROOF: It is enough to show weak normalty of $\tilde{\Delta}F$. Let $\{X_\alpha : \alpha < \lambda\} \subset \tilde{\Delta}F$ be descending. For each $\alpha < \lambda$, pick a descending $\{X_{\alpha,\beta} : \beta < \lambda\} \subset F$ with $X_\alpha = \Delta_{\beta<\lambda} X_{\alpha,\beta}$.

For each $\alpha, \beta < \lambda$, set $Y_{\alpha,\beta} = \bigcap_{\zeta \le \xi_\alpha} X_{\lambda_\zeta,\beta}$. Then $\{Y_{\gamma,\gamma} : \gamma < \lambda\} \subset F$ is descending and $\Delta_{\gamma<\lambda} Y_{\gamma,\gamma} = \{x : \forall\gamma \in x\ x \in Y_{\gamma,\gamma}\} \subset \{x : \forall\alpha,\beta \in x\ x \in Y_{\alpha,\beta}\} = \Delta_{\alpha<\lambda}\Delta_{\beta<\lambda} Y_{\alpha,\beta} \subset \Delta_{\alpha<\lambda}\Delta_{\beta<\lambda} X_{\lambda_{\xi_\alpha},\beta} = \Delta_{\alpha<\lambda} X_{\lambda_{\xi_\alpha}} \subset \Delta_{\alpha<\lambda} X_\alpha$. Thus $\Delta_{\alpha<\lambda} X_\alpha \in \tilde{\Delta}F$. ∎

We have already known

3.2 PROPOSITION ([8]). *$FSF_{\kappa\lambda}$ is not weakly normal.*

Therefore $\tilde{\Delta}FSF_{\kappa\lambda}$ is the minimal weakly normal filter. It is easy to see that for any λ (not necessary $cf\lambda = \kappa$), $X \in \tilde{\Delta}FSF_{\kappa\lambda}$ iff $\{x : \forall\alpha \in x \exists\alpha' \ge \alpha\ f(\alpha') \subset x\} \subset X \subset P_\kappa\lambda$ for some $f : \lambda \to P_\kappa\lambda$. But when $cf\lambda = \kappa$, we have a better expression.

DEFINITION. *$X \in G_{\kappa\lambda}$ iff $\{x : f''x \subset Px\} \subset X \subset P_\kappa\lambda$ for some $f : \lambda \to P_\kappa\lambda$ such that $f(\gamma) \subset f(\gamma')$ for any $\gamma < \gamma' < \lambda$ ("increasing" for short).*

Note that $G_{\kappa\lambda} = FSF_{\kappa\lambda}$ when $cf\lambda \ne \kappa$.

3.3 PROPOSITION. *$G_{\kappa\lambda} = \tilde{\Delta}FSF_{\kappa\lambda}$.*

PROOF: It is enough to show that $G_{\kappa\lambda}$ is the minimal weakly normal filter.

(weak normality) Let $\{X_\alpha : \alpha < \lambda\} \subset G_{\kappa\lambda}$ be descending. For each $\alpha < \lambda$, pick an increasing $f_\alpha : \lambda \to P_\kappa\lambda$ with $\{x : f_\alpha'' x \subset Px\} \subset X_\alpha$ and define $g_\alpha : \lambda \to P_\kappa\lambda$ by $g_\alpha(\beta) = \bigcup_{\zeta \le \xi_\alpha} f_{\lambda_\zeta}(\beta)$.

Now define $h : \lambda \to P_\kappa\lambda$ by $h(\gamma) = g_\gamma(\gamma)$. Then h is increasing and $\{x : h''x \subset Px\} \subset \{x : \forall\alpha,\beta \in x\ g_\alpha(\beta) \subset x\} = \Delta_{\alpha<\lambda}\{x : g_\alpha'' x \subset Px\} \subset \Delta_{\alpha<\lambda}\{x : f_{\lambda_{\xi_\alpha}}'' x \subset Px\} \subset \Delta_{\alpha<\lambda} X_{\lambda_{\xi_\alpha}} \subset \Delta_{\alpha<\lambda} X_\alpha$. Thus $\Delta_{\alpha<\lambda} X_\alpha \in G_{\kappa\lambda}$.

(minimality) Let F be weakly normal. Fix an increasing $f : \lambda \to P_\kappa\lambda$. Then $\{\{x : f(\alpha) \subset x\} : \alpha < \lambda\} \subset F$ is descending. Therefore $\Delta_{\alpha<\lambda}\{x : f(\alpha) \subset x\} = \{x : f''x \subset Px\} \in F$. Thus $G_{\kappa\lambda} \subset F$. ∎

In [8], only the following case was left open.

3.4 THEOREM. *$SCF_{\kappa\lambda}$ is not weakly normal.*

PROOF: Define $f : \lambda^2 \to P_\kappa\lambda$ by $f(\alpha,\beta) = \{\pi_2(\alpha,\lambda_\zeta) : \zeta \le \xi_\beta\}$. For each $\beta < \lambda$, set $X_\beta = \{x : \forall\alpha \in x\ f(\alpha,\beta) \subset x\}$. Then $\{X_\beta : \beta < \lambda\} \subset SCF_{\kappa\lambda}$ is descending. We show that $\Delta_{\beta<\lambda} X_\beta \notin SCF_{\kappa\lambda}$. Otherwise

$$(*)\qquad \{x : g''x \subset Px\} \subset \Delta_{\beta<\lambda} X_\beta = \{x : f''x^2 \subset Px\}$$

for some $g : \lambda \to P_\kappa\lambda$ such that $g(\gamma)$ is closed under π_2^{-1} for any $\gamma < \lambda$.

For each $\gamma < \lambda$, let $x_\gamma \in P_\kappa\lambda$ be the closure of $\{\gamma\}$ under g. Then for any $\gamma < \lambda$ there exists $\xi < \kappa$ such that $\sup x_\gamma < \lambda_\xi$.

Now pick $Y \in [\lambda]^\kappa$ and $\beta < \lambda$ such that $\sup x_\alpha < \lambda_{\xi_\beta}$ for any $\alpha \in Y$. We claim that $\{\pi_2(\alpha, \lambda_{\xi_\beta}) : \alpha \in Y\} \subset x_\beta$, which immediately gives the required contradiction.

Fix $\alpha \in Y$. Then $\pi_2(\alpha, \lambda_{\xi_\beta}) \in f(\alpha,\beta) \subset x_\alpha \cup x_\beta$, since $x_\alpha \cup x_\beta$ is closed under g and since $(*)$. But $\pi_2(\alpha, \lambda_{\xi_\beta}) \notin x_\alpha$. Otherwise $\sup x_\alpha < \lambda_{\xi_\beta} \in x_\alpha$, since x_α is closed under π_2^{-1}. Contradiction. ∎

Therefore $\tilde{\Delta} SCF_{\kappa\lambda}$ is the weakly normal closure of $SCF_{\kappa\lambda}$. It is easy to see that for any λ (not necessary $cf\lambda = \kappa$), $X \in \tilde{\Delta} SCF_{\kappa\lambda}$ iff $\{x : \forall\alpha\in x\exists\alpha' \geq \alpha\forall\beta\in x\ f(\alpha',\beta) \subset x\} \subset X \subset P_\kappa\lambda$ for some $f : \lambda^2 \to P_\kappa\lambda$. But when $cf\lambda = \kappa$, we have a better expression.

Definition. *$X \in H_{\kappa\lambda}$ iff $\{x : f''x^2 \subset Px\} \subset X \subset P_\kappa\lambda$ for some $f : \lambda^2 \to P_\kappa\lambda$ such that $f(\alpha,\beta) \subset f(\alpha,\beta')$ for any $\alpha < \lambda$ and $\beta < \beta' < \lambda$.*

Note that $H_{\kappa\lambda} = SCF_{\kappa\lambda}$ when $cf\lambda \neq \kappa$. Similarly to 3.3, we have

3.5 Proposition. $H_{\kappa\lambda} = \tilde{\Delta} SCF_{\kappa\lambda}$.

$CF_{\kappa\lambda}$ is also a weakly normal filter extending $SCF_{\kappa\lambda}$ and hence may be equal to $\tilde{\Delta} SCF_{\kappa\lambda}$. But this is not the case.

3.6 Theorem. $\tilde{\Delta} SCF_{\kappa\lambda} \subsetneq CF_{\kappa\lambda}$.

Proof: It is enough to show that $\{x : \pi_2''x^2 \subset x\} \notin H_{\kappa\lambda}$. Otherwise

$$(*) \qquad \{x : f''x^2 \subset Px\} \subset \{x : \pi_2''x^2 \subset x\}$$

for some $f : \lambda^2 \to P_\kappa\lambda$ such that $\lambda_{\xi_\beta} \in f(\alpha,\beta) \subset f(\alpha,\beta')$ and $f(\alpha,\beta)$ is closed under π_2^{-1} for any $\alpha < \lambda$ and $\beta < \beta' < \lambda$.

For each $\gamma < \lambda$, let $x_\gamma \in P_\kappa\lambda$ be the closure of $\{\gamma\}$ under f. We show that for any $\gamma < \lambda$ there exists $\xi < \kappa$ with $\sup x_\gamma = \lambda_\xi$.

Otherwise $\lambda_\xi < \sup x_\gamma$, where $\lambda_\xi \leq \sup x_\gamma < \lambda_{\xi+1}$. Then $\sup x_\gamma < \lambda_{\xi+1} \leq \lambda_{\xi_\delta} \in f(\delta,\delta) \subset x_\gamma$ for some $\lambda_\xi < \delta \in x_\gamma$. Contradiction.

Now pick $X \in [\lambda]^{\kappa^+}$ and $\xi < \kappa$ such that $\sup x_\gamma = \lambda_\xi$ for any $\gamma \in X$. We claim that $x_\alpha \cup x_\beta$ is closed under f for any $\alpha, \beta \in X$.

Fix $\delta \in x_\alpha$ and $\eta \in x_\beta$. Then $\eta < \lambda_{\xi_\eta} \leq \sup x_\beta = \sup x_\alpha$, since $\lambda_{\xi_\eta} \in f(\eta,\eta) \subset x_\beta$. Therefore $f(\delta,\eta) \subset f(\delta,\eta') \subset x_\alpha$ for some $\eta < \eta' \in x_\alpha$. Similarly if $\delta \in x_\beta$ and $\eta \in x_\alpha$, then $f(\delta,\eta) \subset x_\beta$. We finish the claim.

Now pick $Y \in [X]^\kappa$ and $\beta \in X - \bigcup_{\alpha\in Y} x_\alpha$. We claim that $\{\pi_2(\alpha,\beta) : \alpha \in Y\} \subset x_\beta$, which immediately gives the required contradiction.

Fix $\alpha \in Y$. Then $\pi_2(\alpha,\beta) \in x_\alpha \cup x_\beta$, since (*). But $\pi_2(\alpha,\beta) \notin x_\alpha$. Otherwise $\beta \in x_\alpha$, since x_α is closed under π_2^{-1}. Contradiction. ∎

Let us summarize the situation below $CF_{\kappa\lambda}$.

normal		non-normal		
weakly normal			non-weakly normal	
$CF_{\kappa\lambda}$	$\supsetneqq$	$\tilde{\Delta} SCF_{\kappa\lambda}$	$\supsetneqq$	$SCF_{\kappa\lambda}$
			$\not\subseteq\ \not\supseteq$	
		$\tilde{\Delta} FSF_{\kappa\lambda}$	$\supsetneqq$	$FSF_{\kappa\lambda}$

4. Case $cf\lambda > \kappa$

Throughout this section, we assume that $cf\lambda > \kappa$.

4.1 Theorem ([3]). *$\tilde{\Delta}^\kappa F$ is the minimal weakly normal filter extending F.*

Proof: It is enough to show weak normality of $\tilde{\Delta}^\kappa F$. Let $\{X_\alpha : \alpha < \lambda\} \subset \tilde{\Delta}^\kappa F = \bigcup_{\beta<\kappa} \tilde{\Delta}^\beta F$ be descending. Pick $\beta < \kappa$ such that $\{X_\alpha : \alpha < \lambda\} \subset \tilde{\Delta}^\beta F$. Then $\Delta_{\alpha<\lambda} X_\alpha \in \tilde{\Delta}^{\beta+1} F \subset \tilde{\Delta}^\kappa F$. ∎

We investigate the minimal α such that $\tilde{\Delta}^\alpha FSF_{\kappa\lambda}$ is weakly normal and show that $\tilde{\Delta}$ may not stop at finite steps in contrast to Δ. We have already known

4.2 Proposition ([8]). *$FSF_{\kappa\lambda}$ is not weakly normal.*

It is easy to see that for any $1 \le n < \omega$ and λ (not necessary $cf\lambda > \kappa$), $X \in \tilde{\Delta}^n FSF_{\kappa\lambda}$ iff
$\{x : \forall\alpha_0 \in x \exists\alpha'_0 \ge \alpha \cdots \forall\alpha_{n-1} \in x \exists\alpha'_{n-1} \ge \alpha_{n-1}\ f(\alpha'_0, \cdots, \alpha'_{n-1}) \subset x\} \subset X \subset P_\kappa\lambda$ for some $f : \lambda^n \to P_\kappa\lambda$. We express this fact formally.

Definition. *Let $y \subset x^n$, where $x \in P_\kappa\lambda$ and $1 \le n < \omega$. It is defined inductively as follows that y is unbounded in x^n; y is unbounded in x^1 iff $\{u(0) : u \in y\}$ is unbounded in x, and y is unbounded in x^{n+1} iff $\{u \mid n : u \in y\}$ is unbounded in x^n and $\{u(n) : u \in y, u \mid n = v \mid n\}$ is unbounded in x for any $v \in y$.*

By induction on n, we have for any λ (not necessary $cf\lambda > \kappa$)

4.3 Proposition. *For any $1 \le n < \omega$, $X \in \tilde{\Delta}^n FSF_{\kappa\lambda}$ iff $\{x : \exists y$ unbounded in $x^n f''y \subset Px\} \subset X \subset P_\kappa\lambda$ for some $f : \lambda^n \to P_\kappa\lambda$.*

4.4 Lemma. *Let $f : \lambda \to P_\kappa\lambda$. Then there exists $z \in P_\kappa\lambda$ such that $\{\gamma : f(\gamma) \cap y \subset z\}$ is stationary in λ for any $y \in P_\kappa\lambda$.*

Proof: Otherwise for any $z \in P_\kappa\lambda$ there exists $y \in P_\kappa\lambda$ such that $\{\gamma : f(\gamma) \cap (y - z) \ne 0\}$ contains club. For each $\xi < \kappa$, define $x_\xi \in P_\kappa\lambda$

inductively such that $C_\xi = \{\gamma : f(\gamma) \cap (x_\xi - \bigcup_{\zeta<\xi} x_\zeta) \neq 0\}$ contains club. Pick $\gamma \in \bigcap_{\xi<\kappa} C_\xi$. Then $f(\gamma) \cap (x_\xi - \bigcup_{\zeta<\xi} x_\zeta) \neq 0$ for any $\xi < \kappa$. Contradiction. ∎

4.5 **Theorem.** *For any $n < \omega$, $\widetilde{\Delta}^n FSF_{\kappa\lambda}$ is not weakly normal.*

Proof: It is enough to show that for any $1 \leq n < \omega$, $\{x : \exists y \text{ unbounded in } x^{n+1}\ \pi''_{n+1} y \subset x\} \notin \widetilde{\Delta}^n FSF_{\kappa\lambda}$. Otherwise

$$(*) \qquad \begin{aligned} &\{x : \exists y \text{ unbounded in } x^n\ f''y \subset Px\} \\ \subset &\{x : \exists z \text{ unbounded in } x^{n+1}\ \pi''_{n+1} z \subset x\} \end{aligned}$$

for some $f : \lambda^n \to P_\kappa\lambda$ such that $f(\alpha_0, \cdots, \alpha_{n-1})$ contains $\{\alpha_0, \cdots, \alpha_{n-1}\}$ and is closed under π_{n+1}^{-1} for any $\alpha_0, \cdots, \alpha_{n-1} < \lambda$.

By 4.4, define $f_m : \lambda^m \to P_\kappa\lambda$ for each $n \geq m \geq 1$ and $f_0 \in P_\kappa\lambda$ inductively by $f_n = f$ and $\{\alpha_{m-1} : f_m(\alpha_0, \cdots, \alpha_{m-1}) \cap y \subset f_{m-1}(\alpha_0, \cdots, \alpha_{m-2})\}$ is stationary for any $\alpha_0, \cdots, \alpha_{m-2} < \lambda$ and $y \in P_\kappa\lambda$.

Set $T = \{t \in \omega^n : \forall i < \forall j < n\ t(i) \leq t(j)\}$ and for each $s, t \in T$, let $s < t$ iff $s(i) < t(i)$, where $i = \max\{j : s(j) \neq t(j)\}$. Then $<$ well-orders T of order type ω. For each $1 \leq m \leq n$, set $T_m = \{t \in T : m \leq \forall i < n\ t(m-1) = t(i)\}$ and $T_0 = 0$. For each $t \in T$ and $1 \leq m \leq n$, let $t^{(m)}$ be the unique $s \in T_m$ with $s|m = t|m$.

For each $t \in T$, define $\gamma_t < \lambda$ inductively by $\sup f_m(\gamma_{s^{(1)}}, \cdots, \gamma_{s^{(m)}}) < \gamma_t$ for any $m \leq n$ and $s < t$, and $f_m(\gamma_{t^{(1)}}, \cdots, \gamma_{t^{(m)}}) \cap \bigcup_{s<t} f_m(\gamma_{s^{(1)}}, \cdots, \gamma_{s^{(m)}}) \subset f_{m-1}(\gamma_{t^{(1)}}, \cdots, \gamma_{t^{(m-1)}})$, where $t \in T_m - T_{m-1}$.

Set $x = \bigcup_{t \in T} f(\gamma_{t^{(1)}}, \cdots, \gamma_{t^{(n)}})$. Then $y = \{(\gamma_{t^{(1)}}, \cdots, \gamma_{t^{(n)}}) : t \in T\}$ is unbounded in x^n, since $\{\gamma_{t^{(m)}} : t \in T, t \mid m = s \mid m\}$ is unbounded in x for any $s \in T$ and $1 \leq m \leq n$. Therefore we can pick z unbounded in x^{n+1} such that for any $u \in z$ there exists $t \in T$ with ran $u \subset f(\gamma_{t^{(1)}}, \cdots, \gamma_{t^{(n)}})$, since $f''y \subset Px$, since $(*)$ and since $f(\gamma_{t^{(1)}}, \cdots, \gamma_{t^{(n)}})$ is closed under π_{n+1}^{-1}.

For each $1 \leq m \leq n+1$, set $z_m = \{u|m : u \in z\}$. We claim that

$$\begin{aligned} \varphi(m) \equiv &\text{ for any } u \in z_m \text{ there exists } t \in T \text{ with} \\ &\text{ran } u \subset f_{m-1}(\gamma_{t^{(1)}}, \cdots, \gamma_{t^{(m-1)}}) \end{aligned}$$

for any $1 \leq m \leq n+1$.

To argue by induction on m, suppose $\varphi(m+1)$. We show $\varphi(m)$. Fix $u \in z_m$ and pick $v \in z_{m+1}$ with $u = v \mid m$. Then we can pick $s \in T$ with ran $v \subset f_m(\gamma_{s^{(1)}}, \cdots, \gamma_{s^{(m)}})$, since $\varphi(m+1)$. Next pick $\bar{s} \in$

$T_m - T_{m-1}$ with $\sup f_m(\gamma_{s(1)}, \cdots, \gamma_{s(m)}) < \gamma_{\bar{s}}$. Then we can pick $t \in T$ with ran $u \cup \{\gamma_{\bar{s}}\} \subset f_m(\gamma_{t(1)}, \cdots, \gamma_{t(m)})$, since $\varphi(m+1)$. Then $s < t$, since $\sup f_m(\gamma_{s(1)}, \cdots, \gamma_{s(m)}) < \gamma_{\bar{s}} \in f_m(\gamma_{t(1)}, \cdots, \gamma_{t(m)})$. Therefore ran $u \subset f_m(\gamma_{s(1)}, \cdots, \gamma_{s(m)}) \cap f_m(\gamma_{t(1)}, \cdots, \gamma_{t(m)}) \subset f_{m-1}(\gamma_{t(1)}, \cdots, \gamma_{t(m-1)})$. We finish the claim.

Thus f_0 is unbounded in x, since z_0 is unbounded in x and since $\varphi(1)$. Contradiction. ∎

We have that $\tilde{\Delta}^2 FSF_{\kappa\lambda} \not\subset \tilde{\Delta} FSF_{\kappa\lambda}$ by putting $n = 1$ in 4.5. Even better, we have

4.6 **Theorem.** $\tilde{\Delta}^2 FSF_{\kappa\lambda} \not\subset SCF_{\kappa\lambda}$.

Proof: It is enough to show that $\{x : \forall\alpha \in x \exists\alpha' \geq \alpha \forall\beta \in x \exists\beta' \geq \beta \ \pi_2(\alpha', \beta') \in x\} \notin SCF_{\kappa\lambda}$. Otherwise

$$(*) \quad \{x : f''x \subset Px\} \subset \{x : \forall\alpha \in x \exists\alpha' \geq \alpha \forall\beta \in x \exists\beta' \geq \beta \ \pi_2(\alpha', \beta') \in x\}$$

for some $f : \lambda \to P_\kappa\lambda$ such that $f(\gamma)$ is closed under π_2^{-1} for any $\gamma < \lambda$.

For each $\gamma < \lambda$, let $x_\gamma \in P_\kappa\lambda$ be the closure of $\{\gamma\}$ under f. By 4.4, we can pick $z \in P_\kappa\lambda$ such that $\{\gamma : x_\gamma \cap y \subset z\}$ is stationary in λ for any $y \in P_\kappa\lambda$.

For each $n < \omega$, define $\gamma_n < \lambda$ inductively by $\sup z < \gamma_0$, $\sup x_{\gamma_m} < \gamma_{n+1}$ for any $m \leq n$ and $x_{\gamma_{n+1}} \cap \bigcup_{m \leq n} x_{\gamma_m} \subset z$.

Set $x = \bigcup_{n<\omega} x_{\gamma_n}$. Then we can pick $\alpha' \geq \gamma_0$ such that for any $\beta \in x$ there exists $\beta' \geq \beta$ and n with $\alpha', \beta' \in x_{\gamma_n}$, since x is closed under f, since $(*)$ and since x_{γ_n} is closed under π_2^{-1}. Fix m with $\alpha' \in x_{\gamma_m}$ and pick $\beta' \geq \gamma_{m+1}$ and n with $\alpha', \beta' \in x_{\gamma_n}$. Then $m < n$, since $\sup x_{\gamma_m} < \gamma_{m+1} \leq \beta' \in x_{\gamma_n}$. Therefore $\sup z < \gamma_0 \leq \alpha' \in x_{\gamma_m} \cap x_{\gamma_n} \subset z$. Contradiction. ∎

Let us present a direct proof of a corollary of 4.6, which have been shown by more complicated argument.

4.7 **Corollary** ([8]). *$SCF_{\kappa\lambda}$ is not weakly normal.*

Proof: For each $\beta < \lambda$, set $X_\beta = \bigcup_{\beta \leq \beta' < \lambda} \{x : \forall\alpha \in x \ \pi_2(\alpha, \beta') \in x\}$. Then $\{X_\beta : \beta < \lambda\} \subset SCF_{\kappa\lambda}$ is descending. We show that $\Delta_{\beta<\lambda} X_\beta \notin SCF_{\kappa\lambda}$. Otherwise

$$(*) \quad \{x : f''x \subset Px\} \subset \Delta_{\beta<\lambda} X_\beta = \{x : \forall\beta \in x \exists\beta' \geq \beta \forall\alpha \in x \ \pi_2(\alpha, \beta') \in x\}$$

for some $f : \lambda \to P_\kappa\lambda$ such that $f(\gamma)$ is closed under π_2^{-1} for any $\gamma < \lambda$.

For each $\gamma < \lambda$, let $x_\gamma \in P_\kappa\lambda$ be the closure of $\{\gamma\}$ under f.

Now pick $\beta < \lambda$ such that $\sup x_\alpha < \beta$ for any $\alpha < \kappa$. We claim that there exists $\{\beta_\alpha : \alpha < \kappa\} \subset \lambda$ with $\{\pi_2(\alpha, \beta_\alpha) : \alpha < \kappa\} \subset x_\beta$, which immediately gives the required contradiction.

Fix $\alpha < \kappa$. Then there exists $\beta_\alpha \geq \beta$ with $\pi_2(\alpha, \beta_\alpha) \in x_\alpha \cup x_\beta$, since $x_\alpha \cup x_\beta$ is closed under f and since $(*)$. But $\pi_2(\alpha, \beta_\alpha) \notin x_\alpha$. Otherwise $\sup x_\alpha < \beta \leq \beta_\alpha \in x_\alpha$, since x_α is closed under π_2^{-1}. Contradiction. ∎

By induction on α, we have

4.8 Proposition. *For any $\alpha \leq \kappa$ and $X \in \tilde{\Delta}^\alpha FSF_{\kappa\lambda}$, there exists $Y \in \tilde{\Delta}^\alpha FSF_{\kappa\lambda}$ such that $Y \subset X$ and for any $x \in Y$ and $y \in P_\kappa\lambda$ such that $x \subset y$ and for any $\beta \in y$ there exists $\gamma \in x$ with $\beta \leq \gamma$, $y \in Y$.*

Note that the following two statements are true but trivial by the results in section 2 and 3, when $cf\lambda \leq \kappa$.

4.9 Theorem. $\tilde{\Delta}^\kappa FSF_{\kappa\lambda} \not\supset SCF_{\kappa\lambda}$.

Proof: Let $\pi : \lambda \to \lambda$ be a bijection such that $\alpha \neq \pi(\alpha) < \kappa$ for any $\alpha < \kappa$. It is enough to show that $\{x : \pi''x \subset x\} \notin \tilde{\Delta}^\kappa FSF_{\kappa\lambda}$. Otherwise there exists $Y \in \tilde{\Delta}^\kappa FSF_{\kappa\lambda}$ such that $Y \subset \{x : \pi''x \subset x\}$ and for any $x \in Y$ and $y \in P_\kappa\lambda$ such that $x \subset y$ and for any $\alpha \in y$ there exists $\beta \in x$ with $\alpha \leq \beta$, $y \in Y$.

Now pick $x \in Y$ with $\kappa \leq \sup x$ and $\alpha < \kappa$ with $\pi(\alpha) \notin x$ and set $y = x \cup \{\alpha\}$. Then $y \in Y$. But y is not closed under π. Contradiction. ∎

4.10 Theorem. *$\tilde{\Delta}SCF_{\kappa\lambda}$ is the minimal weakly normal filter extending $SCF_{\kappa\lambda}$.*

Proof: It is enough to show weak normality of $\tilde{\Delta}SCF_{\kappa\lambda}$, i.e. that for any $f : \lambda^3 \to P_\kappa\lambda$ there exists $g : \lambda^2 \to P_\kappa\lambda$ with

$$\begin{aligned}&\{x : \forall\alpha\in x\exists\alpha'\geq\alpha\forall\beta\in x\ g(\alpha',\beta) \subset x\}\\ \subset&\{x : \forall\alpha\in x\exists\alpha'\geq\alpha\forall\beta\in x\exists\beta'\geq\beta\forall\gamma\in x\ f(\alpha',\beta',\gamma) \subset x\}.\end{aligned}$$

Given $f : \lambda^3 \to P_\kappa\lambda$, define $g : \lambda^2 \to P_\kappa\lambda$ by $g(\alpha, \beta) = f(\alpha, \pi_2^{-1}(\beta)) \cup \{\pi_2(\alpha, \beta)\}$. This g works by the standard argument. ∎

$CF_{\kappa\lambda}$ is also a weakly normal filter extending $SCF_{\kappa\lambda}$, and hence may be equal to $\tilde{\Delta}SCF_{\kappa\lambda}$. But this is not the case for any $\lambda > \kappa^+$.

4.11 Theorem. *For any $\lambda > \kappa^+$, $\tilde{\Delta}SCF_{\kappa\lambda} \subsetneq CF_{\kappa\lambda}$.*

Proof: It is enough to show that $\{x : \pi_2''x^2 \subset x\} \notin \tilde{\Delta}SCF_{\kappa\lambda}$. Otherwise

$$(*)\qquad \{x : \forall\delta\in x\exists\delta'\geq\delta\forall\eta\in x\ f(\delta',\eta) \subset x\} \subset \{x : \pi_2''x^2 \subset x\}$$

for some $f : \lambda \to P_\kappa\lambda$ such that $f(\delta,\eta)$ is closed under π_2^{-1} for any $\delta,\eta < \lambda$.

(Case $cf\lambda > \kappa^+$) For each $n < \omega$, define $X_n \in [\lambda]^{\kappa^+}$ and $\delta_n < \lambda$ inductively by $X_0 = \delta_0 = \kappa^+$, $X_{n+1} = X_n \cup \bigcup f''\{\delta_m : m \leq n\} \times X_n$ and $\delta_{n+1} = \sup X_{n+1}$.

For each $\gamma < \kappa^+$, let $x_\gamma = \bigcup_{n<\omega} x_{\gamma,n}$, where $x_{\gamma,n} \in P_\kappa\lambda$ is defined inductively by $x_{\gamma,0} = \{\gamma\}$ and $x_{\gamma,n+1} = x_{\gamma,n} \cup \bigcup f''\{\delta_m : m \leq n\} \times x_{\gamma,n}$. Then by induction on n, we have that $x_{\gamma,n} \subset X_n$ for any $\gamma < \kappa^+$ and $n < \omega$. We claim that for any $\alpha,\beta < \kappa^+$ and $\delta \in x_\alpha \cup x_\beta$ there exists $\delta' \geq \delta$ such that $f(\delta',\eta) \subset x_\alpha \cup x_\beta$ for any $\eta \in x_\alpha \cup x_\beta$.

Fix $\delta \in x_\alpha$ and m with $\delta \in x_{\alpha,m}$. Then $\delta \leq \sup x_{\alpha,m} \leq \sup X_m = \delta_m$. We show that $f(\delta_m,\eta) \subset x_\alpha \cup x_\beta$ for any $\eta \in x_\alpha \cup x_\beta$.

Fix $\eta \in x_\alpha$ and $n \geq m$ with $\eta \in x_{\alpha,n}$. Then $f(\delta_m,\eta) \subset x_{\alpha,n+1} \subset x_\alpha$. Similarly if $\eta \in x_\beta$, then $f(\delta_m,\eta) \subset x_\beta$. We finish the claim.

Now pick $\beta \in \kappa^+ - \bigcup_{\alpha<\kappa} x_\alpha$. We claim that $\{\pi_2(\alpha,\beta) : \alpha < \kappa\} \subset x_\beta$, which immediately gives the required contradiction.

Fix $\alpha < \kappa$. Then $\pi_2(\alpha,\beta) \in x_\alpha \cup x_\beta$, since $(*)$. But $\pi_2(\alpha,\beta) \notin x_\alpha$. Otherwise $\beta \in x_\alpha$, since x_α is closed under π_2^{-1}. Contradiction.

(Case $cf\lambda = \kappa^+$) For each $\xi < \kappa^+$, define $X_\xi \in [\lambda]^\kappa$ and $\delta_\xi < \lambda$ inductively such that $\{\delta_\xi : \xi < \kappa^+\}$ is unbounded in λ, by $X_0 = \delta_0 = \kappa$, $X_{\xi+1} = X_\xi \cup \bigcup f''\{\delta_\zeta : \zeta \leq \xi\} \times X_\xi$, $\sup X_{\xi+1} \leq \delta_{\xi+1}$, $\delta_\xi < \delta_{\xi+1}$ and $X_\xi = \bigcup_{\zeta<\xi} X_\zeta$, $\delta_\xi = \sup_{\zeta<\xi} \delta_\xi$ when ξ is limit.

Now pick $\beta \in \lambda - \bigcup_{\xi<\kappa^+} X_\xi$. For each $\gamma \in \kappa\cup\{\beta\}$, let $x_\gamma = \bigcup_{n<\omega} x_{\gamma,n}$, where $x_{\gamma,n} \in P_\kappa\lambda$ and $\xi_n < \kappa^+$ are defined inductively by $x_{\gamma,0} = \{\gamma\}$, $\beta \leq \delta_{\xi_0}$, $x_{\gamma,n+1} = x_{\gamma,n} \cup \bigcup f''\{\delta_{\xi_m} : m \leq n\} \times x_{\gamma,n}$, and $\sup x_{\gamma,n+1} \leq \delta_{\xi_{n+1}}$ for any $\gamma \in \kappa\cup\{\beta\}$. We claim that $\{\pi_2(\alpha,\beta) : \alpha < \kappa\} \subset x_\beta$, which immediately gives the required contradiction.

Fix $\alpha < \kappa$. We claim that for any $\delta \in x_\alpha \cup x_\beta$ there exists $\delta' \geq \delta$ such that $f(\delta',\eta) \subset x_\alpha \cup x_\beta$ for any $\eta \in x_\alpha \cup x_\beta$.

Fix $\delta \in x_\alpha$ and m with $\delta \in x_{\alpha,m}$. Then $\delta \leq \sup x_{\alpha,m} \leq \delta_{\xi_m}$. We show that $f(\delta_{\xi_m},\eta) \subset x_\alpha \cup x_\beta$ for any $\eta \in x_\alpha \cup x_\beta$.

Fix $\eta \in x_\alpha$ and $n \geq m$ with $\eta \in x_{\alpha,n}$. Then $f(\delta_{\xi_m},\eta) \subset x_{\alpha,n+1} \subset x_\alpha$. Similarly if $\eta \in x_\beta$, then $f(\delta_{\xi_m},\eta) \subset x_\beta$. We finish the claim.

Therefore $\pi_2(\alpha,\beta) \in x_\alpha \cup x_\beta$, since $(*)$. But $\pi_2(\alpha,\beta) \notin x_\alpha$. Otherwise $\beta \in x_\alpha$, since x_α is closed under π_2^{-1}. Contradiction. ∎

Let us summarize the situation below $CF_{\kappa\lambda}$.

normal		non-normal				
weakly-normal				non-weakly	normal	
$CF_{\kappa\lambda}$	$\supsetneq$	$\tilde{\Delta}SCF_{\kappa\lambda}$	$\supsetneq$	$SCF_{\kappa\lambda}$		
		$\rotatebox{90}{$\subsetneq$}$	$\not\supset$	$\rotatebox{90}{$\not\subset$}$	$\not\supset$	
		$\tilde{\Delta}^{\kappa}FSF_{\kappa\lambda} \cdots$	$\supsetneq$	$\tilde{\Delta}^{2}FSF_{\kappa\lambda}$	$\supsetneq$	$\tilde{\Delta}FSF_{\kappa\lambda}$
					$\subsetneq$	
				$FSF_{\kappa\lambda}$		

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