A Note On Subcontinua of $\beta[0,\infty) - \{0, \infty\}$

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Abstract. Let $M = \bigtimes_{n \in \omega} I_n$ be the topological sum of countably many copies of the unit interval $I$. For any ultrafilter $u \in \omega^*$, we let $M^u = \cap \{cl_{BM}(\{I_n : n \in A\}) : A \in u\}$. It is well-known that $M^u$ is a decomposable continuum with a very nice internal structure (See Mioduszewski[7], Smith[10] and Zhu[11]). In this paper, we show

(1) Every nondegenerate subcontinuum of $\beta[0,\infty) - \{0, \infty\}$ contains a copy of $M^u$ for some $u \in \omega^*$;

(2) There is no non-trivial simple point in Laver's model for Borel conjecture.

The second answers a question posed by Baldwin and Smith[1]

negatively.


Key Words: Stone-Cech remainder, Laver real, continuum.
§0. Introduction. In this paper, we study subcontinua of the Stone-Cech compactification of the reals. We refer to [7] and [11] for background on this topics. Let $M = \bigoplus_{n \in \omega} I_n$ be the topological sum of countably many copies of the unit interval. For any ultrafilter $u \in \omega^*$, we let $M^u = \cap \{cl_{\beta M} \left( \bigcup \{I_n : n \in A \} \right) : A \in u \}$. It is not difficult to prove that $M^u$ is a continuum (See, for example, [4]). If we let $i : M \rightarrow \omega$ be the map defined by $i(r) = n$ for any $r \in I_n$ and $\beta i : \beta M \rightarrow \beta \omega$ be the extension of $i$, it is easy to see that $M^u = \beta i^{-1}(u)$. So every subcontinuum of $\beta M$-M, therefore, every proper subcontinuum of $\beta(0, \omega) - [0, \omega)$, can be embedded into $M^u$ for some $u \in \omega^*$. Moreover, we have

Theorem 1. Every nondegenerate subcontinuum of $\beta(0, \omega) - [0, \omega)$ contains a copy of $M^u$ for some $u \in \omega^*$.

For any map $f \in \omega I$ and $u \in \omega^*$, let $f^u = \{F \subseteq M : F$ is closed and $\{n : f(n) \in F \cap I_n \} \in u \}$ and $P^u = \{f^u : f \in \omega I\}$. It is well known that $f^u$ is a cut point of $M^u$ if $\{n \in \omega : f(n) \neq 0, 1 \} \in u$ ((1) in [7]). It is also well known that there are many indecomposable subcontinua with cardinalities $2^c$ in $M^u$ for any $u \in \omega^*$ ((19) in [7]). Therefore, by our Theorem 1, we have

Corollary. (a) Every subcontinuum of $\beta(0, \omega) - [0, \omega)$ contains an indecomposable subcontinuum;

(b) $\beta(0, \omega)$ does not contain non-degenerate hereditarily indecomposable subcontinuum.
(a) is due to D. P. Bellamy [2]. (b) was proved by M. Smith in [9] (van Douwen also announced it in [3]). The following problem was first posed by van Douwen (See the remarks at the end of [10]).

Question 1. (van Douwen) Is there any cut point of \( M^u \) which is not in \( P^u \)?

Definition 1. A point \( x \in B^M \) is said to be (non-trivial) simple if for any \( F \in x \) there is \( U \in x \) such that \( U \subseteq F \) and \( U \cap I_n = \emptyset \) or \( U \cap I_n \) is a (non-degenerate) interval.

Fact 1. (a) (Corollary in §1 of [11]) If \( x \) is a cut point of \( M^u \) and \( x \in P^u \), then \( x \) is a far point of \( B^M \);

(b) (Theorem 1.1 in [11]) \( x \in M^u \) is a non-trivial simple if and only if \( x \) is a cut point of \( M^u \) and remote point of \( B^M \).

The author [11] proved under \( CH \) that there is \( u \in \omega^* \) such that there is a cut point of \( M^u \) which is not simple. Badlwin and Smith [1] proved that \( MA_{\text{countable}} \) implies that there is a non-trivial simple point. They asked

Question 2. (Baldwin and Smith [1]) Is there any non-trivial simple point in ZFC?

Theorem 2. There is no non-trivial simple point in Laver’s model.
for Borel conjecture.

Question 1 remains open!

§1. Proof of Theorem 1. Let $X=[0,\infty)$ and $K \subset \beta X - X$ be a non-degenerate subcontinuum. The following lemma was proved by M. Smith in [9] for locally compact, locally connected metric spaces. We give a direct proof here.

Lemma 1.1. Let $\{U_0, U_1, \ldots, U_m\}$ be a finite open cover of $K$ in $\beta X$ such that $U_i \cap K \neq \emptyset$ for any $i \leq m$. Then there is a closed interval $H \subset X$ such that $H \cap U_i \neq \emptyset$ for $i \leq m$ and $H \subset \cup \{U_i : i \leq m\}$.

Proof. Let $V = \cup \{U_0, U_1, \ldots, U_m\}$ and $V' = V \cap X$. Then there are disjoint open intervals $\{J_n : n \in \omega\}$ so that $V' = \cup \{J_n : n \in \omega\}$. Let $A_0 = \{n \in \omega : J_n \cap U_0 \neq \emptyset\}$, $V_0 = \cup \{J_n : n \in A_0\}$ and $W_0 = \cup \{J_n : n \in A_0\}$. We have $K \subset \overline{V_0} \cup \overline{W_0}$ and $(\overline{V_0}) \cap (\overline{W_0}) \subset (\overline{V_0}) \cap (\overline{W_0}) = \overline{V_0} \cap \overline{W_0}$, where $\overline{V_0}$ and $\overline{W_0}$ are the closures of $V_0$ and $W_0$ in $X$ respectively. Since $V$ is an open neighbourhood of $K$, we have $K \cap (\overline{V_0} \cap \overline{W_0}) = \emptyset$. Therefore, $K \subset \overline{V_0}$ since $K$ is connected and $K \cap (\overline{V_0}) = K \cap \overline{V_0} = \emptyset$.

If we let $A_i = \{n \in \omega : J_n \cap U_i \neq \emptyset\}$ for $j < i$ and $V_i = \cup \{J_n : n \in A_i\}$ for $i \leq m$, we can easily show by induction that $K \subset \overline{V_i}$ for $i \leq m$. So $A_m = \emptyset$. This completes the proof of Lemma 1.1.

We take $U_0$ and $U_1$ be disjoint open sets of $\beta X$ so that $(\overline{U_0}) \cap (\overline{U_1}) = \emptyset$ and $U_i \cap K \neq \emptyset (i=0,1)$. Let $\emptyset$ be the
collection of closed intervals so that an interval \([a,b]\) belongs to \(\mathcal{I}\) if and only if the following conditions hold:

1. \([a,b]\cap(U_0\cup U_1)=\emptyset\) and \(a\neq b\);
2. \(\{a,b\}\subset\text{Br}(U_0\cap X)\cup\text{Br}(U_1\cap X)\) and \(a\in\text{Br}(U_0\cap X)\) if and only if \(b\in\text{Br}(U_1\cap X)\),

where \(\text{Br}\) denotes the boundary operation in \(X\). Since \(\text{cl}_{\beta X}U_0\) and \(\text{cl}_{\beta X}U_1\) are disjoint, \(\mathcal{I}\) is discrete. We enumerate \(\mathcal{I}\) as \(\{J_n:n\in\omega\}\). We need only to show that there is \(u\in\omega^\ast\) such that \(\bigcap\{\text{cl}_{\beta X}(\bigcup\{J_n:n\in A\})\!:\!A\in\mathcal{U}\}\subseteq K\). Let \(U\) be an open neighbourhood base of \(K\) in \(\beta X\). For \(U\in\mathcal{U}\), we let \(A_U=\{n\in\omega:J_n\subset U\}\).

By Lemma 1.1, we have \(A_U\neq\emptyset\) for \(U\in\mathcal{U}\). Since \(A_U\subset A_V\) for \(U\subset V\) and \(U, V\in\mathcal{U}\), \(\{A_U:U\in\mathcal{U}\}\) has finite intersection property. Let \(M_{\mathcal{U}}=\bigcap\{\text{cl}_{\beta X}(\bigcup\{J_n:n\in A_U\})\!:\!U\in\mathcal{U}\}\).

Then \(M_{\mathcal{U}}\subseteq K\). For, if \(x\in M_{\mathcal{U}}\setminus K\), there is \(U\in\mathcal{U}\) such that \(x\in\text{cl}_{\beta X}U\).

But \(x\in M_{\mathcal{U}}\subseteq\text{cl}_{\beta X}U\). Note that \(\text{cl}_{\beta X}(\bigcup\{J_n:i\leq n\})\cap K=\emptyset\) for \(n\in\omega\). So if \(u\) is an ultrafilter on \(\omega\) and \(\{A_U:U\in\mathcal{U}\}\subset u\), then \(u\in\omega^\ast\) and

\(\bigcap\{\text{cl}_{\beta X}(\bigcup\{J_n:n\in A\})\!:\!A\in\mathcal{U}\}\subseteq K\).

This completes the proof of our Theorem 1.

§2. Proof of Theorem 2. Recall that there is a natural partial order \(<_u\) on \(M^u\) for \(u\in\omega^\ast\) defined as follows: \(x<_uy\) if and only if there are \(F\in x\) and \(H\in y\) such that \(\{n\in\omega:F\cap I_n<H\cap I_n\}\in u\), where \(F\cap I_n<H\cap I_n\) means that \(r<s\) for any \(r\in F\cap I_n\) and \(s\in H\cap I_n\).

It is easily seen that \((F^u,<_u)\) is isomorphic to the ultrapower \((\omega I/u,<_u)\). We consider the relation \(\sim\) on \(M^u\) defined by \(x\sim y\) if and only if \(x\sim y\) or \(x\sim y\) and \(y\sim x\). It is very easy to
verify that \( \sim \) is an equivalence relation. A \( \sim \) equivalence class i.e., a maximal pairwise incomparable subset of \((M^u, <_u)\), is called a layer (this definition of layers is equivalent to Mioduszewski's original one in [7], see Lemma 1.2 in [11]). It can be proved easily from Mioduszewski's [7] that if \( x \) is a cut point of \( M^u \), \( \{x\} \) is a layer (Lemma 1.3 in [11]). For any \( Ac^{\omega}I \) and \( u \in \omega^* \), we let \( A^u = \{ f^u \in P^u : f \in A \} \). We say a pair \( \mathcal{E} = (A, B) \) of subsets of \( \omega^I \) determines a layer \( L \) in \( M^u \) for some \( u \in \omega^* \) if the following two conditions hold:

1. \( A^u < _u B^u \), i.e., \( f^u < _u g^u \) for any \( f \in A \) and \( g \in B \);
2. for any \( x \in M^u \), \( x \in L \) if and only if \( f^u < _u x < _u g^u \) for any \( f \in A \) and \( g \in B \).

If \( L = \{x\} \) is a one point layer, we also say that \( x \) is determined by \( \mathcal{E} \). Note that every layer is determined by a pair of subsets of \( \omega^I \) (See [11], where we say layers are determined by gaps in \((\omega^I \setminus u, <_u)\)).

Let \( \mathcal{P} = \bigcup _n \mathcal{P}_n \) be a collection of closed rational sub-intervals of the unit interval \( I \) such that \( \mathcal{P}_n \) is finite, pairwise disjoint and for any interval \( J \subset I \), if the length of \( J \) is larger than \( 1/n \), then \( |\{ \mathcal{H} \in \mathcal{P}_n : \mathcal{H} \subset J \}| > n \). The following lemma is essentially Proposition 3.1 in [11].

**Lemma 2.1.** Let \( \mathcal{E} = (A, B) \) be a pair of of subsets of \( \omega^I \) and \( A^u < _u B^u \) for some \( u \in \omega^* \). \( \mathcal{E} \) determines a one point layer in \( M^u \) if and only if for any \( h \in \omega \), there are \( f \in A \) and \( g \in B \) such that

\( \{ n \in \omega : \text{there is at most one } J \in \mathcal{P}_h(n) \text{ such that } J \subset \{ f(n), g(n) \} \} \in u. \)
By Lemma 2.1, we easily get

Lemma 2.2. Let $\mathbb{N} \subset \mathbb{M}$ be models of ZFC such that there is $r \in \omega^\omega \mathbb{N}$ dominating every $h \in \omega^\omega \mathbb{N}$ i.e., $h(n) < r(n)$ for all but finitely many $n \in \omega$. Then no one point layer in $\mathbb{M}$ is determined by a pair of subsets of $\omega^1$ in $\mathbb{M}$.

Let $P_{\omega_2}$ be the $\omega_2$ iteration of Laver forcing with countable support and $C_{\omega_2} \mathbb{P}_{\omega_2}$-generic over $V$. We assume that the continuum hypothesis holds in $V$. It is well-known that Laver real dominates every real in the ground model. Therefore, by Lemma 5.10 in [8] and Lemma 11 in [5], we have

Corollary 2.1. There is no cut point in $\mathbb{M}^u$ determined by a pair of subsets of $\omega^1$ with cardinalities $\omega_1$ in $V[C_{\omega_2}]$ for any $u \in \omega^*$. 

The following lemma can be proved by modifying Miller's argument for Mathias forcing in §6 [8].

Lemma 2.3. Suppose that $p \mathbb{P}_{\omega_2} "f: \omega \rightarrow I"$. There are an extension $q$ of $p$ and a sequence $\{c_n : n \in \omega\}$ of codes for closed nowhere dense set in $V$ such that $q \mathbb{P}_{\omega_2} "f(n) \text{ belongs to the set coded by } c_n \text{ for } n \in \omega"$.
Since every non-trivial simple point is a remote point of $\mathfrak{M}$, we can easily see

Corollary 2.2. Let $x \in M^U$ be a non-trivial simple point and $\mathcal{E} = (A, B)$ a pair subsets of $\omega I$ determining $x$. Then in $V[G_{\omega_2}]$, for any $u' \in \omega^*$ and $ucu'$, there is no $f \in \omega I$ such that $[A]_u, [f] < [B]_u$, in $(\omega I/u', <_u)$.

Now we are in a position to complete the proof of Theorem 2. Suppose that there is a non-trivial point $x \in M^U$ in $V[G_{\omega_2}]$. Then there is a pair $\mathcal{E} = (A, B)$ of subsets of $\omega I$ determining $x$. By Lemma 5.10 in [8], there is $\alpha < \omega^*$ such that in $V[G_{\alpha}]$, $x'$ is a non-trivial point of $M^{U'}$ and $\mathcal{E}' = (A', B')$ determines $x'$, where $x' = x \cap V[G_{\alpha}]$, $u' = u \cap V[G_{\alpha}]$, $A' = A \cap V[G_{\alpha}]$ and $B' = B \cap V[G_{\alpha}]$. By Lemma 11 in [5] and Corollary 2.2, $\mathcal{E}'$ determines $x$ in $V[G_{\omega_2}]$. This is impossible by Lemma 2.2.
References


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