<table>
<thead>
<tr>
<th>Title</th>
<th>Hypergeometric functions and modular embeddings (Special Differential Equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Wolfart, J.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1991), 773: 96-105</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82394">http://hdl.handle.net/2433/82394</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Hypergeometric functions and modular embeddings

J. Wolfart, Frankfurt a.M.

I. Discontinuous groups acting on irreducible complex symmetric domains of dim > 1 with finite covolume are arithmetically defined with the possible exception of groups on the complex ball \( B_N \)

\[ 12_1^1 + \ldots + 12_N^1 < 12_0^1 \]

(Conjecture of Selberg, proven by Margolinsi and Selberg)

Morton: Examples of non-arithmetic groups acting on \( B_2 \) and \( B_3 \)

History: Picard 1885
Terada 1973/83

Deligne - Morton and Morton 1986
Hirzebruch - Höfer - Yoshida 1983 - 87
Sauter 1990

"Picard - Terada - Morton - Deligne " groups
PTM D - groups \( \Delta \)
Construction of $\Delta$ as monodromy groups of the Appell - Lauricella - functions

From now on $N = 2$

$\mu_0, \mu_1, \ldots, \mu_4 \in \mathbb{Q} \cap \mathbb{N}, \mathbb{R}$, $\mu_0 + \ldots + \mu_4 = 2$

$$F_4(x, y) := \sum_{\omega} \frac{\prod_{i=1}^{\infty} u^{-\mu_0(x-i)} \prod_{i=0}^{\infty} (u-x)^{-\mu_2} (u-y)^{-\mu_3} du}{\omega}$$

solution of a system of linear PDE's holomorphic outside the "characteristic surfaces"

$x = y$ and $x, y = 0, 1, \infty.$

$Q := \mathbb{C}^2 - \{ \text{characteristic surfaces} \}$

fundamental solutions e.g. $\frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial y}$

Integration paths avoiding other singularities of $\omega$, can be shown as cycles on the Riemann surface of $\omega$

"Pochhammer cycles"

$\Delta$ can be calculated moving integration paths

[Felix Klein .... Yoshida] $\Rightarrow \Delta$ is induced by some automorphism group of $H_4$ of the Riemann surface of $\omega$. 

2
**Theorem (P-T-H-D):**

\[ Q \rightarrow \mathbb{P}^2(C) : (x, y) \mapsto (\bar{z}_0, \bar{z}_1, \bar{z}_2) \]

defines a \( \mathbb{P} \mathcal{L}_3(C) \) - multivalent, locally biholomorphic map \( \psi \) onto a dense subset of a complex ball \( B \cong B_2 \). The non-uniqueness of \( \psi \) is described by the action of \( \Delta \) on \( B \). This action is discontinuous if e.g.

\[
(1 - \mu_i - \mu_j)^{-1} \in \begin{cases} \frac{1}{2} \mathbb{Z} \cup \{\infty\} & \text{if } \mu_i = \mu_j, \\ \mathbb{Z} \cup \{\infty\} & \text{otherwise} \end{cases} \quad i, j \in \{0, 1, 2\}
\]

In the second case, \( \psi \) is the inverse of the canonical projection \( B \rightarrow \mathbb{D}^B \).

---

**II. Main result (P.Cohen, J.W.)** For any PTHD group \( \Delta \) there is an arithmetic group \( \Gamma \) acting on a power \( B^m \) of the ball and a "modular embedding" consisting of two compatible injections

\[
h : \Delta \hookrightarrow \Gamma \quad (\text{group homomorphism})
\]
\[
F : B \hookrightarrow B^m \quad (\text{analytic})
\]

with \( F(\gamma x) = h(\gamma) F(x) \) for all \( x \in B \) and \( \gamma \in \Delta \).

\( F \) induces a morphism of algebraic varieties

\[
\overline{F} : \mathbb{D}^B \rightarrow \mathbb{G} \quad (\text{compactified if necessary})
\]

defined over \( \overline{\mathbb{Q}} \).
III. Elements of the proof

Easy part: Construction of \( \Delta \)

\[ d := d e n (\mu_0, \ldots, \mu_4) \Rightarrow \Delta \subset \text{PSU}(2,1) \oplus \mathbb{Z}[L_d] \]

By direct calculation of generators, \( L_d := \exp \frac{2\pi i}{d} \).

Often \( \Gamma = \text{PSU}(2,1) \oplus \mathbb{Z}[L_d] \), \( d = \text{id} \).

E.g., in the example:

\[ \mu_0 = \frac{\pi}{12}, \mu_1 = \frac{5}{12}, \mu_2 = \frac{6}{12}, \mu_3 = \mu_4 = \frac{3}{12} \]

\( \Gamma \) acts on \( B^2 = B \times B \) discontinuously by:

\[ (\tau_1, \tau_2) \rightarrow (\gamma \tau_1, \gamma^5 \tau_2) \]

where \( \gamma^5 : \tau_{12} \rightarrow \tau_{12} \)

How to construct \( F \)?

Scheme:

Digression to the easiest case \( N = 1 \) of triangle groups \( \Delta \).

Example: Signature \([5, \infty, \infty]\)

\[ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \]

Construction of \( f \)

by Riemann theorem and Schwarz' reflection principle

\[ \begin{pmatrix} 1 & 0 \\ -3 + \frac{\sqrt{5}}{2} & 1 \end{pmatrix} \]

\[ \text{Wanted: A modular embedding } f : \mathbb{Q}_5 \rightarrow \mathbb{Q}_5 \times \mathbb{Q}_5 \]

\( f(\tau) = (\tau, f(\tau)) \) with

\[ f : \mathbb{Q}_5 \rightarrow \mathbb{Q}_5 \text{ disclo and } \gamma \mapsto \gamma^5 \text{ induced by } \sqrt{5} \mapsto -\sqrt{5} \]

\[ \gamma \mapsto \gamma^5 \text{ induced by } \sqrt{5} \mapsto -\sqrt{5} \]

\[ \begin{pmatrix} 1 & 0 \\ -3 + \frac{\sqrt{5}}{2} & 1 \end{pmatrix} \]
using triangle functions \( f = D_3 \circ D_1^{-1} \)

or (projectively, neglecting constants and
\( \text{PSL}_2 \)-transformations)

\[
\left( \frac{\omega}{0}, \frac{\omega}{0} \right) \mapsto \left( \frac{\omega}{0}, \frac{\omega}{0}; \frac{\omega_3}{0}, \frac{\omega_3}{0} \right)
\]

where

\[
\omega = u^{-\frac{3}{5}} \left( u-1 \right)^{-\frac{3}{5}} \left( u-x \right)^{-\frac{7}{5}} du = \frac{du}{w^3}
\]

on the curve \( w^5 = u^3 \left( u-1 \right)^3 \left( u-x \right)^2 \)

\( \omega_3 = \ldots = \frac{u(u-1)(u-x)}{w^3} du \) on the same curve

Digression: Number-theoretic motivation.

There are generating \( \Delta \)-automorphic functions
with Taylor expansions

\[
j(\tau) = \sum_{n \geq 0} c_n \tau^n \left( \frac{\tau-\tau_0}{\tau-\overline{\tau}_0} \right)^n, \text{ all } c_n \in \mathbb{Q}
\]

and \( \tau = \frac{B(\frac{3}{5}, \frac{2}{5})}{B(\frac{3}{5}, \frac{3}{5})} \).

The same constants play the same role for Hilbert modular functions at the corresponding fixed point of \( \text{PSL}_2 \mathbb{Q}_{15} \). Why beta-values?

End of the digression, back to the \( N=2 \) example: For \( F : B \rightarrow B \times B \) take

\[
\left( \frac{\omega}{0}, \frac{\omega}{0}, \frac{\omega}{0} \right) \mapsto \left( \frac{\omega}{0}, \frac{\omega}{0}, \frac{\omega}{0}; \frac{\omega_5}{0}, \frac{\omega_5}{0}, \frac{\omega_5}{0} \right)
\]

with differentials.
\[ \omega = u^{-\frac{3}{4}2} (u-1)^{-\frac{5}{4}2} (u-x)^{-\frac{3}{4}2} (u-y)^{-\frac{3}{4}2} \, du \]

on the curve \( X_5(x,y) \) given by

\[ w^{12} = u^7 (u-1)^5 (u-x)^6 (u-y)^3 \]

and

\[ w_5 = u^{-\frac{3}{4}2} (u-1)^{-\frac{3}{4}2} (u-x)^{-\frac{3}{4}2} (u-y)^{-\frac{3}{4}2} \]

\[ \frac{u^2(u-1)^2(u-x)^2(u-y)}{w_5^5} \, du \quad \text{on the same curve.} \]

---

IV. Principles behind this construction.

Let \( X(x,y) \) a non-singular projective model of \( X_5(x,y_1) \),
Jac \( X(x,y) \) its Jacobian, \( w_4 \) and \( w_3 \) morphisms of \( \text{Jac } X(x,y) \) on other Jac's induced by

\[ X_5(x,y_1) \quad \rightarrow \quad w^4 = u^7 (u-1)^5 \ldots \]

\[ X_5(x,y_1) \quad \rightarrow \quad w^6 = u^7 (u-1)^5 \ldots \]

and \( T(x,y) := \text{connected component of } 0 \) of \( \text{Ker } w_4 \cap \text{Ker } w_3 \)

\( T(x,y) \) is a pp abelian variety of dimension 6

\( (\frac{3}{2} q(d) \quad \text{d} = 12) \):

\[ \chi : X_5(x,y) \rightarrow X_5(x,y) : (u,w) \mapsto (u, b_{12}^{-1} w) \]

induces \( \mathbb{Z}[b_{12}] \subset \text{End } T(x,y) \).

\( H^0(T(x,y), \Omega) \) splits into \( \chi \)-Eigenpaces

\[ V_n := \{ \omega \text{ (first kind)} \mid c_{12}\omega = b_{12}^n \cdot \omega \} \]

with \( n \in (\mathbb{Z}/12\mathbb{Z})^* \). The dimensions \( r_n = \dim V_n \)
can be calculated by an old theorem of Chevalley and Weil:

$$\tau_n = -1 + \sum_{i=0}^{4} \langle m_i \rangle$$

where $\langle \alpha \rangle$ denotes the fractional part $\alpha - [\alpha]$ of $\alpha \in \mathbb{R}$. In our example

$$\tau_4 = \tau_5 = 1 \quad \tau_{-4} = \tau_{-5} = 2$$

(always $\tau_m + \tau_{-m} = 3$, so $\dim T(x,y) = \frac{3}{2} \chi(d)$)

$\omega$ and $\omega_5$ generate $V_4$ and $V_5$

(if $\dim V_m = 1$, it has a generator on $X_3(x,y)$

$$u - \langle m_0 \rangle (u-1) - \langle m_1 \rangle (u-x) - \langle m_2 \rangle (u-y) - \langle m_3 \rangle du$$

$T(x,y)$ belongs to a family of $p.p.$ abelian varieties with "generalized complex multiplication" by $\mathbb{Q}(\sqrt{12})$ and "type"

$$\sum \tau_m \sigma_m = 1 \cdot \sigma_4 + 1 \cdot \sigma_5 + 2 \sigma_{-5} + 2 \sigma_{-4}$$

[Siegel / Shimura]: This family is parametrized by $B^m$,

$$m = \# V_m$$

of dimension 1, i.e. $m=2$

in our case, its coordinates are given by

$$\begin{align*}
\{(w_1, w_2, w_3, w_4, w_5, y_0, y_1, y_2) \mid \psi(x,y) \in B \} \\
\{(w_5, y_0, y_1, y_2) \mid \psi_5(x,y) \in B \}
\end{align*}$$

(neglecting linear transformations) where

$\omega_1 = \omega$ and $\omega_5$ generate the $\dim - 1$ - eigen-space of $H^0(\cdot, \omega)$

7
and $\gamma_0, \gamma_1, \gamma_2$ generate the cycles of the abelian variety as $\mathbb{Z}[\Gamma_4]$ - module. So

$$F: \psi(x,y) \mapsto (\psi(x,y), \psi_5^4 \psi(x,y)), \quad \text{at least in } \psi Q < B.$$ 

$F$ is clearly injective and holomorphic.

Since $\Delta$ only changes the $\mathbb{Z}[\Gamma_4]$ - basis of $H_1(\Lambda, \mathbb{Z})$, $T(x,y)$ remains the same, only its coordinates in $\mathbb{B}^\mathbf{2}$ change $\Rightarrow \Delta$ is in a natural way a subgroup of the modular group for the family of abelian varieties considered. This modular group $\Gamma$ is always arithmetic.

\[\text{V. Singularities.}\]

$\mathbb{B} = \psi Q =$ images of "stable singular points" under (a continuous extension of) $\psi$

e.g. of $y = 0$ ($\mu_0 + \mu_3 = \frac{10}{12} < 1$)

= locally finite union of analytic subsets of $\mathbb{B}$ of codimension $\geq 1$

components of $F$ holomorphic and bounded outside $\Rightarrow$ singularities removable (Riemann).

Behaviour of $T(x,y)$ in the characteristic surfaces:

In $y = 0$

$$\omega = u^{\frac{-10}{12}} (u-1)^{-\frac{5}{12}} (u-x)^{-\frac{2}{3}} du$$

same procedure as before leads to a family $T(x)$ of abelian varieties with CM by $\mathbb{Q}(\zeta_{142})$ and
of type $1 \cdot \mathfrak{S}_1 + 2 \cdot \mathfrak{S}_5 + 1 \cdot \mathfrak{S}_4$ and $\dim = 4$

( cf(d) in general ) belonging to Gauss hyper-

geometric functions with arithmetic (!) monodromy

group $A_{y=0}$ of signature $[3, 4, 12]$ )

$\Rightarrow$ On $y = 0$, $T(x, y) = T(x) \oplus A_{y=0}$

with a constant p.p. abelian variety with

$CH$ by $\mathcal{O}(b_{12})$ and type $1 \cdot \mathfrak{S}_5 + 1 \cdot \mathfrak{S}_4$

(in the narrow sense of [Shimura - Taniyama] )

and periods of first kind

$B(\mu_0, \mu_3) = B(\frac{3}{12}, \frac{5}{12})$ and $B(-5\mu_0, -5\mu_3) = B(\frac{5}{12}, \frac{9}{12})$

In $(x, y) = (1, 0)$ $T(x, y)$ splits into three abelian

varieties of $CH$-type.

Shimura $\Rightarrow$ Their periods occur in the Taylor

expansions of suitably normalized $\Gamma$-automorphic

functions

$\Rightarrow$ ... $F$ defined over $\overline{\mathbb{Q}}$.

$x = 0$ is non-stable ($\mu_0 + \mu_2 = \frac{43}{42} > 1$)

$\psi$ blows down $x = 0$ to a $\Delta$-orbit of points in $B$

$\Rightarrow T(x, y) \cong A \oplus A' \oplus A'$, all of $CH$ type

$\Rightarrow F_4(0, y)$ is an algebraic hypergeometric function;

its monodromy group $A_{x=0}$ (tetrahedral) in the

fixgroup in $\Delta$ for $\psi(0, y)$. 
Literature

Paula Cohen, Jürgen Wolfart:

Modular embeddings for some non-arithmetic Fuchsian groups,
Acta Arithmetica 56 (1990), 93 - 110
(N = 1 case)

- "- : Fonctions hypergéométriques en
plusieurs variables et espaces de modules de
variétés abéliennes, preprint.
(32 p., not in final form, but copies can be made)

- "- : Monodromie des fonctions d'Appell,
variétés abéliennes et plongement modulaire,
preprint MPI Bonn 1989 - 80, to appear in
the proceedings of the Journées Arithmétiques
Luminy 1989 in Astérisque.
(6 p., very short form)

- "- : Algebraic Appell-Lauricella Functions,
to appear in the proceedings of the Katata workshop.