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A Note On Subcontinua of $\beta\{0,\infty\}\setminus\{0,\infty\}$

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Abstract. Let $M=\bigoplus_{n\in\omega}I_n$ be the topological sum of countably many copies of the unit interval $I$. For any ultrafilter $u\in\omega^*$, we let $M^u=\cap\{\text{cl}_{\beta M}(\cup\{I_n:n\in A\}):A\in u\}$. It is well-known that $M^u$ is a decomposable continuum with a very nice internal structure (See Mioduszewski[7], Smith[10] and Zhu[11]). In this paper, we show

(1) Every nondegenerate subcontinuum of $\beta\{0,\infty\}\setminus\{0,\infty\}$ contains a copy of $M^u$ for some $u\in\omega^*$;

(2) There is no non-trivial simple point in Laver's model for Borel conjecture.

The second answers a question posed by Baldwin and Smith[1]

negatively.


Key Words: Stone-Cech remainder, Laver real, continuum.
§0. Introduction. In this paper, we study subcontinua of the Stone-Čech compactification of the reals. We refer to [7] and [11] for background on this topics. Let $M = \bigoplus_{n \in \omega} I_n$ be the topological sum of countably many copies of the unit interval. For any ultrafilter $u \in \omega^*$, we let $M^u = \bigcap \{cl_{\beta M}(\bigcup \{I_n : n \in A\}) : A \in u\}$. It is not difficult to prove that $M^u$ is a continuum (see, for example, [4]). If we let $i : M \rightarrow \omega$ be the map defined by $i(r) = n$ for any $r \in I_n$ and $\beta i : \beta M \rightarrow \beta \omega$ be the extension of $i$, it is easy to see that $M^u = \beta i^{-1}(u)$. So every subcontinuum of $\beta M - M$, therefore, every proper subcontinuum of $\beta[0, \omega) - [0, \omega)$, can be embedded into $M^u$ for some $u \in \omega^*$. Moreover, we have

Theorem 1. Every nondegenerate subcontinuum of $\beta[0, \omega) - [0, \omega)$ contains a copy of $M^u$ for some $u \in \omega^*$.

For any map $f \in \omega I$ and $u \in \omega^*$, let $f^u = \{F \subseteq M : F$ is closed and $\{n : f(n) \in F \cap I_n\} \in u\}$ and $P^u = \{f^u : f \in \omega I\}$. It is well known that $f^u$ is a cut point of $M^u$ if $\{n \in \omega : f(n) \neq 0, 1\} \in u$ ((1) in [7]). It is also well known that there are many indecomposable subcontinua with cardinalities $2^C$ in $M^u$ for any $u \in \omega^*$ ((19) in [7]). Therefore, by our Theorem 1, we have

Corollary. (a) Every subcontinuum of $\beta[0, \omega) - [0, \omega)$ contains an indecomposable subcontinuum;

(b) $\beta[0, \omega)$ does not contain non-degenerate hereditarily indecomposable subcontinuum.
(a) is due to D. P. Bellamy [2]. (b) was proved by M. Smith in [9] (van Douwen also announced it in [3]). The following problem was first posed by van Douwen (See the remarks at the end of [10]).

Question 1. (van Douwen) Is there any cut point of $M^U$ which is not in $p^U$?

Definition 1. A point $x \in \beta M$ is said to be (non-trivial) simple if for any $F \in x$ there is $U \in x$ such that $U \cap F$ and $U \cap I_n = \emptyset$ or $U \cap I_n$ is a (non-degenerate) interval.

Fact 1. (a) (Corollary in §1 of [11]) If $x$ is a cut point of $M^U$ and $x \in p^U$, then $x$ is a far point of $\beta M$;

(b) (Theorem 1.1 in [11]) $x \in M^U$ is a non-trivial simple if and only if $x$ is a cut point of $M^U$ and remote point of $\beta M$.

The author [11] proved under CH that there is $u \in \omega^*$ such that there is a cut point of $M^U$ which is not simple. Badlwin and Smith [1] proved that $\text{MA}_{\text{countable}}$ implies that there is a non-trivial simple point. They asked

Question 2. (Baldwin and Smith [1]) Is there any non-trivial simple point in ZFC?

Theorem 2. There is no non-trivial simple point in Laver's model.
for Borel conjecture.

Question 1 remains open!

§1. Proof of Theorem 1. Let $X = (0, \infty)$ and $K \subset \beta X - X$ be a non-degenerate subcontinuum. The following lemma was proved by M. Smith in [9] for locally compact, locally connected metric spaces. We give a direct proof here.

Lemma 1.1. Let $\{U_0, U_1, \ldots, U_m\}$ be a finite open cover of $K$ in $\beta X$ such that $U_i \cap K \neq \emptyset$ for any $i \leq m$. Then there is a closed interval $H \subset X$ such that $H \cap U_i \neq \emptyset$ for $i \leq m$ and $H \subset \bigcup \{U_i : i \leq m\}$.

Proof. Let $V = \bigcup \{U_0, U_1, \ldots, U_m\}$ and $V' = V \cap X$. Then there are disjoint open intervals $\{J_n : n \in \omega\}$ so that $V' = \bigcup \{J_n : n \in \omega\}$. Let $A_0 = \{n \in \omega : J_n \cap U_0 \neq \emptyset\}$, $V_0 = \bigcup \{J_n : n \in A_0\}$ and $W_0 = \bigcup \{J_n : n \in A_0\}$. We have $K \subset (\text{cl}_{\beta X} V_0) \cup (\text{cl}_{\beta X} W_0)$ and $(\text{cl}_{\beta X} V_0) \cap (\text{cl}_{\beta X} W_0) \subset (\text{cl}_{\beta X} V_0) \cap (\text{cl}_{\beta X} W_0) = \text{cl}_{\beta X} (\bar{V}_0 \cap \bar{W}_0)$, where $\bar{V}_0$ and $\bar{W}_0$ are the closures of $V_0$ and $W_0$ in $X$ respectively. Since $V$ is an open neighbourhood of $K$, we have $K \cap (\text{cl}_{\beta X} (\bar{V}_0 \cap \bar{W}_0)) = \emptyset$. Therefore, $K \subset \text{cl}_{\beta X} V_0$ since $K$ is connected and $K \cap (\text{cl}_{\beta X} V_0) \supset K \cap U_0 = \emptyset$.

If we let $A_j = \{n \in \omega : J_n \cap U_j \neq \emptyset\}$ for $j \leq i$ and $V_i = \bigcup \{J_n : n \in A_i\}$ for $i \leq m$, we can easily show by induction that $K \subset \text{cl}_{\beta X} V_i$ for $i \leq m$. So $A_m \neq \emptyset$. This completes the proof of Lemma 1.1.

We take $U_0$ and $U_1$ be disjoint open sets of $\beta X$ so that $(\text{cl}_{\beta X} U_0) \cap (\text{cl}_{\beta X} U_1) = \emptyset$ and $U_i \cap K \neq \emptyset (i = 0, 1)$. Let $\emptyset$ be the
collection of closed intervals so that an interval \([a, b]\) belongs to \(\mathcal{F}\) if and only if the following conditions hold:

1. \([a, b] \cap (U_0 \cup U_1) = \emptyset\) and \(a \neq b\);
2. \([a, b] \subset \text{Br}(U_0 \cap X) \cup \text{Br}(U_1 \cap X)\) and \(a \in \text{Br}(U_0 \cap X)\) if and only if \(b \in \text{Br}(U_1 \cap X)\),

where \(\text{Br}\) denotes the boundary operation in \(X\). Since \(\text{cl}_X U_0\) and \(\text{cl}_X U_1\) are disjoint, \(\mathcal{F}\) is discrete. We enumerate \(\mathcal{F}\) as \(\{J_n : n \in \omega\}\). We need only to show that there is \(u \in \omega^*\) such that

\[\bigcap \{\text{cl}_X (\cup \{J_n : n \in A\} : A \in \mathcal{U})\} \subseteq K.\]

Let \(\mathcal{U}\) be an open neighbourhood base of \(K\) in \(X\). For \(U \in \mathcal{U}\), we let

\[A_U = \{n \in \omega : J_n \subset U\}.\]

By Lemma 1.1, we have \(A_U \neq \emptyset\) for \(U \in \mathcal{U}\). Since \(A_U \cap A_V\) for \(U \cap V\) and \(U, V \in \mathcal{U}\), \(\{A_U : U \in \mathcal{U}\}\) has finite intersection property. Let

\[M_\mathcal{U} = \bigcap \{\text{cl}_X (\cup \{J_n : n \in A_U\}) : U \in \mathcal{U}\}.\]

Then \(M_\mathcal{U} \subseteq K\). For, if \(x \in M_\mathcal{U} \setminus K\), there is \(U \in \mathcal{U}\) such that \(x \in \text{cl}_X U\).

But \(x \in M_\mathcal{U} \subseteq \text{cl}_X U\). Note that \(\text{cl}_X (\cup \{J_i : i \leq n\}) \cap K = \emptyset\) for \(n \in \omega\). So if \(u\) is an ultrafilter on \(\omega\) and \(\{A_U : U \in \mathcal{U}\} \subseteq u\), then \(u \in \omega^*\) and

\[\bigcap \{\text{cl}_X (\cup \{J_n : n \in A\}) : A \in u\} \subseteq K.\]

This completes the proof of our Theorem 1.

§2. Proof of Theorem 2. Recall that there is a natural partial order \(<_u\) on \(M^u\) for \(u \in \omega^*\) defined as follows: \(x <_u y\) if and only if there are \(F \in x\) and \(H \in y\) such that \(\{n \in \omega : F \cap I_n < H \cap I_n\} \in u\),

where \(F \cap I_n < H \cap I_n\) means that \(r < s\) for any \(r \in F \cap I_n\) and \(s \in H \cap I_n\).

It is easily seen that \((F^u, <_u)\) is isomorphic to the ultrapower \((\omega I/u, <_u)\). We consider the relation \(\sim\) on \(M^u\) defined by \(x \sim y\) if and only if \(x = y\) or \(x < y\) and \(y < x\). It is very easy to
verify that \( \sim \) is an equivalence relation. A \( \sim \) equivalence class i.e., a maximal pairwise incomparable subset of \( (\mathbb{M}_u^u, <_u) \), is called a layer (this definition of layers is equivalent to Mioduszewski's original one in [7], see Lemma 1.2 in [11]). It can be proved easily from Mioduszewski's [7] that if \( x \) is a cut point of \( \mathbb{M}_u^u \), \( \{x\} \) is a layer (Lemma 1.3 in [11]). For any \( A \subseteq \omega \) and \( u \in \omega^* \), we let \( A^u = \{f^u \in P^u : f \in A\} \). We say a pair \( \varepsilon = (A, B) \) of subsets of \( \omega^\omega \) determines a layer \( L \) in \( \mathbb{M}_u^u \) for some \( u \in \omega^* \) if the following two conditions hold:

1. \( A^u <_u B^u \), i.e., \( f^u <_u g^u \) for any \( f \in A \) and \( g \in B \);
2. for any \( x \in \mathbb{M}_u^u \), \( x \in L \) if and only if \( f^u <_u x <_u g^u \) for any \( f \in A \) and \( g \in B \).

If \( L = \{x\} \) is a one point layer, we also say that \( x \) is determined by \( \varepsilon \). Note that every layer is determined by a pair of subsets of \( \omega^\omega \) (See [11], where we say layers are determined by gaps in \( (\omega^\omega \setminus u, <_u) \)).

Let \( \mathcal{P} = \bigcup_{n \in \omega} \mathcal{P}_n \) be a collection of closed rational subintervals of the unit interval \( I \) such that \( \mathcal{P}_n \) is finite, pairwise disjoint and for any interval \( J \subseteq I \), if the length of \( J \) is larger than \( 1/n \), then \( |\{H \in \mathcal{P}_n : H \subseteq J\}| > n \). The following lemma is essentially Proposition 3.1 in [11].

**Lemma 2.1.** Let \( \varepsilon = (A, B) \) be a pair of subsets of \( \omega^\omega \) and \( A^u <_u B^u \) for some \( u \in \omega^* \). \( \varepsilon \) determines a one point layer in \( \mathbb{M}_u^u \) if and only if for any \( h \in \omega^\omega \), there are \( f \in A \) and \( g \in B \) such that

\[
\{n \in \omega : \text{there is at most one } J \in \mathcal{P}_h(n) \text{ with } J \subseteq \{f(n), g(n)\}\} \in u.
\]
By Lemma 2.1, we easily get

Lemma 2.2. Let $\mathbb{M} \subseteq \mathbb{N}$ be models of ZFC such that there is $r \in\omega_1 \mathbb{M}$ dominating every $h \in\omega \mathbb{M}$, i.e., $h(n) < r(n)$ for all but finitely many $n \in \omega$. Then no one point layer in $\mathbb{M}$ is determined by a pair of subsets of $\omega_1$ in $\mathbb{M}$.

Let $\mathbb{P}_{\omega_2}$ be the $\omega_2$ iteration of Laver forcing with countable support and $\mathcal{G}_{\omega_2}$ generic over $V$. We assume that the continuum hypothesis holds in $V$. It is well-known that Laver real dominates every real in the ground model. Therefore, by Lemma 5.10 in [8] and Lemma 11 in [5], we have

Corollary 2.1. There is no cut point in $\mathbb{M}^u$ determined by a pair of subsets of $\omega_1$ with cardinalities $\omega_1$ in $V[\mathcal{G}_{\omega_2}]$ for any $u \in \omega^*$. 

The following lemma can be proved by modifying Miller's argument for Mathias forcing in §6 [8].

Lemma 2.3. Suppose that $p \Vdash_{\omega_2} "f: \omega \rightarrow I"$. There are an extension $q$ of $p$ and a sequence $\{c_n : n \in \omega\}$ of codes for closed nowhere dense set in $V$ such that $q \Vdash_{\omega_2} "f(n) \text{ belongs to the set coded by } c_n \text{ for } n \in \omega"$. 

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Since every non-trivial simple point is a remote point of $\mathcal{B}M$, we can easily see

Corollary 2.2. Let $x \in M^U$ be a non-trivial simple point and $\mathcal{C} = (A, B)$ a pair subsets of $\omega I$ determining $x$. Then in $V[G_{\omega_2}]$, for any $u' \in \omega^*$ and $ucu'$, there is no $f \in \omega I$ such that $[A]_u,<[f]<[B]_u$, in $(\omega I/u',<_u')$.

Now we are in a position to complete the proof of Theorem 2. Suppose that there is a non-trivial point $x \in M^U$ in $V[G_{\omega_2}]$. Then there is a pair $\mathcal{C} = (A, B)$ of subsets of $\omega I$ determining $x$. By Lemma 5.10 in [8], there is $\alpha < \omega_2$ such that in $V[G_{\alpha}]$, $x'$ is a non-trivial point of $M^{U'}$ and $\mathcal{C}' = (A', B')$ determines $x'$, where $x' = x \cap V[G_{\alpha}]$, $u' = u \cap V[G_{\alpha}]$, $A' = A \cap V[G_{\alpha}]$ and $B' = B \cap V[G_{\alpha}]$. By Lemma 11 in [5] and Corollary 2.2, $\mathcal{C}'$ determines $x$ in $V[G_{\omega_2}]$. This is impossible by Lemma 2.2.
References


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